RESONANCES AND SCATTERING IN THE CLASSICAL LIMIT

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We consider the Schrödinger operator on $L^2(R^+)$

$$H = -\hbar^2 \frac{d^2}{dx^2} + V(x)$$

with Dirichlet boundary condition at $x = 0$ when V has the form depicted in fig. 1 ; more precisely we assume :

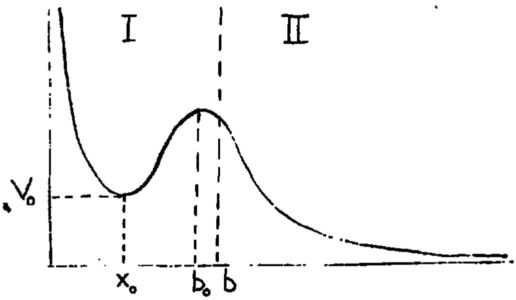
A) 1) $V \in C^2(R^+ \setminus \{0\})$

2) $V \geqslant 0$

3) $\lim_{x \to \infty} V(x) = 0$

4) $\exists ! x_0 , 0 < x_0$ such that

$V'(x_0) = 0 , V''(x_0) > 0.$



It is a common belief that the well around x_0 will be responsible for long lifetime resonances as $\hbar \to 0$. We have started our investigations on this problem in ([1] , [2] , [3]) where additional information can be found. We also refer to G. Jona-Lasinio, F. Martinelli and E. Scoppola [4] for a different approach using stochastic methods,

H. Baümgartel [5], M.S. Asbaugh, E.R. Harrell II ([19]), and R. Lavine [6] for a very promising approach using the local spectral density concept. On an elementary mathematical level this can be understood by adding a fictitious Dirichlet boundary condition at some point b separating the well (region I) from the tail of V (region II) ; for example one could take $b = b_0$ where V reaches its maximum on $(0,\infty)$. Then one gets a new operator H_D having the direct sum form $H_D = H_D^I \oplus H_D^{II}$. The part H_D^I has pure point spectrum ; the lowest eigenvalues have the well-known asymptotic behaviour (see e.g. [7], [8], [9]).

$$1) \qquad E_D^{(n)} = V_0 + \hbar(n+\tfrac{1}{2})\sqrt{V''(x_0)} + O(\hbar^2), n = 0,1,..$$

The operator H_D^{II} has absolutely continuous spectrum $(0,\infty)$ so that H_D has mixed spectrum with all eigenvalues embedded in the continuum. One expects that as the Dirichlet perturbation is removed these eigenvalues will disappear but that some of them will turn into resonances. A perturbative description of this phenomenon requires at first a suitable operator setting for Dirichlet perturbations. Instead of Green's formula we found it more convenient to use a related form of it due to Krein [10] :

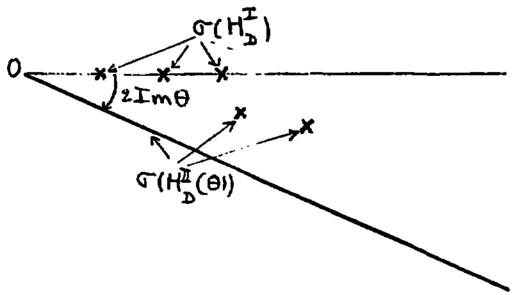
$$2) \qquad (H-a)^{-1} - (H_D-a)^{-1} = \lambda(a)\ P(a)$$

where P(a) is a rank one operator with kernel F(x,b;a)F(b,y;a), with F(x,y;a) the Green's function of H, and $\lambda(a) = F(b,b;a)^{-1}$. For a mathematical description of resonances it is convenient, in the situation analyzed here, to use the notion of exterior complex scaling [12]. Consider the mapping :

$$x \to \begin{cases} x, 0 \leq x \leq b \\ b + e^{\theta}(x-b) \ , \ x > b \end{cases} \quad , \theta \in R$$

It induces a unitary mapping $\mathcal{U}_\theta$ on $L^2(\mathbb{R}^+)$; the operator valued function $H(\theta) = \mathcal{U}_\theta H \mathcal{U}_\theta^{-1}$ is analytic in the strip $S_\alpha = \{\theta / |\mathrm{Im}\,\theta| < \alpha\}$ under the condition :

B) V restricted to region II is dilation analytic in S_α ([13])
Then the statement about analyticity of $H(\theta)$ follows from the fact that $H_D(\theta)$ obviously has such a property and from the extension of 2) to complex values of θ (see also [11]). The spectrum of $H_D(\theta)$ is the union of the spectra of H_D^I and $H_D^{II}(\theta)$ (fig. 2)



So it consists of a set of isolated real eigenvalues which are those of H_D^I , a continuum $e^{-2\mathrm{Im}\,\theta}\,\mathbb{R}^+$ and possibly some complex eigenvalues in the sector $\{z, -2\mathrm{Im}\,\theta \leqslant \mathrm{Arg}\, z \leqslant 0\}$. Such eigenvalues correspond to resonances due to the tail of the potential V in region II. In order not to have to deal with the complications of degenerate perturbation theory we will impose some condition or V in order that such resonances are not "too close" to $\mathbb{R}^+$ in the classical limit ; in contrast the "shape resonances" arising from thc perturbation of the real eigenvalues of H_D^I have exponentially small imaginary parts as expected from the order of magnitude of the tunnel effect and as the theorem below shows. The condition on V is essentially a non-trapping condition on its tail which reads as follows :

C) $\exists\ \delta(b)\ ,\ 0 < \delta(b) \leq 2$ such that

$\delta(b)\ V(x) + xV'(x) \leq 0$ for all $x > b$

A condition of this genre is known to imply absence of positive energy bound states ([14]). Also for $\delta = 2$ the quantity on the l.h.s. of the inequality is proportional to the virial of V ; then condition C) implies a negative time delay which already strongly indicates absence of close resonances. It turns out that if V in region II is positive the condition C) is a stronger requirement which jointly with B) implies that the numerical range of $H_D^{II}(\theta)$, $\mathrm{Im}\,\theta > 0$, lies outside a sector around the positive real axis of the form :

$$S_h = \left\{ z \,\middle|\, - c h^{\frac{7}{2}} < \mathrm{Arg}\, z < 0 \right\}$$

This last result (not optimal in general) opens the way to a perturbative treatment of Eq. 2) through the Weinstein-Aronsjan determinant formula. Let $e_D^{(n)} = (E_D^{(n)} - a)^{-1}$ be an eigenvalue of $(H_D^I - a)^{-1}$; then the corresponding perturbed eigenvalue of $(H-a)^{-1}$ satisfies the equation

$$3) \qquad 1 + \lambda(a)\ \mathrm{Trace}\ (P(a)(r^D - z)^{-1}) = 0$$

where $r_D = (H_D - a)^{-1}$.

Existence of a solution in a complex neighbourhood of $e_D^{(n)}$ is provided by Rouché's theorem and Lagrange's formula using the analyticity properties induced by assumption B). Our main result is the following :

<u>Theorem</u>

Let V satisfy conditions A, B and C and let $E_D^{(n)}$ be the n^{th} eigenvalue of H_D^I, $P_D^{(n)}$ the corresponding spectral projection operator. Then there exists a complex number θ with $0 < \mathrm{Im}\,\theta < \alpha$ such that $H(\theta)$ has an eigenvalue $E^{(n)}$ given by the convergent series

$$E^{(n)} = E_D^{(n)} + \sum_{p=1}^{\infty} \frac{t^p}{p!} C_p$$

for $\hbar$ small enough where $t = \lambda(a)$ Trace $(P(a)\, P_D^{(n)})$ and the C_p's are h dependent. The C_p's are polynomialy bounded in $\hbar^{-1}$ and the "Tunneling parameter" t obeys (see [3]) :

$$t = 0(\exp - 2\beta h^{-1} \int_{x_0}^{b} \sqrt{V(s)}\, ds \quad (\beta < 1)$$

The proof of this theorem requires in particular estimates on the boundary values of $(H_D^{II} - z)^{-1}$ as $\mathrm{Im}z \to 0$. For $\hbar \neq 0$ existence of such boundary values as bounded mappings between suitable weighted Sobolev spaces is a well-known problem of Scattering Theory (see eg. [15]). Obviously such boundary values, whose existence requires some type of ellipticity for H, don't exist at $\hbar = 0$. It has been shown by D. Robert and H. Tamura [16], in connection with the problem of semi-classical asymptotics for the scattering phase, that such boundary values behave like $\hbar^{-1}$(i.e. as for V=0) provided some non-trapping condition is satisfied. To deal with this difficulty we have adopted a variant of a commutator method due to R. Lavine [17] and further developped by E. Mourre [18]. It allows a rather detailed qualitative analysis of the local as well as global properties for solutions of $(H_D^{II} - z)\, u = \psi$ (in particular of scattering solutions when $\mathrm{Im}z \to 0$). The basic equation reads :

$$\langle u,\ f(E-V)' \cdot u \rangle + h^2 \langle u', f' u' \rangle = 2\mathrm{Re} \langle f u', (H-E)u \rangle$$

where $E = \mathrm{Re}z$, $\mathrm{Im}z \neq 0$ and f is an arbitrary C^1 function with $f' \geqslant 0$.

This allows to get estimates on $\| f'^{1/2} u \|$ if e.g.

$$4) \qquad 2f'(E-V) - fV' \geq \alpha f' \text{ for some } \alpha > 0$$

It can be shown that under assumption C there exists an f such that 4) holds for all positive E. From this one can recover a more general form of the $O(\hbar^{-1})$ behaviour result of Robert and Tamura for the boundary values of $(H_D^{II} - z)^{-1}$ and show that the estimates on such boundary values are in fact improved with respect to the situation with V = 0 ! this is another indication that the non-trapping potentials satisfying C) accelerate particles.

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