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# On the Laurent polynomial rings

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ABSTRACT : We describe some properties of the Laurent polynomial rings in a finite number of indeterminates over a commutative unitary ring. We study some sub-rings of the Laurent polynomial rings. We finally obtain two cancellation properties.

A ring $R = A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$, where A is a commutative unitary ring and $X_1, \dots, X_n$ are indeterminates over A, is called a <u>Laurent polynomial ring</u> (sometimes it is called a torus extension). A Laurent polynomial ring can be realized as the ring of quotients of an usual polynomial ring with respect to the multiplicative system generated by the indeterminates $X_1, \dots, X_n$. Therefore it is natural to study what properties of the polynomial rings can be extended to the Laurent polynomial rings. We also shall obtain some cancellation properties of these rings.

<u>1.Lemma</u>. <u>Let A be a domain and $X_1, \dots, X_n$ indeterminates over A. If</u> $f \in A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ <u>is invertible in</u> $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ <u>then there are</u> $a \in A$ <u>invertible in A and</u> $a_1, \dots, a_n \in \underline{Z}$ <u>such that</u> $f = aX_1^{a_1} \dots X_n^{a_n}$.

<u>Proof</u>. Let's first remark that it suffices to prove the lemma for $n = 1$. Let $X = X_1$ and let $f, g \in A[X, X^{-1}]$ be such that $fg = 1$. Since 1 is homogeneous in $A[X, X^{-1}]$ it follows that $f, g$ are homogeneous. Therefore $f = aX^m$, $g = bX^n$, where $a, b \in A$, $m, n \in \underline{Z}$. Hence $ab = 1$ and $n = -m$.

<u>Remark</u>. The proof of the lemma 1 is standard. We included it here because it is usefull for the rest of the paper.

<u>2.Proposition</u>. <u>Let A be a domain and X an indeterminate over A. Then</u> $\mathrm{Aut}_A(A[X, X^{-1}]) = \{ t : A[X, X^{-1}] \longrightarrow A[X, X^{-1}] \;;\; t(X) = aX^m$, <u>where a is invertible in A and</u> $m \in \{-1, 1\} \}$ .

<u>Proof</u>. Let $t \in \mathrm{Aut}_A(A[X, X^{-1}])$ and let $s = t^{-1}$. We may suppose that

$$t(X) = \frac{a_0+\ldots+a_m X^m}{X^u}\ , \quad s(X) = \frac{b_0+\ldots+b_n X^n}{X^v}\ ,$$

where $a_i, b_j \in A$, $m,n,u,v \in \underline{N}$, $a_m \neq 0$, $b_n \neq 0$. Therefore

$$X.t(X)^v = b_0 + b_1 t(X) + \ldots + b_n t(X)^n \qquad (1)$$

If $\deg t(X) \geqslant 0$ from (1) it follows that $\deg s.\deg t = 1$. Therefore $\deg s = \deg t = 1$. If $\mathrm{ord}\ t = 1$ then $t(X) = aX$ and from (1) it follows that $s(X) = bX$ and $ab = 1$ $(a,b \in A)$. If $\mathrm{ord}\ t \leqslant 0$ from (1) it would follow that $1 = (n-v)\mathrm{ord}\ t$, which is impossible.

If $\deg t < 0$ from (1) it follows that $1+v.\mathrm{ord}\ t = n.\mathrm{ord}\ t$. Therefore $1 = (n-v).\mathrm{ord}\ t$, hence $\mathrm{ord}\ t = -1$. Since $\deg t < 0$ it follows that $\mathrm{ord}\ t = \deg t = -1$. Then $t(X) = \frac{a}{X}$ , where $a \in A \setminus \{0\}$. From $t(s(X)) = X$ it follows that $a$ is invertible in $A$.

*3. Lemma.* *Let A be a commutative unitary ring, X an indeterminate over A and* $Y \in A[X,X^{-1}]$ *such that* $\frac{A[X,X^{-1}]}{\underline{N}(A[X,X^{-1}])} = \frac{A[Y,Y^{-1}]}{\underline{N}(A[Y,Y^{-1}])}$ . *Then* $A[X,X^{-1}] = A[Y,Y^{-1}]$ *and* $t \in \mathrm{Aut}_A(A[X,X^{-1}])$, *where* $t(X) = Y$.

*Proof.* Let $a,b \in A[Y,Y^{-1}]$, $u_j,v_j \in A[X,X^{-1}]$, $c_j,d_j \in \underline{N}(A)$ such that

$$X = a(Y) + \sum_{j=1}^{s} c_j u_j(X)\ ,$$

$$X^{-1} = b(Y) + \sum_{j=1}^{t} d_j v_j(X).$$

Let $\underline{n}$ be the ideal generated by $c_1,\ldots,c_s,d_1,\ldots,d_t$ in A. Since $\underline{n}$ is a nilpotent ideal and $A[X,X^{-1}] \subset A[Y,Y^{-1}] + \underline{n}A[X,X^{-1}]$ it follows that $A[X,X^{-1}] = A[Y,Y^{-1}]$.

It remains to prove that Y is an indeterminate over A. Let $a_0,\ldots,a_m \in A$ be such that $a_0+a_1Y+\ldots+a_mY^m = 0$. Let B be the subring of A generated by $1_A, a_0,\ldots,a_m$, the coefficients of Y in $A[X,X^{-1}]$ and the coefficients of X in $A[Y,Y^{-1}]$. Let $t\colon B[X,X^{-1}] \longrightarrow B[X,X^{-1}]$ be the B-morphism defined by $t(X) = Y$. Since B is noetherian then $B[X,X^{-1}]$ is also noetherian and the surjection t is also an injection.

4. <u>Proposition</u>. <u>Let A be a commutative unitary ring and X an indeterminate over A. If</u> $t \in \mathrm{Aut}_A(A[X,X^{-1}])$ <u>then</u>

$$t(X) = \sum_{i=-m}^{n} a_iX^i,$$

<u>where</u> $a_i \in \underline{N}(A)$ <u>for</u> $i \neq s$, $a_s$ <u>is invertible in A and</u> $s = \pm 1$.

<u>Proof</u>. Let $A^* = \dfrac{A}{\underline{N}(A)}$ and let's consider the canonical map $u\colon \mathrm{Aut}_A(A[X,X^{-1}]) \longrightarrow \mathrm{Aut}_{A^*}(A^*[X,X^{-1}])$. If $s \in \mathrm{Aut}_{A^*}(A^*[X,X^{-1}])$ let $y \in A[X,X^{-1}]$ be such that $y \bmod \underline{N}(A) = s(X)$. Therefore $\dfrac{A[X,X^{-1}]}{\underline{N}(A[X,X^{-1}])} = \dfrac{A[y,y^{-1}]}{\underline{N}(A[y,y^{-1}])}$. It follows that there is $g \in \mathrm{Aut}_A(A[X,X^{-1}])$ such that $g(X) = y$. Then $u(g) = s$, so the map u is onto.

Let's remark that $\ker u = \{n \in \mathrm{Aut}_A(A[X,X^{-1}]);\ n(X) = a_{-m}X^{-m}+\ldots+a_0+a_1X+\ldots+a_nX^n,\ a_i \in \underline{N}(A)\}$. Since u is onto it follows that $Aut_{A^*}(A^*[X,X^{-1}]) \simeq \dfrac{Aut_A(A[X,X^{-1}])}{\ker u}$. Therefore, if $t \in Aut_A(A[X,X^{-1}])$ then $t(X) = g(X) + n(X)$, where $g \in Aut_{A^*}(A^*[X,X^{-1}])$ and $n(X) = \sum_{i=-m}^{n} a_i X^i$, $a_i \in \underline{N}(A)$ $(i = -m, -m+1, \ldots, n)$.

Remark. The proposition 4 shows that the structure of $Aut_A(A[X,X^{-1}])$ is not far from that of $Aut_A(A[X])$ (cf. with [3]).

5.Lemma. Let A,B be domains, $K = Q(A)$, $A \subset B$. If there is $f \in B \setminus A$ such that $B \subset K[f,f^{-1}]$ then f is invertible in B or f is irreducible in B.

Proof. Let's suppose that $f$ is not invertible in B.

If there are $g,h \in B$ such that

$$f = g.h \tag{1}$$

then $g = \dfrac{g_1}{f^u}$, $h = \dfrac{h_1}{f^v}$, where $g_1, h_1 \in K[f]$, $u,v \in \underline{N}$.

From (1) it follows that

$$g_1 h_1 = f^{1+u+v} \tag{2}$$

Since $f$ is irreducible in $K[f]$ there are $m,n \in \underline{N}$ such that $g_1 = f^m$, $h_1 = f^n$ (modulo the multiplication by an invertible element). Therefore $g = f^{m-u}$, $h = f^{n-v}$. Since $f^{-1} \notin B$ it follows that $m-u \geqslant 0$, $n-v \geqslant 0$. From (1) we deduce that $m-u = 0$ or $n-v = 0$, i.e. g or h is invertible in B. Therefore $f$ is irreducible in B.

6.Proposition. Let A,B be domains, $K = Q(A)$ and $X_1, \ldots, X_n$ indeterminates over A such that $A \subset B \subset A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. If B is a GCD-domain and there is $f \in B \setminus A$ such that $B \subset$

$K[f,f^{-1}]$ <u>and</u> $af^{-1} \notin B$ <u>for every</u> $a \in A \setminus \{0\}$ <u>then</u> $B = A[f]$.

<u>Proof</u>. From the lemma 5 it follows that f is irreducible in B.

Let $b \in B \subset K[f,f^{-1}]$. Then there are $a, a_0, \ldots, a_m \in A$, $a \neq 0$, $t, m \in \underline{N}$ such that

$$abf^t = a_0 + a_1 f + \ldots + a_m f^m \qquad (1)$$

If $t > 0$ then f divides $a_0$ in B. Therefore there is $h \in B \subset K[f,f^{-1}]$ such that

$$a_0 = f.h \qquad (2)$$

Since we may suppose that $b \neq 0$, we may assume that $a_0 \neq 0$ if $t > 0$. Therefore from (2) it would follow that $h = a_0 f^{-1} \in B$, in contradiction with the hypothesis. Therefore $t = 0$ and the relation (1) becomes

$$ab = a_0 + a_1 f + \ldots + a_m f^m \qquad (3)$$

We shall prove by induction over m that in (1), a divides $a_0, a_1, \ldots, a_m$.

If $m = 0$ then $ab = a_0$, therefore $b \in K \cap A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = A$.

Let's prove that "$m \leq s$" implies "$m = s+1$".

Since $t = 0$ it follows that $b \in K[f]$. But $K[f] = K[f+a]$ for every $a \in K$, so we may suppose that f is without constant term. Then there is $c \in A$ such that

$$a_0 = ac \qquad (4)$$

It follows that

$$a(b - c) = f(a_1 + \ldots + a_m f^{m-1}) \qquad (5)$$

Since $f$ is irreducible in $B$ and $B$ is a GCD-domain then there is $d$ $B$ such that

$$b - c = df \tag{6}$$

Then

$$da = a_1 + \dots + a_m f^{m-1} \tag{7}$$

By the induction it follows that $a_i = ac_i$, where $c_1,\dots,c_m \in A$. Therefore

$$a(b - c - c_1 f - \dots - c_m f^m) = 0 \tag{8}$$

Since $a \neq 0$ and $B$ is a domain it follows that $b = c + c_1 f + \dots + c_m f^m \in A[f]$. We deduce that $B = A[f]$.

<u>Remark</u>. If $A \subset B \subset A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$, $B \subset K[f, f^{-1}]$ and there is $a \in A \setminus \{0\}$ such that $af^{-1} \in B$ then it is possible that $B \neq A[f]$. For instance, let $A = \underline{Z}$, $B = \underline{Z}[\frac{2X_1}{X_2}, \frac{X_2}{X_1}]$, $f = \frac{2X_1}{X_2}$. Then $\underline{Z} \subset B \subset \underline{Z}[X_1, X_1^{-1}, X_2, X_2^{-1}]$, $B \subset \underline{Q}[f, f^{-1}]$, $f^{-1} = \frac{X_2}{2X_1} \notin B$, $2f^{-1} = \frac{X_2}{X_1} \in B$ and $B \neq A[f] = \underline{Z}[\frac{2X_1}{X_2}]$.

7. <u>Proposition</u>. <u>Let $A, B$ be domains, $K = Q(A)$ and $X$ an indeterminate over $A$ such that</u> $A \subset B \subset A[X, X^{-1}]$. <u>If $B$ is a GCD-domain and there is</u> $f \in B \setminus A$ <u>such that</u> $B \subset K[f, f^{-1}]$ <u>then</u> $B = A[f]$ <u>or</u> $B = A[aX^m, bX^{-m}]$, <u>where</u> $a, b \in A \setminus \{0\}$, $m \in \underline{N}^*$.

<u>Proof</u>. If $f^{-1} \in B$ then $f = aX^m$, where $a$ is invertible in $A$ and $m \in \underline{Z}$. Since $f \notin A$ we have $m \in \underline{N}^*$. Therefore

$$A[aX^m, a^{-1}X^{-m}] \subset B \subset K[aX^m, a^{-1}X^{-m}].$$

Since $B \subset A[X, X^{-1}]$ it follows that $B = A[aX^m, a^{-1}X^{-m}]$.

If $f^{-1} \notin B$ from the lemma 5 it follows that $f$ is irreducible in $B$. If $af^{-1} \notin B$ for every $a \in A \setminus \{0\}$ then $B = A[f]$

by proposition 6. Let's suppose that there is $\alpha \in A \setminus \{0\}$ such that $\alpha f^{-1} \in B$. We reexamine under this hypothesis the relations (1) and (2) from the proof of the proposition 6. If $B \neq A[f]$ then we may suppose that $a_o \neq 0$, $h \in B$ and

$$a_o = f.h \qquad (2)$$

Since $a_o$ is homogeneous it follows that $f,h$ are homogeneous in $A[X,X^{-1}]$, hence $f = cX^s$, $h = dX^{-s}$, where $c,d \in A \setminus \{0\}$, $cd = a_o$, $s \in \underline{Z} \setminus \{0\}$. The relation (1) from the proof of the proposition 6 becomes

$$b = \frac{1}{a} \sum_{i=0}^{m} a_i f^{i-t} = \sum_{i=0}^{m} \frac{a_i c^{i-t}}{a} X^{s(i-t)} \qquad (1')$$

But from $b \in A[X,X^{-1}]$ it follows that $\frac{a_i c^{i-t}}{a} \in A$ for every $i = 0,1,\ldots,m$. Therefore we may suppose that there are $c,d \in A \setminus \{0\}$, $s \in \underline{N}^*$ such that

$$A[cX^s,dX^{-s}] \subset B \subset A[X^s,X^{-s}] \qquad (*)$$

Let $e_1X^s$, $e_2X^s \in B \setminus \{0\}$. Then $(e_1,e_2)X^s \in B \setminus \{0\}$, where $(e_1,e_2)$ is the greatest common divisor of $e_1$ and $e_2$ in B.

Let $M = \{b \in B;\ b = e_iX^s,\ e_i \in A \setminus \{0\}\}$, $N = \{b \in B;\ b = f_jX^{-s},\ f_j \in A \setminus \{0\}\}$.

From the above it follows that there are $e,f \in A$ such that $M \subset A[eX^s]$, $N \subset A[fX^{-s}]$. From (*) it follows that

$$A[eX^s,fX^{-s}] \subset B \subset A[eX^s,fX^{-f}] \qquad (**)$$

Hence $B = A[eX^s,fX^{-s}]$.

<u>8.Proposition</u>. <u>Let k be a commutative field</u>, $X_1,\ldots,X_n$ <u>indeterminates over</u> k <u>and</u> A <u>a domain such that</u> $k \subset A \subset k[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$. <u>If</u> $\dim A = 1$ <u>then there is an indeterminate</u> Y

over k such that $k \subset A \subset k[X,X^{-1}]$.

Proof. Let $R = k[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$ and let $\underline{p} = (X_1-1, \ldots ,X_n-1)R \cap A$. Because $(X_1-1,\ldots,X_n-1)R$ is a maximal ideal in R it follows that $\underline{p} \in \mathrm{Spec}\, A$. The rest of the proof is similar to that of 5.6 from [1].

9.Corollary. Let k be a commutative field, $X_1,\ldots,X_n$ indeterminates over k and A a GCD-domain such that $k \subset A \subset k[X_1, X_1^{-1},\ldots,X_n,X_n^{-1}]$. If dim A = 1 then A is isomorphic to $k[X_1]$ or to $k[X_1,X_1^{-1}]$.

Proof. By proposition 8 there is an indeterminate X over k such that $k \subset A \subset k[X,X^{-1}]$. From proposition 7 it follows that $A = k[X]$ or $A = k[aX^s,bX^{-s}]$, where $a,b \in k \setminus \{0\}$ , $s \in \underline{N}^*$. Since $k[X^s,X^{-s}] = k[aX^s,bX^{-s}]$ it follows that A is isomorphic to $k[X_1]$ or to $k[X_1,X_1^{-1}]$.

Remark. If k is algebraically closed, A a normal affine ring of dimension 1 and n = 1 our corollary 9 follows also from 4.1 of [8].

10.Proposition. Let A be a domain, X an indeterminate over A and B an UFD such that $A \subset B \subset A[X,X^{-1}]$. Then B = A or B is isomorphic to $A[X]$ or to $A[cX,dX^{-1}]$, where $c,d \in A \setminus \{0\}$ .

Proof. Let's suppose that $A \subsetneq B$. Let K = Q(A). Therefore

$$K \subset K \otimes_A B \subset K[X,X^{-1}].$$

Since $K \otimes_A B$ is a GCD-domain from the corollary 9 it follows that there is $f \in K \otimes_A B \setminus K$ such that $K \otimes_A B = K[f]$ or $K \otimes_A B = K[f,f^{-1}]$.

Let $S = A \setminus \{0\}$ . Then $K = S^{-1}A$.

Let's suppose that $K\otimes_A B = K[f]$. Because $K[f] = K[sf]$ for every $s$ $S$ we may suppose that $f \in A[X,X^{-1}]$. Therefore $f \in A[X,X^{-1}] \cap (K\otimes_A B) = A[X,X^{-1}] \cap S^{-1}B = B$, hence $f \in B$. Since $K\otimes_A B = K[f]$ it follows that $B$ $K[f]$. If $f^{-1} \notin B$ it follows that $f$ is irreducible in B. From the proof of the proposition 6 it follows that $B = A[f]$, hence B is isomorphic to $A[X]$.

If $f^{-1} \in B$ then $B \subset K[f,f^{-1}]$.

Let's consider now the case $K\otimes_A B = K[f,f^{-1}]$. Because $K[f,f^{-1}] = K[sf,(sf)^{-1}]$ for every $s \in S$, we may suppose that $f \in A[X,X^{-1}]$. Therefore $f \in A[X,X^{-1}] \cap (K\otimes_A B) = B$, hence $f \in B$. Since $K\otimes_A B = K[f,f^{-1}]$ then $B \subset K[f,f^{-1}]$. Therefore $B = A[f]$ (and B is isomorphic to $A[X]$) or there are $c,d \in A \setminus \{0\}$, $m \in \mathbb{N}^*$ such that $B = A[cX^m,dX^{-m}]$, hence B is isomorphic to $A[cX,dX^{-1}]$.

<u>11.Corollary</u>. <u>Let A be a domain, X an indeterminate over A and B a noetherian UFD such that</u> $A \subset B \subset A[X,X^{-1}]$. <u>Then A is a UFD</u>.

<u>Proof</u>. It suffices to remark that B is a graduate ring, $B = \sum_{i \in \mathbb{Z}} B_i$, where $B_0 = A$.

<u>12.Proposition</u>. <u>Let A be a domain</u>, $X_1,\ldots,X_n$ <u>indeterminates over A and B a UFD. If</u> $A \subset B \subset A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$ <u>and</u> $\mathrm{tr.deg.}_A B \leq 1$ <u>then</u> $B = A$ <u>or B is isomorphic to</u> $A[X]$ <u>or to</u> $A[cX,dX^{-1}]$, <u>where X is an indeterminate over A</u>, $c,d \in A \setminus \{0\}$ .

<u>Proof</u>.

Let $K = Q(A)$. Then

$K \subset K\otimes_A B \subset K[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$.

If $\mathrm{tr.deg.}_A B = 0$ then $\dim(K\otimes_A B) \leq \mathrm{tr.deg.}_K(K\otimes_A B) = 0$.

Hence $B \subset K$, therefore $B = A$.

Let's suppose that $\text{tr.deg.}_A B = 1$. Then $\dim(K\otimes_A B) \leq 1$. If $\dim(K\otimes_A B) = 0$ then $B = A$. Therefore we may suppose that $\dim(K\otimes_A B) = 1$. From the corollary it follows that $K\otimes_A B$ is isomorphic to $K[X_1]$ or to $K[X_1, X_1^{-1}]$. Then there is an indeterminate $Y$ over $K$ such that $K\otimes_A B = K[Y]$ or $K\otimes_A B = K[Y, Y^{-1}]$.

If $K\otimes_A B = K[Y]$ then $B \subset K[Y]$, hence $B$ is isomorphic to $B[Y]$.

If $K\otimes_A B = K[Y, Y^{-1}]$ then $B \subset K[Y, Y^{-1}]$, hence $B$ is isomorphic to $A[Y]$ or to $A[cY, dY^{-1}]$, where $c, d \in A \setminus \{0\}$ .

13. Proposition. Let $A$ be a UFD, $A \subset B$ a ring extension, $X, Y, Z$ indeterminates over $A$ and $T$ an indeterminate over $B$. If $B[T, T^{-1}] = A[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]$ and $A[X, X^{-1}]$ $B$ then $B$ is isomorphic to one of the following rings: $A[X, X^{-1}]$, $A[X, X^{-1}, Y]$, $A[X, X^{-1}, cY, dY^{-1}]$, where $c, d \in A \setminus \{0\}$ .

Proof. Since $A$ is a UFD it follows that $B[T, T^{-1}] = A[X, X^{-1}, Y, Y^{-1}, Z, Z^{-1}]$ is a UFD, hence $B$ is a UFD.

We may suppose that $Z \notin B$. Indeed, if $Y, Z \in B$ then

$$B[T, T^{-1}] = A[X, X^{-1}, Y, Z][Y^{-1}, Z^{-1}] \subset B[Y^{-1}, Z^{-1}] = S^{-1}B \quad (1)$$

where $S$ is the multiplicative system generated by $Y, Z$.

From (1) it would follow that $1 + \dim B \leq \dim B[T, T^{-1}] \leq \dim S^{-1}B \leq \dim B$, a contradiction.

Since $Z \notin B$ it follows that

$$B \cap (Z - 1)B[T, T^{-1}] = (0) \quad (2)$$

Indeed, let $u \in B \cap (Z-1)B[T, T^{-1}]$. Because $Z-1 \in B[T, T^{-1}] \setminus \{0\}$ there is $h(T) \in B[T, T^{-1}] \setminus \{0\}$ such that $Z-1 = h(T)$. Therefore there is $g(T) \in B[T, T^{-1}]$ such that $u = h(T)g(T)$. If

$u \neq 0$ then $g(T) \neq 0$. Since $u \in B$ it would follow that $h(T) = b_1T^m$, $g(T) = b_2T^{-m}$, where $b_1, b_2 \in B \setminus \{0\}$, $m \in \underline{Z}$. Therefore $Z-1 = b_1T^m$. But $Z$ is homogeneous in $B[T,T^{-1}]$. Therefore $m = 0$, hence $Z \in B$, a contradiction. Therefore $u = 0$ and (2) is proved.

It follows that we have the following ring extension

$$B \subset \frac{B[T,T^{-1}]}{(Z-1)B[T,T^{-1}]} = \frac{A[X,X^{-1},Y,Y^{-1},Z,Z^{-1}]}{(Z-1)A[X,X^{-1},Y,Y^{-1},Z,Z^{-1}]} = A[X,X^{-1},Y,Y^{-1}].$$

Therefore

$$A[X,X^{-1}] \subset B \subset A[X,X^{-1}][Y,Y^{-1}] \qquad (3)$$

Since $B$ is a UFD the proposition results from (3) and the proposition 10.

<u>Remarks</u>. i) If in the proposition 13 the ring $A$ is noetherian then $B$ is isomorphic to $A[Y,X,X^{-1}]$ or to $A[X,X^{-1},cY,dY^{-1}]$, where $c,d \in A \setminus \{0\}$.

ii) The proposition 13 is similar to an analogous result on polynomial rings.([9]).

<u>Definition.</u> A commutative unitary ring $A$ is called <u>strongly n-torus invariant</u> if from the equality of Laurent polynomial rings $A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}] = B[Y_1,Y_1^{-1},\ldots,Y_n,Y_n^{-1}]$ it follows that $A = B$.

We shall give two sufficient conditions for the strongly torus invariance.

For the first result we need the following lemma.

<u>14. Lemma</u>. <u>Let $A,B$ be commutative domains</u>, $X_1,\ldots,X_n$ <u>indeterminates over $A$ and</u> $Y_1,\ldots,Y_n$ <u>indeterminates over $B$. If</u> $A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}] = B[Y_1,Y_1^{-1},\ldots,Y_n,Y_n^{-1}]$ <u>then there are ho-</u>

*mogeneous forms* $u_{ij} \in A[X_1, X_1^{-1}, \ldots, \widehat{X_i, X_i^{-1}}, \ldots, X_n, X_n^{-1}]$, $v_{ij} \in B[Y_1, Y_1^{-1}, \ldots, \widehat{Y_j, Y_j^{-1}}, \ldots, Y_n, Y_n^{-1}]$ *and* $e_{ij}, f_{ij} \in \mathbb{Z}$ *such that* $X_i = v_{ij} Y_j^{e_{ij}}$, $Y_j = u_{ij} X_i^{f_{ij}}$.

*Proof*. Let $A_i = A[X_1, X_1^{-1}, \ldots, \widehat{X_i, X_i^{-1}}, \ldots, X_n, X_n^{-1}]$, $B_j = B[Y_1, Y_1^{-1}, \ldots, \widehat{Y_j, Y_j^{-1}}, \ldots, Y_n, Y_n^{-1}]$. Then $A_i[X_i, X_i^{-1}] = B_j[Y_j, Y_j^{-1}]$ and we apply lemma 1.

*15. Proposition. Let A, B be commutative domains,* $X_1, \ldots, X_n$ *be indeterminates over A and* $Y_1, \ldots, Y_n$ *be indeterminates over B. If* $Q(A) \subseteq Q(B)$ *and* $A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$ *then A = B.*

*Proof*. Let's first remark like in the proof of 3.1 of [8] that from the hypothesis it follows that $A \subseteq B$. Indeed, let $x \in A$, hence $x \in Q(B)$. Then $x = \frac{b}{b'}$, where $b, b' \in B$, $b' \neq 0$. Since $b, b'$ are homogeneous of degree zero in $B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$ we have $\deg_{Y_1, \ldots, Y_n} x = 0$, hence $x \in B$. Therefore $A \subseteq B$.

Since $A \subseteq B$ we have $B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}] = A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \subseteq B[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \subseteq B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$, because $X_1, X_1^{-1}, \ldots, X_n, X_n^{-1} \in B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$ by the hypothesis. Therefore

$$B[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}], \text{ hence}$$

$$A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = B[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] = B[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}] \qquad (1)$$

We shall prove by induction over n that A = B.

If n = 1 let $b \in B$, hence $b = \sum a_i Y_1^i$. Since $X_1$ is algebraically independent over B and $b, a_i \in B$ it follows that $b = a_0$, hence $b \in A$. Therefore $B \subset A$, i.e. $B = A$.

Let's suppose the result proved for n-1 indeterminates and let's prove it for n indeterminates. Because $A \subset B$ we have

$$A[X_1,X_1^{-1},\ldots,X_{n-1},X_{n-1}^{-1}] \subset B[X_1,X_1^{-1},\ldots,X_{n-1},X_{n-1}^{-1}].$$

Since $A[X_1,X_1^{-1},\ldots,X_{n-1},X_{n-1}^{-1}][X_n,X_n^{-1}] = B[X_1,X_1^{-1},\ldots,X_{n-1},X_{n-1}^{-1}][X_n,X_n^{-1}]$ it follows that $A[X_1,X_1^{-1},\ldots,X_{n-1},X_{n-1}^{-1}] = B[X_1,X_1^{-1},\ldots,X_{n-1},X_{n-1}^{-1}]$, hence by induction $A = B$.

<u>16.Corollary</u>. <u>Let A be a domain and A' its integral closure. If A' is strongly n-torus invariant then A is strongly n-torus invariant</u>.

<u>Remark</u>. For n= 1 our proposition 15 and corollary 16 are proved in 3.1 and 3.2 from [8]. Our proof of the proposition 15 is diverse from that of 3.1 from [8]. The corollary 16 can be deduced from the proposition 15 like 3.2 is deduced from 3.1 in [8].

<u>17.Theorem</u>. <u>Let k be a commutative infinite field and A a k-algebra which is a domain. If</u> $\mathrm{card}(\mathrm{Aut}_k(A)) < \mathrm{card}(k)$ <u>then A is strongly n-torus invariant</u>.

<u>Proof</u>. Let $R = A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}] = B[Y_1,Y_1^{-1},\ldots,Y_n,Y_n^{-1}]$. For $m \in k \setminus \{0\}$ we define $F_m : R \longrightarrow R$ as follows:

$$F_m(Y_j) = mY_j \ (j = 1,2,\ldots,n), \quad F_m(b) = b, \ b \in B.$$

Let's remark that $F_m \in \mathrm{Aut}(R)$. We also have

$F_m(X_i) = F_m(v_{ij}Y_j^{e_{ij}}) = m^{e_{ij}} . Y_j^{e_{ij}} . F_m(v_{ij})$. Since $v_{ij}$ is homogeneous in $Y_1,\ldots,Y_{j-1},Y_{j+1},\ldots,Y_n$ it follows that we have $F_m(v_{ij}) = m^{\deg v_{ij}} . v_{ij}$. Therefore

$F_m(X_i) = m^{e_{ij}+\deg v_{ij}}.v_{ij}Y_j^{e_{ij}} = m^{\deg_Y X_i} = m^{e_i}.X_i$, where $e_i = \deg_{Y_1,\ldots,Y_n} X_i$. It follows that

$$R = F_m(A)[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}].$$

Let $p\colon A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}] \longrightarrow A$ be defined by $p(X_1) = \ldots = p(X_n) = 1$, $p(a) = a$, $a \in A$ and let $i\colon A \hookrightarrow A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$ be defined by $i(a) = a$, $a \in A$. We define $s_m\colon A \longrightarrow A$ by $s_m = p\circ F_m\circ i$. Therefore $s_m \in \mathrm{End}(A)$.

We shall show that $s_m$ is surjective. Let $a \in A$. Since $a \in A \subset R = F_m(A)[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$ there are $a_{j_1\ldots j_n} \in A$ such that

$$a = \sum_{\text{finite}} F_m(a_{j_1\ldots j_n})X_1^{j_1}\ldots X_n^{j_n}. \text{ Then } a = p(a) = \sum p(F_m(a_{j_1\ldots})).$$

Let $a' = \sum a_{j_1\ldots j_n} \in A$. It follows that $s_m(a') = p(F_m(a')) =$

$$\sum p(F_m(a_{j_1\ldots j_n})) = a,$$ hence $s_m$ is onto.

The map $s_m$ is also injective. For proving this let's remark that $\ker p = (X_1-1,\ldots,X_n-1)R$. But $F_m^{-1}(\ker p \cap F_m(A)) = (m^{-e_1}X_1-1,\ldots,m^{-e_n}X_n-1)R \cap A = (0)$, hence $\ker p \cap F_m(A) = (0)$. Therefore $\ker s_m = (0)$, i.e. $s_m$ is injective.

Let $S = \{s_m ; m \in k \setminus \{0\}\}$. We shall prove that if $A \neq B$ then $\mathrm{card}\, S > \mathrm{card}\, k$, which is in contradiction with the hypothesis.

Let $u'_{ij} = p(u_{ij}) \in A$, $v'_{ij} = p(v_{ij}) \in A$. Since $X_i =$ $v_{ij}Y_j^{e_{ij}} = v_{ij}(u_{ij}X_i^{f_{ij}})^{e_{ij}} = v_{ij}u_{ij}^{e_{ij}}X_i^{f_{ij}e_{ij}}$ it follows that $1 = v'_{ij}u'^{e_{ij}}_{ij}$, hence $u'_{ij} \neq 0$, $v'_{ij} \neq 0$.

Because $F_m(u_{ij}) = F_m(v_{ij}^{-f_{ij}}.Y_j^{1-e_{ij}f_{ij}}) =$ $m^{-f_{ij}\cdot\deg v_{ij}}.v_{ij}^{-f_{ij}}.m^{1-e_{ij}f_{ij}}.Y_j^{1-e_{ij}f_{ij}} = m^{1-e_if_{ij}}.u_{ij}$ it follows that $s_m(u'_{ij}) = m^{1-e_if_{ij}}.u'_{ij}$.

If there is $(i,j)$ such that $1-e_if_{ij} \neq 0$ then card $S$ $\geqslant$ card $k$, in contradiction with the hypothesis.

If $1-e_if_{ij} = 0$ for all $(i,j)$ then $e_if_{ij} = 1$. Hence $Y_j = u_{ij}X_i^{f_i}$, where $f_i = \pm 1$ and $f_i = f_{ij} = e_i$. Therefore $B =$ $A[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}] = B[Y_1,Y_1^{-1},\ldots,Y_n,Y_n^{-1}] = B[u_{11}X_1^{f_1},(u_{11}X_1^{f_1})^{-1},$ $\ldots,u_{nn}X_n^{f_n},(u_{nn}X_n^{f_n})^{-1}]$.

If $A \neq B$ then from the proposition 15 it follows that there is $a \in A \notin B$. Therefore there are $b_{j_1\ldots j_n} \in B$ such that

$$a = \sum b_{j_1\ldots j_n}(u_{11}X_1^{f_1})^{j_1}\ldots(u_{nn}X_n^{f_n})^{j_n} \qquad (1)$$

Since $a \in A \quad B$ we have $a \neq 0$, therefore $s_m(a) \neq 0$ because $s_m$ is injective. Therefore there is $(j_1,\ldots,j_n)$ such that $p(b_{j_1\ldots j_n}) \neq 0$. Since $u'_{11},\ldots,u'_{nn} \neq 0$ and the field $k$ is infinite from (1) and the expression of $s_m(a)$ it follows that card$(S)$ $\geqslant$ card $k$.

Therefore we must have $A = B$ and thus $A$ is strongly n-torus invariant.

<u>18.Corollary</u>. <u>Let $k$ be a commutative field and $A$ a</u>

<u>domain that contains k. If $Aut_k(A)$ is infinite then A is strongly n-torus invariant.</u>

<u>Remark</u>. For n = 1 our corollary 18 follows from 3.8 of [8].

Another sufficient condition for the strongly torus invariance can be obtained with the help of the below defined pseudo-locally nilpotent derivations.

<u>Definition</u>. Let A be a ring. We call a <u>pseudo-locally nilpotent derivation</u> on A an infinite sequence of endomorphisms $D = (D_i)_{i \in \mathbb{Z}}$ defined on the additive group (A,+) such that:

i) For every $a \in A$ there are $N(a), M(a) \in \mathbb{Z}$ such that $D_n(a) = 0$ if $n > N(a)$ or $n < M(a)$.

ii) $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$ for every $a,b \in A, n \in \mathbb{Z}$.

A pseudo-locally nilpotent derivation $D = (D_i)_{i \in \mathbb{Z}}$ is called <u>trivially</u> if $D_0 = id_A$ and $D_i = 0$ for $i \neq 0$.

<u>Remark</u>. The former definition is similar to that of the locally nilpotent derivations from [7].

<u>19.Proposition</u>. <u>Let A be a commutative ring that contains an infinite field. If on A there are no pseudo-locally not trivially nilpotent derivations then A is strongly n-torus invariant for every n.</u>

<u>Proof</u>. Let's suppose that there were a commutative ring $B \neq A$, $X_1,\ldots,X_n$ indeterminates over A and $Y_1,\ldots,Y_n$ indeterminates over B such that

$$A[X_1,X_1^{-1},\dots,X_n,X_n^{-1}] = B[Y_1,Y_1^{-1},\dots,Y_n,Y_n^{-1}].$$

Let $a \in A \setminus B$. Then there is $f(Y_1,\dots,Y_n) \in B[Y_1,Y_1^{-1},\dots,Y_n,Y_n^{-1}] \setminus B$ such that

$$a = f(Y_1,\dots,Y_n).$$

We may suppose that $f(Y_1,\dots,Y_n) = \sum_{j=m_1}^{m_2} g_j(Y_2,\dots,Y_n)Y_1^j$,

where $g_j(Y_2,\dots,Y_n) \in B[Y_2,Y_2^{-1},\dots,Y_n,Y_n^{-1}]$, $m_1 \leq m_2$, $g_{m_1}(Y_2,\dots,Y_n) \neq 0$, $g_{m_2}(Y_2,\dots,Y_n) \neq 0$, $m_1$ or $m_2 \neq 0$.

Let $T$ be a new indeterminate and let's consider the morphism $G: B[Y_1,Y_1^{-1},\dots,Y_n,Y_n^{-1}] \longrightarrow B[Y_1,Y_1^{-1},\dots,Y_n,Y_n^{-1},T,T^{-1}]$ defined as follows: $G(Y_1) = Y_1T$, $G(Y_2) = Y_2,\dots,G(Y_n) = Y_n$. Then

$$G(a) = \sum_{j=m_1}^{m_2} g_j(Y_2,\dots,Y_n)Y_1^jT^j.$$

Let $g_j(Y_2,\dots,Y_n)Y_1^j = h_j(X_1,\dots,X_n) \in A[X_1,X_1^{-1},\dots,X_n,X_n^{-1}]$. Then $h_{m_1}(X_1,\dots,X_n) \neq 0$, $h_{m_2}(X_1,\dots,X_n) \neq 0$.

If $m_2 \neq 0$ let $a_1,\dots,a_n \in A \setminus \{0\}$ be such that $h_{m_2}(a_1,\dots,a_n) \neq 0$ (this choice is possible because $A$ contains an infinite field). If $m_2 = 0$ then $m_1 \neq 0$ (because $m_1 = m_2 = 0$ would imply $a \in B$) and let $a_1,\dots,a_n \in A \setminus \{0\}$ be such that $h_{m_1}(a_1,\dots,a_n) \neq 0$.

Let $t: A[X_1,X_1^{-1},\dots,X_n,X_n^{-1},T,T^{-1}] \longrightarrow A[T,T^{-1}]$ be the $A$-morphism defined as follows: $t(X_1) = a_1,\dots,t(X_n) = a_n$, $t(T) = T$. (We may suppose that $a_1,\dots,a_n$ are non-zero elements from an infinite field included in $A$, Hence that $a_1,\dots,a_n$ are invertible). Let $i: A \hookrightarrow A[X_1,X_1^{-1},\dots,X_n,X_n^{-1},T,T^{-1}]$ be the canonical inclusion and let's consider $f : A \longrightarrow A[T,T^{-1}]$, where $f := t \circ G \circ i$.

Then $f$ is a ring morphism and we have

$$f(a) = \sum_{j=m_1}^{m_2} h_j(a_1,\ldots,a_n)T^j \in A[T,T^{-1}] \setminus A \text{ because } h_{m_2}(a_1,\ldots,a_n)$$

$\neq 0$ or $h_{m_1}(a_1,\ldots,a_n) \neq 0$ ( if $m_2 \neq 0$ or $m_2 = 0, m_1 \neq 0$).

$$\text{Let } f(\alpha) = \sum_{\text{finite}} \alpha_i T^i. \text{ Let's consider } D_i(\alpha) := \alpha_i.$$

Since $f$ is a ring morphism it follows that $(D_i)_{i \in \underline{Z}}$ is a pseudo locally nilpotent derivation on $A$. Because $f(a) \notin A$ it is not trivially. Therefore $B = A$, and $A$ is strongly n-torus invariant for every n.

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