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FAST AND FLEXIBLE VERTEX FIT

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Abstract :

A least squares method is proposed to fit the geometrical parameters of a set of curved tracks assumed to originate in a common vertex : the parameters measured independently for each track are first extrapolated with their weight matrix to a point close to the expected vertex position ; then a local parabolic parametrization of the trajectories is used in a fast fitting procedure, where all parameters (vertex coordinates and track parameters) are modified at each iteration : the global amount of computation is roughly proportional to the number of tracks. Moreover this formalism is well suited to add a track to an existing vertex, or to remove a track from it.

Resumé :

On propose une méthode par moindres carrés pour ajuster les paramètres géométriques d'un ensemble de traces courbées supposées provenir d'un vertex commun : les paramètres mesurés indépendamment pour chaque trace sont d'abord extrapolés avec leur matrice de poids jusqu'à un point proche du vertex prévu ; ensuite une procédure rapide d'ajustement, avec une paramétrisation parabolique des trajectoires, modifie tous les paramètres (position du vertex et paramètres des traces) à chaque itération ; le volume global de calcul est à peu près proportionnel au nombre total de traces. De plus ce formalisme permet aisément d'ajouter ou de retrancher une trace à un vertex existant.

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1. Introduction :

Vertex fitting procedures are used to obtain a precise determination of the momenta of charged particles (or neutral ones decaying in charged mode), and possibly a discrimination between different possible topologies. They are applied to a set of tracks previously individually fitted ; the information on each track is summarized by geometrical parameters (5 for 3 D tracks curved in a magnetic field) and their weight matrix.

The simplest approach is to search for the vertex as the point closest to all trajectories, according to their weights, and then to determine the direction cosines of each particle at the point on its trajectory closest to that vertex. To improve the accuracy on the track parameters (especially the curvature), each trajectory can be fitted again, including the vertex as an additional point ; however this is not quite optimal, because the vertex thus defined depends partly on the points measured on this track, so it is correlated to them : the correlation is negligible if the weight of this track is small w.r.t. the total weight of all tracks involved.

In sect. 2 we propose a method to perform an optimal fit of globally all parameters : vertex coordinates and 3-momenta of all particle. Rather than a constrained fit (difficult to implement) we consider a parametric fit, where the trajectories are determined by 3 general parameters (vertex coordinates) and 3 particular parameters for each track (e.g : two direction cosines and the curvature, or the components of the momentum). This "hierarchical" parametrization (cf. ref. 1) allows a fast resolution of the linear system to be solved at each iteration, using operations on (3x3) matrices ; the number of elementary operations is merely proportional to the number of tracks.

Moreover we show in sect. 3 that this algorithm is well suited to the addition of a track to a vertex already fitted with other tracks, or to the subtraction of a track from a vertex : such an operation is simplified by linear approximation of the variables as functions of the parameters.

2. Global fitting algorithm :

2.1. Formalism :

Let us consider a vertex with n tracks ($n \geq 2$). The track i was initially described by 5 parameters $q_{i_1}\, q_{i_2} \cdots q_{i_5}$ and their weight matrix W_i. The q_{i_j} are now variables depending on the parameters X, Y, Z (coord. of the vertex) and $p_{i_1}, p_{i_2}, p_{i_3}$ (defining the track i at the vertex) :

$$q_{i_j}^{R} = F_j\,(X, Y, Z,\ p_{i_1}, p_{i_2}, p_{i_3})$$

We want to find $V = (X, Y, Z)$ and the p_{i_j} which minimize the χ^2 :

$$\chi^2 = \sum_i \sum_{j,k} (W_i)_{jk} \underbrace{\left[q_{i_j} - F_j(V, p_i)\right]}_{\Delta q_{i_j}} \underbrace{\left[q_{i_k} - F_k(V, p_i)\right]}_{\Delta q_{i_k}}$$

or, with matrix notations :

$$\chi^2 = \sum_i \Delta q_i^{\,t}\, W_i\, \Delta q_i$$

To do this, we linearize F around starting values of the parameters :

i.e. $F\,(V + \delta V,\ p_i + \delta p_i) = F\,(V, p_i) + D_i\,\delta V + E_i\,\delta p_i$

The equations for the minimum are then :

$$\left(\sum_i D_i^t W_i D_i\right)\delta V + \sum_i (D_i^t W_i E_i)\,\delta p_i = \sum_i D_i^t W_i \Delta q_i \qquad (2.1)$$

and, for each i : $(E_i^t W_i D_i)\,\delta V + (E_i^t W_i E_i)\,\delta p_i = \sum_i E_i^t W_i \Delta q_i \qquad (2.2)$

where : $\delta V = (\delta X, \delta Y, \delta Z)$ and $\delta p_i = (\delta p_{i_1}, \delta p_{i_2}, \delta p_{i_3})$

are the variations of the parameters (*unknowns to be calculated*).

. D_i is the matrix of derivatives of q_i w.r.t. V

$$(D_i)_{jk} = \frac{\partial F_i}{\partial V_k}(V, p_i)$$

. E_i is the matrix of derivatives of q_i w.r.t. p_i

$$(E_i)_{jk} = \frac{\partial F_i}{\partial p_{ik}}(V, p_i)$$

So we obtain 3 equations involving all parameters :

$$A\ \delta V + \sum_i B_i\ \delta p_i = T \qquad (2.1')$$

with $A = \sum_i D_i^t W_i D_i$ and $B_i = D_i^t W_i E_i$

and 3 equations involving X, Y, Z and $p_{i_1}, p_{i_2}, p_{i_3}$ for each of the n values of i :

$$B_i^t\ \delta V + C_i\ \delta p_i = U_i \qquad (2.2')$$

Eq. (2.2') gives δp_i as a function of δV :

$$\delta p_i = C_i^{-1}\left(U_i - B_i^t\ \delta V\right) \qquad (2.3)$$

with these expressions, (2.1') becomes :

$$\left(A - \sum_i B_i C_i^{-1} B_i^t\right)\ \delta V = T - \sum_i B_i C_i^{-1} U_i \qquad (2.4)$$

This is a system of 3 equations giving δX, δY, δZ, hence δp_i through (2.3).
Some extra algebra provides the covariance matrix of the parameters :

$\mathrm{cov}(V, V) = (A - \sum_i B_i C_i^{-1} B_i^t)^{-1}$
$\mathrm{cov}(V, p_i) = -\mathrm{cov}(V,V).\ B_i C_i^{-1}$
$\mathrm{cov}(p_i, p_j) = \delta_{ij}\ C_i^{-1} + C_i^{-1} B_i^t\ \mathrm{cov}(V,V)\ B_j C_j^{-1}$

All parameters are now correlated.

2.2. Parametrization :

We need only a local parametrization of the trajectories from the vertex to the point where the quantities q_{ij} are defined (generally the first point measured on the track). If the distance is too long, we can in a first step extrapolate the q_{ij} to a point close to the expected vertex, and propagate W_i into $\mathcal{D}_i^t W_i \mathcal{D}_i$, where $\mathcal{D}_i$ is the derivative matrix of the initial q_{ij} w.r.t. the extrapolated ones. Exact and approximate expressions of $\mathcal{D}_i$ are given in app.I for some simple cases. This propagation can also account for multiple scattering between the vertex and the track detector.

Around the vertex we can use, either a helix parmetrization, or, simpler, a second order expansion of the trajectory. As an example, if the q_{ij} are the position (y_i and z_i), the direction (slopes $u_i = \frac{dy_i}{dx}$ and $v_i = \frac{dz_i}{dx}$) and the curvature at a given value of x, we choose as parameters p_{ik} the slopes $U_i = \frac{dy_i}{dx}$, $V_i = \frac{dz_i}{dx}$ at the vertex (X, Y, Z), and the curvature (which is independent of the point chosen).

The trajectory is defined locally by :

$$
\begin{aligned}
y_i &= Y + U_i (x - X) + \frac{\alpha_i}{2} (x - X)^2 \\
z_i &= Z + V_i (x - X) + \frac{\beta_i}{2} (x - X)^2 \\
u_i &= U_i + \alpha (x - X) \\
v_i &= V_i + \beta (x - X)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_i &= \rho_i \sqrt{1 + U_i^2 + V_i^2} \, \left[V_i B_x + U_i V_i B_y - (1 + U_i^2) B_z \right] \\
\beta_i &= \rho_i \sqrt{1 + U_i^2 + V_i^2} \, \left[- U_i B_x + (1 + V_i^2) B_y - U_i V_i B_z \right]
\end{aligned}
$$

B_x, B_y, B_z are the local magnetic field components, and ρ_i is the signed ratio of electric charge to momentum.

Hence the derivatives (with $\Delta x = x - X$) :

$\partial\downarrow / \partial\rightarrow$	X	Y	Z	U_i	V_i	ρ_i
y_i	$-U_i - \alpha_i \Delta x$	1	0	$\Delta x + \frac{\partial\alpha_i}{\partial U_i} \cdot \frac{\Delta x^2}{2}$	$\frac{\partial\alpha_i}{\partial V_i} \cdot \frac{\Delta x^2}{2}$	$\frac{\alpha_i}{\rho_i} \cdot \frac{\Delta x^2}{2}$
z_i	$-V_i - \beta_i \Delta x$	0	1	$\frac{\partial\beta_i}{\partial U_i} \cdot \frac{\Delta x^2}{2}$	$\Delta x + \frac{\partial\beta_i}{\partial V_i} \cdot \frac{\Delta x^2}{2}$	$\frac{\beta_i}{\rho_i} \cdot \frac{\Delta x^2}{2}$
u_i	$-\alpha_i$	0	0	$1 + \frac{\partial\alpha_i}{\partial U_i} \cdot \Delta x$	$\frac{\partial\alpha_i}{\partial V_i} \cdot \Delta x$	$\frac{\alpha_i}{\rho_i} \cdot \Delta x$
v_i	$-\beta_i$	0	0	$\frac{\partial\beta_i}{\partial U_i} \cdot \Delta x$	$1 + \frac{\partial\beta_i}{\partial V_i} \cdot \Delta x$	$\frac{\beta_i}{\rho_i} \cdot \Delta x$
ρ_i	0	0	0	0	0	1
	matrix D_i			matrix E_i		

For short distances many terms containing Δx or Δx^2 can be neglected. When the distances are also negligible w.r.t. the radius of curvature, D_i and E_i have very simple expressions :

$$D_i = \begin{pmatrix} -U_i & 1 & 0 \\ -V_i & 0 & 1 \\ -\alpha_i & 0 & 0 \\ -\beta_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_i = \begin{pmatrix} [\Delta x] & 0 & 0 \\ 0 & [\Delta x] & 0 \\ 1 & 0 & [\alpha_i/\rho_i \Delta x] \\ 0 & 1 & [\beta_i/\rho_i \Delta x] \\ 0 & 0 & 1 \end{pmatrix}$$

Terms into brackets can be omitted when the ratios (error or position)/(error or slope) and (error or slope)/(error or curvature) are large w.r.t. the range of the parametrization. If the extrapolation length between the first measured point and the vertex is not too large these ratios are of the order of the measured length, so in most cases the above condition is fulfilled.

3. Updating the vertex : addition and subtraction of tracks :

3.1. Linear approximation formalism :

The linear approximation of the variables as functions of the parameters :

$$F(V+\delta V,\ p_i+\delta p_i) = F(V,p_i) + D_i\,\delta V + E_i\,\delta p_i$$

is applicable provided that the parameters do not vary largely when adding or subtracting a single track.

Let us suppose that we want to add a (n+1)-th track to a vertex already constructed with n tracks. We take as starting values : V and p_i (i = 1 to n) previously fitted, and p_{n+1} so as to have small differences Δq_{n+1} between the values deduced from V and p_{n+1} through (2.1), and the measured ones (e.g. choosing the measured curvatures, and the direction at the vertex so that the direction at the first measured point coincides with the measured one).

Within the linear approximation, the contribution of the first n tracks to the χ^2 is a quadratic function of δV and δp_i :

$$\chi^2_{min} + \sum_{i=1}^{n} \left(\delta V^t D_i^t + \delta p_i^t E_i^t\right) W_i \left(D_i\,\delta V + E_i\,\delta p_i\right)$$

The contribution of the (n + 1)-th track is :

$$\left(\Delta q_{n+1} + D_{n+1}\,\delta V + E_{n+1}\,\delta p_{n+1}\right)^t W_{n+1} \left(\Delta q_{n+1} + D_{n+1}\,\delta V + E_{n+1}\,\delta p_{n+1}\right)$$

Minimizing the global gives :

$$\left\{\begin{array}{lr}
\left(\sum_{i=1}^{n+1} D_i^t W_i D_i\right)\delta V + \sum_{i=1}^{n+1}\left(D_i^t W_i E_i\right)\delta p_i = D_{n+1}^t W_{n+1}\,\Delta q_{n+1} & (3.1.)\\
\text{for } i=1 \text{ to } n \ : \ \left(E_i^t W_i D_i\right)\delta V + \left(E_i^t W_i E_i\right)\delta p_i = 0 & (3.2.)\\
\left(E_{n+1}^t W_{n+1} D_{n+1}\right)\delta V + \left(E_{n+1}^t W_{n+1} E_{n+1}\right)\delta p_{n+1} = & (3.3.)\\
\qquad E_{n+1}^t W_{n+1}\,\Delta q_{n+1} &
\end{array}\right.$$

With the notations used in (2.1') and (2.2'), plus :

$$A_{n+1} = D_{n+1}^{t}\ W_{n+1}\ D_{n+1}$$
$$T_{n+1} = D_{n+1}^{t}\ W_{n+1}\ \Delta q_{n+1}$$

we obtain :

$$\begin{cases} (A + A_{n+1})\ \delta V + \sum_{i=1}^{n+1} B_i\ \delta p_i = T_{n+1} & (3.1') \\ \text{for } i = 1 \text{ to } n : \quad B_i^t\ \delta V + C_i\ \delta p_i = 0 & (3.2') \\ B_{n+1}^t\ \delta V + C_{n+1}\ \delta p_{n+1} = U_{n+1} & (3.3') \end{cases}$$

Eqs (3.2') and (3.3') give δp_i ($i = 1$ to $n + 1$) as functions of δV :

$$\begin{cases} \delta p_i = - C_i^{-1}\ B_i^t\ \delta V \qquad (i = 1 \text{ to } n) & (3.4) \\ \delta p_{n+1} = C_{n+1}^{-1}\ (U_{n+1} - B_{n+1}^t\ \delta V) & (3.5) \end{cases}$$

whence, with (3.1') :

$$\left(A + A_{n+1} - \sum_{i=1}^{n+1} B_i\, C_i^{-1}\, B_i^t\right)\delta V = T_{n+1} - B_{n+1}\, C_{n+1}^{-1}\, U_{n+1} \qquad (3.6)$$

This system of 3 equations gives $\delta V = (\delta X, \delta Y, \delta Z)$, and then δp_i through (3.4) and (3.5).

Since $A - \sum_{i=1}^{n} B_i\, C_i^{-1}\, B_i^t$ was already calculated in the n-track fit, one has to compute the right-hand side and an additional term to the left-hand side. Also $C_i^{-1} B_i^t$ are already known, so that the amount of calculation needed in this formalism is much smaller than a new fit with n + 1 tracks (even if limited to only one iteration).

The same algorithm can be applied to remove a track from a vertex, giving a negative weight $-W_i$ to this track.

3.2. Discussion :

The main point to discuss is the validity of the linear approximation, which allows to consider the derivatives matrices D_i and E_i as constant in the range of variation of the parameters. In principle this range should not exceed largely the uncertainties on the track measurements : for usual detectors this corresponds to negligible variations of D_i and E_i . If a track added to a vertex modifies strongly its parameters, it is likely not issued from this vertex. If that is the case when removing a track, this track should have been rejected from this vertex before the fit by preliminary cuts.

Eqs (3.1.) and (3.2.) give a quick procedure for updating of the global and also the contribution of each track to its variation, whence a probability criterion to accept a new track, or, if needed, to reject an old track, in order to find a better topological assignment.

4. Conclusion :

The vertex fitting procedure proposed in this paper has many advantages :

. It uses in an optimal way the informfation provided by the individual fit of the tracks.

. It relies on few elementary operations on (3x3) and (3x5) matrices ; so it can be coded with high efficiency.

. In many cases, the calculations are speeded up by reasonable approximations on slightly curved tracks. The modular structure of the algorithm allows to apply different treatments to tracks in the same vertex.

. In the linear approximation, addition or subtraction of tracks are fast operations compared to the whole fit. They could be used as tools to construct a vertex track by track, and to examine quickly different possible topologies.

Appendix

weight matrix propagation

We want to calculate the matrix $\mathcal{D}$ of derivatives of 5 quantities q_j^F defining the track at a given point F (for example the first measured point), w.r.t. 5 quantities q_j^V at a point V extrapolated near to the vertex. As in 2.2. we choose for q_j : y, z, u dy/dx, v dz/dx at fixed x and ρ signed ratio of charge to momentum.

Before any calculation let us remark that, contrary to the accuracy needed for trajectory extrapolation, we can accept an approximation of $\mathcal{D}$ for weight propagation. So we suppose hereafter the magnetic field to be uniform and the energy loss to be negligible in the range of propagation, in order to use a helix parametrization.

In most cases, the extrapolation length (projected onto a plane perpendicular to the field) is small w.r.t. the radius of the helix, so that the rotation angle from V to F is small, and we can use the same second order parametrization as in 2.2., and an approximation of $\mathcal{D}$ at first order in this angle. In this approximation we obtain, with $\Delta x = x^F - x^V$

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & \Delta x & 0 & 0 \\ 0 & 1 & 0 & \Delta x & 0 \\ 0 & 0 & 1 & 0 & \frac{\alpha}{\rho}\Delta x \\ 0 & 0 & 0 & 1 & \frac{\beta}{\rho}\Delta x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(expressions of α and β were given in 2.2.).

If the rotation angle is not negligible, we need an exact helix parametrization. Let us examine first the case where the magnetic field is along x-axis : we define for convenience $q_1^{V'} \dots q_5^{V'}$ = $y'^V, z'^V, \alpha^V, t^V, c^V$ at fixed x^V so that $y'^V = t^V \cos\alpha^V$, $z'^V = t^V \sin\alpha^V$, $c^V = 1/R = \varrho B_x$ (R is signed) and in the same way $q_1^{F'} \dots q_5^{F'}$ at fixed x^F (t is the slope of the track w.r.t. a plane perpendicular to the magnetic field, and α is the angle w.r.t. y axis in projection onto this plane).

The propagation along the helix from x^V to x^F is expressed by :

$$\begin{cases} c^F = c^V \\ t^F = t^V \\ \Delta\alpha = \alpha^F - \alpha^V = c\,t\,(x^F - x^V) \\ \Delta y = y^F - y^V = \dfrac{\sin\alpha^F - \sin\alpha^V}{c} \\ \Delta z = z^F - z^V = \dfrac{\cos\alpha^V - \cos\alpha^F}{c} \end{cases}$$

From these expressions we deduce the matrix of derivatives $D_{F'/V'}$

$\partial\downarrow / \partial\rightarrow$	y^V	z^V	α^V	t^V	c^V
y^F	1	0	$-\Delta z$	$\Delta x \cos\alpha^F$	$\dfrac{t\,\Delta x \cos\alpha^F - \Delta y}{c}$
z^F	0	1	Δy	$\Delta x \sin\alpha^F$	$\dfrac{t\,\Delta x \sin\alpha^F - \Delta z}{c}$
α^F	0	0	1	$c\,\Delta x$	$t\,\Delta x$
t^F	0	0	0	1	0
c^F	0	0	0	0	1

Hence the matrix :

$$\mathcal{D} = D_{F/F'}\, D_{F'/V'}\, D_{V'/V}$$

where $D_{F/F'}$ is the matrix of derivatives of q^F w.r.t. $q^{F'}$ and $D_{V'/V}$ the matrix of derivatives of $q^{V'}$ w.r.t. q^V :

$$D_{F/F'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -t\sin\alpha^F & \cos\alpha^F & 0 \\ 0 & 0 & t\cos\alpha^F & \sin\alpha^F & 0 \\ 0 & 0 & 0 & 0 & 1/B_x \end{pmatrix} \qquad D_{V'/V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\sin\alpha^V}{t} & \dfrac{\cos\alpha^V}{t} & 0 \\ 0 & 0 & \cos\alpha^V & \sin\alpha^V & 0 \\ 0 & 0 & 0 & 0 & B_x \end{pmatrix}$$

When the magnetic field is along an axis perpendicular to x-axis (for example along z-axis) we define $q^{V''}$ at point V and $q^{F''}$ at point F, as y, z, $s = \frac{u}{\sqrt{1+u^2}}$, $t = \frac{v}{\sqrt{1+u^2}}$ at fixed x, and c $\rho B_z = \frac{1}{R}$ (t has the same signification as above, and s is the sine of the angle α w.r.t. x-axis in projection). The propagation along the helix gives now :

$$\begin{cases} c^F = c^V \\ t^F = t^V \\ s^F - s^V = c\,(x^F - x^V) \\ y^F - y^V = R\,(\cos\alpha^V - \cos\alpha^F) = \dfrac{\sqrt{1-(s^V)^2} - \sqrt{1-(s^F)^2}}{c} \\ z^F - z^V = R\,t\,(\alpha^F - \alpha^V) = \dfrac{t}{c}\,(\arcsin s^F - \arcsin s^V) \end{cases}$$

whence the matrix of derivatives $D_{F''/V''}$:

$\partial\downarrow/\partial\rightarrow$	y^V	z^V	s^V	t^V	c^V
y^F	1	0	$\frac{\Delta(\tan\alpha)}{c}$	0	$\frac{\Delta(\cos\alpha)}{c^2} + \frac{\Delta x \tan\alpha^F}{c}$
z^F	0	1	$\frac{t}{c}\,\Delta\left(\frac{1}{\cos\alpha}\right)$	$\frac{\Delta\alpha}{c}$	$-\frac{t\,\Delta\alpha}{c^2} + \frac{\Delta x}{c\cos\alpha^F}$
s^F	0	0	1	0	Δx
t^F	0	0	0	1	0
c^F	0	0	0	0	1

(i.e. : $\Delta(\cos\alpha) = \cos\alpha^F - \cos\alpha^V$, and so on)

and $\mathcal{D} = D_{F/F''}\, D_{F''/V''}\, D_{V''/V}$

with $D_{F/F''} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\cos^3\alpha^F} & 0 & 0 \\ 0 & 0 & \frac{t\sin\alpha^F}{\cos^3\alpha^F} & \frac{1}{\cos\alpha^F} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{B_z} \end{pmatrix}$

$$D_{V''/V} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos^3\alpha^V & 0 & 0 \\ 0 & 0 & -t\sin\alpha^V\cos\alpha^V & \cos\alpha^V & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix}$$

We come now to the most general case : the magnetic field is no more along a coordinate axis. We can overcome this difficulty with a rotation of the frame at both F and V, defining q^{Vr} (or q^{Fr}) as y^r, z^r, u^r $\frac{dy^r}{dx^r}$, v $\frac{dz^r}{dx^r}$ at fixed x^r in the rotated frame, and ρ .

If the field is along x-axis (resp. z-axis), we can write, with the same notations as above :

$$\mathcal{L} \quad D_{F/Fr}\, D_{Fr/Fr'}\, D_{Fr'/Vr'}\, D_{Vr'/Vr}\, D_{Vr/V}$$

$$(\text{resp. } \mathcal{L} \quad D_{F/Fr}\, D_{Fr/Fr''}\, D_{Fr''/Vr''}\, D_{Vr''/Vr}\, D_{Vr/V})$$

We have only to evaluate $D_{F/Fr}$, matrix of derivatives of q^F w.r.t. q^{Fr} and $D_{Vr/V}$. If the errors on the position are negligible w.r.t. the radius of curvature, we can consider the trajectory as a straight line in the range corresponding to its possible fluctuations around F or V ; so we calculate only the derivatives of y, z, u, v at fixed x w.r.t. y, z, u , v at fixed x, and vice versa.

The local equations of the trajectory are :

$$\begin{cases} y = y_0 + u x \\ z = z_0 + v x \end{cases}$$

The rotated frame is defined by a matrix R (R^{-1} R^t)

$$\begin{pmatrix} x^r \\ y^r \\ z^r \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} \begin{pmatrix} x^r \\ y^r \\ z^r \end{pmatrix}$$

We want to describe the trajectory by equations in this frame :

$$\begin{cases} y^r = y_o^r + u^r x^r \\ z^r = z_o^r + v^r x^r \end{cases}$$

Using the orthogonality of R, we find that these equations are equivalent to the first ones when :

$$\begin{cases} y_o^r = S\left[(R_{33} - v R_{31})\, y_o + (u R_{31} - R_{32})\, z_o \right] \\ z_o^r = S\left[(v R_{21} - R_{23})\, y_o + (R_{22} - u R_{21})\, z_o \right] \\ u^r = S\left(R_{21} + u R_{22} + v R_{23} \right) \\ v^r = S\left(R_{31} + u R_{32} + v R_{33} \right) \end{cases}$$

with $\quad S = 1 / (R_{11} + u R_{12} + v R_{13})$

Hence the derivatives (taken at $y_o = z_o = 0$)

$\partial \downarrow / \partial \rightarrow$	y	z	u	v
y^r	$S(R_{33} - v R_{31})$	$S(u R_{31} - R_{32})$	0	0
z^r	$S(v R_{21} - R_{23})$	$S(R_{22} - u R_{21})$	0	0
u^r	0	0	$S^2(R_{33} - v R_{31})$	$S^2(u R_{31} - R_{32})$
v^r	0	0	$S^2(v R_{21} - R_{23})$	$S^2(R_{22} - u R_{21})$

Introducing the unit vectors $\vec{R}_1, \vec{R}_2, \vec{R}_3$ of the rotated frame, and the unit vector $\vec{U}$ along the trajectory, we obtain a geometrical interpretation of these derivatives :

$\partial\downarrow/\partial\rightarrow$	y	z
y^r	$\dfrac{(\vec{R}_3 \times \vec{U})_y}{\vec{R}_1 \cdot \vec{U}}$	$\dfrac{(\vec{R}_3 \times \vec{U})_z}{\vec{R}_1 \cdot \vec{U}}$
z^r	$\dfrac{(\vec{R}_2 \times \vec{U})}{\vec{R}_1 \cdot \vec{U}}$	$\dfrac{(\vec{R}_2 \times \vec{U})_z}{\vec{R}_1 \cdot \vec{U}}$

$\partial\downarrow/\partial\rightarrow$	u	v
u^r	the same matrix divided by $\vec{R}.\vec{U} \sin\lambda$	
	λ being the angle of $\vec{U}$ w.r.t. x-axis	
v^r		

Reference :

Pierre Billoir "Méthode d'ajustement dans un problème à paramétrisation hiérarchisée". LPC 84-30