CHERN-SIMONS TERMS AND COCYCLES IN PHYSICS AND MATHEMATICS

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Formulas for the finite 2-cocycle appearing in eqs. (2.9), (2.15) and (2.18) are wrong and should be deleted.

I thank L. Faddeev for pointing out my error.

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I. INTRODUCTION

Long before gauge theories became the focus of research for many of us, Professor E. S. Fradkin recognized their importance and took the first steps in unravelling their intricacies. These days he is blazing new trails in supersymmetric thickets, but I thought he might like to know where his old subject stands today. In this essay, dedicated to Fradkin for his sixtieth birthday, I review contemporary topological research in Yang-Mills theory, emphasizing the Chern-Simons terms and their relatives, which currently are widely studied.

Topological analysis of *classical* gauge theories began when A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin pointed out that the four-dimensional Chern-Pontryagin density $P \equiv -\frac{1}{16\pi^2} tr\, {}^*F^{\mu\nu} F_{\mu\nu} = -\frac{1}{32\pi^2}\, tr\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$, already familiar to physicists – but not under that name – as the anomalous divergence of the axial vector current, has a well-known and important place in mathematics. Specifically, its four-dimensional integral is a topological invariant, whose non-vanishing value is a signal for topologically non-trivial properties of the gauge fields. The study of instantons, the $U(1)$ problem and the vacuum angle put physicists in touch with mathematicians, who had come to similar investigations from their own subject. The particular point of contact is the Atiyah-Singer index theorem, which is now recognized in its local form as the axial anomaly equation. Physicists benefitted from the interaction with mathematicians by learning and using their topological methods, which greatly aided in establishing non-perturbative results about the *quantized* gauge theory. It turns out that the Chern-Simons structure [secondary characteristic class] – a relative of the Chern-Pontryagin quantity – is the more useful, and will concern us here.

Thus far there are three roles for the Chern-Simons term in physical theory: (1) it helps understanding gauge theories in even dimensional space-time; (2) it can contribute to gauge field dynamics in odd dimensional space-time; (3) it is used in a mathematically coherent description of [even-dimensional] gauge theories with chiral fermions that are apparently inconsistent, owing to chiral anomalies. These three applications are described in three Sections which follow Sections II and III, where mathematical preliminaries are explained and exemplified in simple quantum mechanical settings.

II. MATHEMATICS

A. Chern – Pontryagin and the Dimensional Ladder to its Relatives

Since the integral of P is a topological invariant, one expectes that P can be written as a total divergence, so that $\int d^4x\, P$ can be cast onto an integral over the surface at infinity and determined by the long-range properties of the gauge fields. This is indeed so, and the following formula is easily verified.[1]

$$P \equiv -\frac{1}{32\pi^2}\, tr\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \partial_\mu \Omega_0^\mu \qquad (2.1)$$

$$\Omega_0^\mu(A) = -\frac{1}{8\pi^2}\, tr\, \epsilon^{\mu\alpha\beta\gamma}\left(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma\right) . \qquad (2.2)$$

Ω_0^μ is called the *Chern-Simons* density.

Let us observe the following propertgies of the Chern-Simons density:

1. The Chern-Simons density naturally lives in one dimension lower than the Chern-Pontryagin, in the following sense. Owing to the ϵ symbol, picking one component of Ω^μ, say $\mu = \mu_0$, forces the remaining components of the derivatives and gauge potentials occurring in (2.2) to be other than μ_0 – *i.e.*, they belong to the three-dimensional subspace of the four-dimensional space that is complementary to μ_0. As a consequence, we suppress one dimension and write the Chern-Simons density as a 3-dimensional object.

$$\Omega_0(A) = -\frac{1}{8\pi^2}\, tr\, \epsilon^{\alpha\beta\gamma}\left(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma\right) \qquad (2.3)$$

2. The passage from P to Ω_0 is ambiguous. More specifically, Ω_0 is gauge dependent, while P is gauge invariant. When $\int d^4x P$ is non-zero, Ω_0 will possess gauge dependent singularities and/or slow long-range fall-off.

3. The gauge variation of Ω_0 is given by

$$\Delta\Omega_0 \equiv \Omega_0(A^g) - \Omega_0(A) = \frac{1}{8\pi^2}\, tr\, \epsilon^{\alpha\beta\gamma}\, \partial_\alpha(a_\beta A_\gamma) + \frac{1}{24\pi^2}\, tr\, \epsilon^{\alpha\beta\gamma} a_\alpha a_\beta a_\gamma$$
$$a_\alpha \equiv \partial_\alpha g\, g^{-1} \qquad (2.4)$$

Moreover, the integral of Ω_0, $\omega_0(A) \equiv \int d^3x\, \Omega_0$, changes by the winding number of the gauge transformation. [This is related to point 2, above.]

$$\omega_0(A^g) = \omega_0(A) + n(g)$$
$$n(g) \equiv \int d^3x \frac{1}{24\pi^2}\, tr\, \epsilon^{\alpha\beta\gamma} a_\alpha a_\beta a_\gamma \quad . \qquad (2.5)$$

For well-behaved gauge transformations, n is an integer characterizing the homotopy class to which g belongs, and is non-trivial when Π_3 of the non-Abelian gauge group $= Z$, the group of integers under addition.

4. The descent from four dimensions to three dimensions, *i.e.*, from the Chern-Pontryagin density to the Chern-Simons density, may be continued since $\Delta\Omega_0$ is again a total divergence, $\Delta\Omega_0 = \partial_\alpha \Omega_1^\alpha(A; g)$, where $\Omega_1^\alpha(A;g)$ again may be singular and/or possess slow fall-off at infinity. It is manifestly true that the next-to-last term in (2.4) is a total divergence. The fact that the last term may also be so written is not self-evident, but it is true. For example, in an $SU(2)$ model, one finds

$$\Omega_1^\alpha(A;g) = \frac{1}{8\pi^2} tr\, \epsilon^{\alpha\beta\gamma} \partial_\beta g g^{-1} A_\gamma + \frac{1}{4\pi^2} tr\, \epsilon^{\alpha\beta\gamma} \log g\, \partial_\beta \log g\, \partial_\gamma \log\left(\frac{1}{\theta^3} - \frac{\sin\theta}{\theta^3}\right) \qquad (2.6)$$

with $\theta^2 \equiv -2tr(\ln g)^2$. Similar formulas hold for other groups.[3] Evidently, Ω_1^α lives in one dimension lower than Ω_0, i.e., in two, and again the index α may be suppressed. We shall write

$$\Omega_1(A;g) = \frac{1}{8\pi^2}\,\epsilon^{\alpha\beta}\,\partial_\alpha g g^{-1} A_\beta + \frac{\partial^{-1}}{24\pi^2}\,tr\,\epsilon^{\alpha\beta\gamma} a_\alpha a_\beta a_\gamma \qquad (2.7)$$

were the last term is a symbolic representation of the [complicated] last term in (2.6) or analogous expressions for groups other than $SU(2)$.

5. The descent may be continued further.[3] We define the Δ operation on Ω_1 by

$$\Delta\Omega_1 \equiv \Omega_1(A^{g_1};g_2) - \Omega_1(A;g_1g_2) + \Omega_1(A;g_1) \qquad (2.8)$$

and verify that $\Delta\Omega_1$ is again a total divergence of an object which lives in one dimension: $\Delta\Omega_1 = \partial_\alpha\Omega_2^\alpha(A;g_1,g_2)$. With the index α suppressed, we have

$$\Omega_2(A;g_1,g_2) = -\frac{1}{16\pi^2}\,tr\left[\ln g_1\frac{d}{dx}\ln g_2 - \frac{d}{dx}\ln g_1 \ln g_2\right] \ . \qquad (2.9)$$

6. When an arbitrary infinitesimal variation is made on the vector potential $A_\mu \to A_\mu + \delta A_\mu$, the Chern-Simons density varies as

$$\delta\Omega_0(A) = -\frac{1}{8\pi^2}\,tr\,\epsilon^{\alpha\beta\gamma}\,F_{\alpha\beta}\delta A_\gamma + \frac{1}{8\pi^2}\,\epsilon^{\alpha\beta\gamma}\partial_\alpha(A_\beta\delta A_\gamma) \qquad (2.10)$$

so that

$$\frac{\delta\omega_0(A)}{\delta A_\alpha^a} = -\frac{1}{8\pi^2}\,tr\,T^a\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = \frac{1}{16\pi^2}\,\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}^a \ . \qquad (2.11)$$

When the variation is a gauge transformation, $\delta A_\mu = D_\mu\theta$, $\delta\Omega_0$ may be read off equation (2.4) with $a_\alpha = \partial_\alpha\theta + \dots$

$$\begin{aligned}\delta\Omega_0(A)\,|_{\delta A_\mu = D_\mu\theta} &= \frac{1}{8\pi^2}\,tr\,\epsilon^{\alpha\beta\gamma}\partial_\alpha(\partial_\beta\theta A_\gamma)\\ &= -\frac{1}{8\pi^2}\,\epsilon^{\alpha\beta\gamma}\partial_\alpha(\theta\partial_\beta A_\gamma)\end{aligned} \qquad (2.12a)$$

or eqivalently from (2.7)

$$\Omega_1(A,I+\theta)\dots) = \frac{1}{8\pi^2}\,\epsilon^{\alpha\beta}\partial_\alpha\theta A_\beta + \dots \qquad (2.12b)$$

Defining the two-dimensional integral of Ω_1 by $\omega_1(A;g) \equiv \int d^2x\,\Omega_1(A;g)$, we find

$$\omega_1(A;\ I+\theta+\dots) = -\frac{1}{8\pi^2}\int d^2x\ tr\ \theta\epsilon^{\alpha\beta}\partial_\alpha A_\beta + \dots \qquad (2.13)$$

It will be important for the subsequent to note that the variation (2.11) of $\omega_0(A)$ is gauge covariant. Furthermore, if the right hand side in (2.11) is written as a two-dimensional object, *i.e.*, if it is considered in the plane orthogonal to the direction of variation - $\frac{1}{16\pi^2}\,\epsilon^{\alpha\beta}F_{\alpha\beta}^a$ - then it and the two-dimensional quantity in (2.13) - $-\frac{1}{8\pi^2}\,tr\,\theta\epsilon^{\alpha\beta}\partial_\alpha A_\beta$ - have a place in two dimensional physics: they are various forms of the axial vector anomaly in two-dimensional gauge theories. This will be further explained in Section VI.

The series of formulas (2.1) - (2.13) may be compactly presented with the help of differential forms. In our notation, we depart from the conventional use of the wedge product; rather, the forms dx^μ are taken as anti-commuting variables $dx^\mu dx^\nu = -dx^\nu dx^\mu$.[4] Thus, we have

$$A \equiv A_\mu dx^\mu,\quad F \equiv \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu,\quad F = dA + A^2 \qquad (2.14)$$

The ladder of descent, beginning with the four-dimensional Chern-Pontryagin term, is

$$\begin{aligned}P &= -\frac{1}{2!(2\pi)^2}\,tr\,F^2 = d\Omega_0(A)\\ \Omega_0(A) &= -\frac{1}{2!(2\pi)^2}\,tr(dAA + \frac{2}{3}A^3)\\ \Delta\Omega_0 &\equiv d\Omega_1(A;g)\\ \Omega_1(A;g) &= \frac{1}{8\pi^2}\,tr\,aA + \frac{d^{-1}}{24\pi^2}\,tr\,a^3\\ \Delta\Omega_1 &\equiv d\Omega_2(A;g_1,g_2)\\ \Omega_2(A;g_1,g_2) &= -\frac{1}{16\pi^2}\,tr\Big(\ln g_1\,d\ln g_2 - d\ln g_1 \ln g_2\Big)\end{aligned} \qquad (2.15)$$

Here $a \equiv dgg^{-1}$ and the Δ operation is defined as in (2.4) and (2.8). The infinitesimal variation reads

$$\delta\Omega_0(A) = -\frac{1}{4\pi^2}\,tr\,F\delta A + \frac{1}{8\pi^2}\,d\,tr(A\delta A) \qquad (2.16a)$$

while for infinitesimal gauge transformations we have

$$\Omega_1(A;\ I+\theta+\dots) = \frac{1}{8\pi^2}\,tr\,d\theta A + \dots \qquad (2.16b)$$

$$\Omega_2(A;\ I+\theta_1+\dots,\ I+\theta_2+\dots) = -\frac{1}{16\pi^2}\,tr(\theta_1 d\theta_2 - d\theta_1\theta_2) \ . \qquad (2.16c)$$

One may consider similar chains of descent beginning in any even dimension.

Thus, starting in two, we obtain

$$\begin{aligned} \mathcal{P} &= \frac{i}{2\pi}\, tr\, F = d\Omega_0(A) \\ \Omega_0(A) &= \frac{i}{2\pi}\, tr\, A \\ \Delta\Omega_0 &= d\Omega_1(A;g) \\ \Omega_1(A;g) &= \frac{i}{2\pi}\, tr \ln g \end{aligned} \tag{2.17}$$

The six-dimensional dimensional ladder, which will play an important role in Section VI, reads

$$\begin{aligned} \mathcal{P} &= \frac{i}{3!(2\pi)^3}\, tr\, F^3 = d\Omega_0(A) \\ \Omega_0(A) &= \frac{i}{3!(2\pi)^3}\, tr[(dA)^2 A + \frac{3}{2} dA\, A^3 + \frac{3}{5} A^5] \\ \Delta\Omega_0 &= d\Omega_1(A;g) \\ \Omega_1(A;g) &= \frac{i}{12(2\pi)^3}\, tr[dAAa + AdAa + A^3 a - \frac{1}{2}(Aa)^2 - Aa^3 + \frac{d}{5}^{-1} a^5 \\ \Delta\Omega_1 &= d\Omega_2(A;g_1,g_2) \\ \Omega_2(A;g_1,g_2) &= \frac{i}{12(2\pi)^3}\, tr\Big[A\Big(dg_1 dg_2 g_2^{-1} g_1^{-1} - g_1 dg_2 g_2^{-1} g_1^{-1} dg_1 g_1^{-1}\Big) \\ &\quad - d^{-1}\Big(d \ln g_1 (d \ln g_1\, d \ln g_2 + \frac{1}{2}\, d \ln g_2\, d \ln g_1 + d \ln g_2\, d \ln g_2)\, d \ln g_2\Big)\Big] \end{aligned} \tag{2.18}$$

Again, the infinitesimal quantities need be recorded, as they will be needed later.

$$\delta\Omega_0(A) = \frac{i}{16\pi^3}\, tr\, F^2 \delta A - \frac{i}{48\pi^3} d tr(FA + AF - \frac{1}{2} A^3)\delta A \tag{2.19a}$$

$$\Omega_1(A;\ I + \theta + \ldots) = \frac{i}{12(2\pi)^3}\, tr(dAA + AdA + A^3) d\theta + \ldots \tag{2.19b}$$

$$\Omega_2(A;\ I + \theta_1 + \ldots,\ I + \theta_2 + \ldots) = \frac{i}{12(2\pi)^3}\, tr\, A(d\theta_1 d\theta_2 - d\theta_2 d\theta_1) + \ldots \tag{2.19c}$$

B. Cochains, Cocycles and the Coboundary Operation

Next, we take note of a series of mathematical concepts which derive from representation theory for transformation groups.[3,5] Let us consider a transformation group with elements g and composition law $g_1 g_2 = g_{12}$ acting on quantities q: $q \xrightarrow[g]{} q^g$. Next, consider functions $\Psi(q)$ defined on q, and represent the group action on these functions by an operator $U(g)$. In the simplest case, we have

$$U(g)\Psi(q) = \Psi(q^g) \tag{2.20}$$

and the composition law for the operators follows that of the group.

$$U(g_1)\, U(g_2) = U(g_{12}) \tag{2.21}$$

However, various generalizations are possible. The first generalization consists of allowing a phase in (2.20).

$$U(g)\Psi(q) = e^{i2\pi\omega_1(q;g)}\Psi(q^g) \tag{2.22}$$

Imposing (2.21) shows that ω_1 must satisfy

$$\omega_1(q^{g_1}; g_2) - \omega_1(q; g_{12}) + \omega_1(q; g_1) = 0 \text{ (mod integer)} \tag{2.23}$$

which may also be written as [compare (2.8)] $\Delta\omega_1 = 0$. A quantity depending on one group element and satisfying (2.23) is called a *1-cocycle*. It may be that ω_1 can be witten as

$$\omega_1(q;g) = \alpha_0(q^g) - \alpha_0(q) \tag{2.24}$$

or [compare (2.4)] $\omega_1(q;g) = \Delta\alpha_0(q)$, for some quantity of α_0. In that case, the 1-cocycle is called *trivial*, and may be removed by rewriting (2.22) as

$$e^{i2\pi\alpha_0(q)} U(g) e^{-i2\pi\alpha_0(q)} e^{i2\pi\alpha_0(q)}\Psi(q) = e^{i2\pi\alpha_0(q^g)}\Psi(q^g) \tag{2.25}$$

i.e., by defining new functions

$$e^{i2\pi\alpha_0(q)}\Psi(q) \equiv \Psi'(q) \tag{2.26a}$$

and new, conjugated operators

$$e^{i2\pi\alpha_0(q)} U(g) e^{-i2\pi\alpha_0(q)} \equiv U'(g) \tag{2.26b}$$

The primed quantities satisfy the simple rules (2.20) and (2.21).

Another generalizaton occurs when the action of U on Ψ requires a further operator.

$$U(g)\Psi(q) = A_1(q;g)\Psi(q^g) \tag{2.27}$$

[For example, Ψ may possess components and A_1 mixes them.] If (2.21) is satisfied, then A_1 must satisfy the composition law

$$A_1(q;g_1) A_1(q^{g_1};g_2) = A_1(q;g_{12}) \tag{2.28}$$

which can be called the *operator* 1-cocycle condition. However, it can happen that a phase occurs in (2.28):

$$A_1(q;g_1) A_1(q^{g_1};g_2) = e^{i2\pi\omega_2(q;g_1,g_2)} A_1(q;g_{12}) \tag{2.29}$$

which means that we are dealing with a projective representation, *i.e.*, (2.21) is modified to

$$U(g_1) U(g_2) = e^{i2\pi\omega_2(q;g_1,g_2)} U(g_{12}) \tag{2.30}$$

Associativity of the triple product imposes the condition

$$\omega_2(g^{g_1};g_2,g_3)-\omega_2(g;g_{12},g_3)+\omega_2(g;g_1,g_{23})-\omega_2(g;g_1,g_2)=0 \text{ (mod integer)} \quad (2.31)$$

which by definition we write as $\Delta\omega_2 = 0$, and $\omega_2(g;g_1,g_2)$ is called a *2-cocycle*. Once again, if ω_2 can be written as Δ of some quantity of α_1.

$$\omega_2(g;g_1,g_2) = \alpha_1(g^{g_1},g_2) - \alpha_1(g;g_{12}) + \alpha_1(g;g_1) = \Delta\alpha_1 \quad (2.32)$$

then (2.29) may be presented as

$$\begin{aligned} e^{-i2\pi\alpha_1(g;g_1)}\, A_1(g;g_1)\, e^{-i2\pi\alpha_1(g^{g_1};g_2)}\, A_1(g^{g_1};g_2) \\ = e^{-i2\pi\alpha_1(g;g_{12})}\, A_1(g;g_{12}) \end{aligned} \quad (2.33)$$

and the modified operators

$$e^{-i2\pi\alpha_1(g;g)}\, A_1(g;g) \equiv A_1'(g;g) \quad (2.34)$$

satisfy the naive composition law (2.28); the 2-cocycle may be removed. It is clear that if the quantities A_1 are numbers, rather than operators, the 2-cocycle is always trivial.

The next generalization, involving 3-cocycles arises when the representation behaves truly anomalously in that the operators implementing the transformation do not associate. This means that they cannot be linear operators on a vector or Hilbert space, because such operators associate, by definition.

We suppose that the group action involves the operator A_1 as in (2.27), but that the composition law for A_1 is not associative: different ways of associating a triple product differ by a phase.

$$\begin{aligned} &\Big(A_1(g;g_1)A_1(g^{g_1};g_2)\Big)A_1(g^{g_{12}};g_3) \\ &= e^{i2\pi\omega_3(g;g_1,g_2,g_3)}A_1(g;g_1)\Big(A_1(g^{g_1};g_2)A_1(g^{g_{12}};g_3)\Big) \end{aligned} \quad (2.35)$$

Furthermore, four-fold products are taken to satisfy

$$\begin{aligned} &\bigg(\Big(A_1(g;g_1)A_1(g^{g_1};g_2)\Big)A_1(g^{g_{12}};g_3)\bigg)A_1(g^{g_{123}};g_4) \\ &= e^{i2\pi\omega_3(g;g_{12},g_3,g_4)}\Big(A_1(g;g_1)A_1(g^{g_1};g_2)\Big)\Big(A_1(g^{g_{12}};g_3)A_1(g^{g_{123}};g_4)\Big) \end{aligned} \quad (2.36a)$$

$$\begin{aligned} &\Big(A_1(g;g_1)A_1(g^{g_1};g_2)\Big)\Big(A_1(g^{g_{12}};g_3)A_1(g^{g_{123}};g_4)\Big) \\ &= e^{i2\pi\omega_3(g;g_1,g_2,g_{34})}A_1(g;g_1)\bigg(A_1(g^{g_1};g_2)\Big(A_1(g^{g_{12}};g_3)A_1(g^{g_{123}};g_4)\Big)\bigg) \end{aligned} \quad (2.36b)$$

$$\begin{aligned} &\bigg(A_1(g;g_1)\Big(A_1(g^{g_1};g_2)A_1(g^{g_{12}};g_3)\Big)\bigg)A_1(g^{g_{123}};g_4) \\ &= e^{i2\pi\omega_3(g;g_1,g_{23},g_4)}A_1(g;g_1)\bigg(\Big(A_1(g^{g_1};g_2)A_1(g^{g_{12}};g_3)\Big)A_1(g^{g_{123}};g_4)\bigg) \end{aligned} \quad (2.36c)$$

One may derive (2.36) from (2.34) if one generalizes (2.29) to

$$A_1(g;g_1)A_1(g^{g_1};g_2) = A_2(g;g_1,g_2)A_1(g;g_{12}) \quad (2.37)$$

and correspondingly (2.30) to

$$U(g_1)U(g_2) = A_2(g;g_1,g_2)U(g_{12}) \quad . \quad (2.38)$$

It is assumed that A_2, an operator 2-cocycle, commutes and associates with all the other operators. By substituting (2.37) into (2.36), using (2.35) to change the association in the resulting triple product, and finally eliminating A_2 with the help of (2.37) again, establises (2.36) as consequence of (2.35) and (2.37).

In order that (2.35) be consistent with non-vanising ω_3, that phase must satisfy a condition, which is found, by multiplying (2.35) on the right by $A_1(g^{g_{123}};g_4)$, and repeatedly using (2.35) and (2.36) to bring the association of the four factors in both elements of the equality into the same form. We then find that ω_3 must satisfy the *3-cocycle* condition.

$$\begin{aligned} \Delta\omega_3 \equiv\ & \omega_3(g^{g_1};g_2,g_3,g_4) - \omega_3(g;g_{12},g_3,g_4) + \omega_3(g;g_1,g_{23},g_4) \\ & - \omega_3(g;g_1,g_2,g_{34}) + \omega_3(g;g_1,g_2,g_3) = 0 \text{ (mod integer)} \end{aligned} \quad (2.39)$$

A 3-cocycle is trivial it if can be written as

$$\omega_3(g;g_1,g_2,g_3) = \alpha_2(g^{g_1};g_2,g_3) - \alpha_2(g;g_{12},g_3) + \alpha_2(g;g_1,g_{23}) - \alpha_2(g;g_1,g_2) = \Delta\alpha_2 \quad (2.40)$$

where α_2 is an arbitrary quantity. When (2.40) holds, ω_3 may be removed by redefining A_2. It is clear that if A_2 is a number rather than an operator, ω_3 is trivial.

Finally, we see that the non-associativity may also be described by

$$\Big(U(g_1)U(g_2)\Big)U(g_3) = e^{i2\pi\omega_3(g;g_1,g_2,g_3)}U(g_1)\Big(U(g_2)U(g_3)\Big) \quad (2.41)$$

We shall call the above a *non-associative representation* of the group.

Next, we examine the implication of all this for the infinitesimal, algebraic relations when the transformation group is a continuous Lie group, and the finite transformation may be expressed in terms of infinitesimal generators. The group element is represented by $g = e^{\theta^a T^a}$, where θ^a is the infinitesimal parameter. The occurrence of a 2-cocycle, as in (2.30) or (2.37), manifests itself in the infinitesimal formulation by the fact that the Lie algebra of the generators does not follow the

Lie algebra of the group; rather, there is an extension. Moreover, with a 3-cocycle, the Jacobi identity *fails*.[5]

Let me now set down some mathematical terminology. Quantities that depend on n group elements are called *n-cochains*. The Δ operation, which has been presented for $n = 0, 1, 2, 3$ is called the coboundary operation and can be given a general definition.

$$\begin{aligned}\Delta\omega_n \equiv\ & \omega_n(q^{g_1}; g_2, \ldots, g_{n+1}) - \omega_n(q; g_{12}, g_3, \ldots, g_{n+1}) + \ldots + \\ & (-1)^m \omega_n(q; g_1, \ldots, g_{m\, m+1}, \ldots, g_{n+1}) + \ldots + (-1)^{n+1}\omega_n(q; g_1, \ldots, g_n)\end{aligned} \tag{2.42}$$

Evidently, operating on an n-cochain, Δ creates an $n+1$-cochain, and one sees that $\Delta^2 = 0$. An n-cochain which can be written as Δ of an $n-1$ cochain is an $n-$*coboundary*, while a cochain ω_n satisfying $\Delta\omega_n = 0$ (mod integer) is an $n-$*cocycle*, which is trivial if it is a coboundary, *i.e.*, if $\omega_n = \Delta\omega_{n-1}$.

III. COCYCLES IN QUANTUM MECHANICS

While the structures introduced above have a general mathematical setting, they possess particular significance in quantum mechanics which naturally concerns itself with unitary operators $U(g)$ that implement transformations g of dynamical variables q on wave functions $\Psi(q)$. Our principal interest here is quantum gauge field theory, where q corresponds to the spatial components of the vector potential $\mathbf{A}$ – the dynamical variable in a canonical/Hamiltonian description – while the states are [in the Schrödinger representation] wave functionals of $\mathbf{A}$, $\Psi(\mathbf{A})$, and the *group elements* g are local gauge functions depending on $\mathbf{x}$. Indeed, we see that the previously defined $\Omega_n(\mathbf{A}; g_1, \ldots, g_n)$ $n = 0, 1, 2$ are examples of cochains and that their integrals, $\omega_n \equiv \int \Omega_n$, are cocycles since they satisfy $\Delta\omega_n = \int \Delta\Omega_n = \int d\Omega_{n+1} = 0$ (mod integer). However, before delving into the quantum field theoretic application of these ideas, it is useful to exemplify them in the much simpler context of quantum mechanics of point particles. Indeed, quantum gauge field theory behaves exactly in the same way, except that one is dealing with gauge transformations, whose effect ultimately must be unobservable, while in the quantum mechanical examples discussed below the transformations describe actual changes in the physical system.

A. 1 – Cocycle

A 1-cocycle occurs in quantum mechanics whenever one is dealing with a transformation which is a symmetry operation of the action, but not of the Lagrangian. Specifically, if we consider a transformation specified by

$$q \to F(q) \tag{3.1a}$$

or in infinitesimal form

$$\delta q = f(q) \tag{3.1b}$$

which does not leave the Lagrangian invariant, but changes it by a total derivative,

$$L \to L - \frac{d}{dt}\omega \tag{3.2a}$$

$$\delta L = -\frac{d}{dt}\chi \tag{3.2b}$$

then Noether's theorem gives the infinitesimal generator as

$$C = \frac{\partial L}{\partial \dot q}\delta q + \chi = p\delta q + \chi \tag{3.3}$$

and the unitary operator effecting the finite transformation

$$U = e^{iC} \tag{3.4}$$

acts on wave functions with a 1-cocycle, which is just the quantity that appears in the finite transformation of the Lagrangian.

$$U\Psi(q) = e^{i\omega}\Psi(F(q)) \ . \tag{3.5}$$

Moreover, if the cocycle is trivial, $\omega = \Delta\alpha$, then it can be removed by adjusting the phase of the wave function, as in (2.26). A phase change in a wave function corresponds to a canonical transformation, which in general changes the Lagrangian by a total time-derivative. In the present case, the new Lagrangian will read

$$L' = L + \frac{d}{dt}\alpha \tag{3.6}$$

and it will have the property that it is invariant under the transformation. [In this Section cocycles are defined without the 2π factor.]

These remarks are well-illustrated in the example of a Galilean transformation, $q \to q - vt$, in a free theory governed by the Lagrangian $L = \frac{1}{2}\dot q^2$. We have $L \to L - \frac{d}{dt}(qv - \frac{1}{2}v^2 t)$ and $\delta q = -vt$, $\delta L = -\frac{d}{dt}qv$. The constant of motion $C = -\dot q vt + qv$ gives rise to the unitary operator $U(v) = e^{iC} = e^{-iv(pt-q)}$ whose effect on wave functions $\psi(q)$ can be easily evaluated with the Baker-Hausdorff lemma, $U(v)\psi(q) = e^{i\omega_1(q;v)}\psi(q - vt)$. One recognizes the 1-cocycle $\omega_1(q; v) = qv - \frac{1}{2}v^2 t$, which is also seen in the finite change of L. The 1-cocycle condition, $\Delta\omega_1 = \omega_1(q - v_1 t; v_2) - \omega_1(q; v_1 + v_2) + \omega_1(q; v_1) = 0$, is easily verified. Moreover, the 1-cocycle is trivial since it can be written as $\omega_1(q; v) = \alpha_0(q - vt) - \alpha_0(q) = \Delta\alpha_0$, $\alpha_0 = -q^2/2t$. To remove the 1-cocycle, we redefine the wave functions as in (2.26a), $e^{-iq^2/2t}\psi(q) = \psi'(q)$. To see how the Lagrangian changes, we first compute the Hamiltonian relevant to ψ' from the Schrödinger equation for ψ.

$$i\frac{\partial}{\partial t}(e^{iq^2/2t}\psi') = \frac{1}{2}p^2\left(e^{iq^2/2t}\psi'\right) \Longrightarrow i\frac{\partial}{\partial t}\psi' = \left(\frac{1}{2}(p + \frac{q}{t})^2 - \frac{q^2}{2t^2}\right)\psi' \ .$$

Hence, the new Hamiltonian $H' = \frac{1}{2}\left(p + \frac{q}{t}\right)^2 - \frac{q^2}{2t^2}$ leads to the new Lagrangian $L' = \frac{1}{2}\dot{q}^2 - \frac{q\dot{q}}{t} + \frac{q^2}{2t^2} = \frac{1}{2}\dot{q}^2 - \frac{d}{dt}\frac{q^2}{2t} = L + \frac{d}{dt}\alpha_0$, and one verifies that L' is indeed invariant against Galilean transformations.

B. 2 – Cocycle

A quantum mechanical 2-cocycle arises in the representation of translations on phase space [coordinates and momenta]. The translation generators are $\mathbf{r}$ and $\mathbf{p}$, whose Lie algebra is non-commutative, possessing a central extension – the Heisenberg commutator $i[p^i, q^j] = \delta^{ij}$. The finite translations are implemented by $U(\mathbf{a},\ \mathbf{b}) = e^{i(\mathbf{a}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{q})}$ which composes as $U(\mathbf{a}_1,\ \mathbf{b}_1)U(\mathbf{a}_2,\ \mathbf{b}_2) = e^{\frac{i}{2}(\mathbf{a}_1\cdot\mathbf{b}_2 - \mathbf{a}_2\cdot\mathbf{b}_1)}U(\mathbf{a}_1 + \mathbf{a}_2,\ \mathbf{b}_1 + \mathbf{b}_2)$. These operators also serve as coherent state creation operators.[6]

C. 3 – Cocycle

Quantum mechanics makes use of a 3-cocycle as well, which thus far as not been seen in quantum field theory. When translations are represented on configuration space $\{\mathbf{q}\}$. Conventionally, the finite translation operator is taken to be $U(\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{p}}$, and no cocycles occur in this representation. However, in the presence of a mangetic field $\mathbf{B}$, $\mathbf{p}$ is not gauge invariant, but the velocity operator $\mathbf{v} = \mathbf{p} - e\mathbf{A}$ is, where $\mathbf{B} = \nabla \times \mathbf{A}$. Since $\mathbf{v}$ satisfies the same commutation relation with $\mathbf{r}$ as $\mathbf{p}$, we may use

$$U(\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{v}} \tag{3.7}$$

as the translation operator. However, it does not represent the Abelian translation group trivially, since the components of $\mathbf{v}$ do not commute.

$$[v^i,\ v^j] = ie\,\epsilon^{ijk}B^k \tag{3.8}$$

Moreover, the triple commutator

$$\Big[[v^1, v^2], v^3\Big] + \Big[[v^2, v^3], v^1\Big] + \Big[[v^3, v^1], v^2\Big] = -e\nabla\cdot\mathbf{B} \tag{3.9}$$

is non-vanishing in the presence of a magnetic point monopole of strength g, located at $\mathbf{r}_0$, for which the divergence is non-vanishing.

$$e\nabla\cdot\mathbf{B} = 4\pi g e\,\delta(\mathbf{r} - \mathbf{r}_0) \tag{3.10}$$

When the Jacobi identities fails, we anticipate the occurrence of a 3-cocycle, and it remains to understand why the finite quantities (3.7) do not associate in the presence of a magnetic monopole.[5]

Before proceeding, let me discuss the numerical coefficient in (3.10). According to Dirac, a consistent quantum dynamics for the monopole requires that eg be

quantized in half integer units. Hence, the coefficient in (3.10) is in fact $2\pi n$. For the moment, let us ignore this, and remain with arbitrary value for eg.

To recognize the non-associativity, we write [compare (2.27)]

$$U(\mathbf{a})\Psi(\mathbf{r}) = \mathcal{A}_1(\mathbf{r};\mathbf{a})\Psi(\mathbf{r}+\mathbf{a}) \tag{3.11}$$

where

$$\mathcal{A}_1(\mathbf{r};\mathbf{a}) = e^{i\mathbf{a}\cdot\mathbf{v}}e^{-i\mathbf{a}\cdot\mathbf{p}} = \exp -ie\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{a}} d\mathbf{s}\cdot\mathbf{A}(\mathbf{s}) \tag{3.12}$$

with the line integral running along the straight line joining $\mathbf{r}$ and $\mathbf{r}+\mathbf{a}$. Furthermore, from [compare (2.37)]

$$\mathcal{A}_1(\mathbf{r};\mathbf{a}_1)\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2) = \mathcal{A}_2(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2)\mathcal{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2) \tag{3.13}$$

we see that

$$\mathcal{A}_2(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2) = e^{ie\Phi} \tag{3.14}$$

where Φ is the outward [direction $\mathbf{a}_1 \times \mathbf{a}_2$] flux through the triangle with vertices $(\mathbf{r}, \mathbf{r}+\mathbf{a}_1, \mathbf{r}+\mathbf{a}_1+\mathbf{a}_2)$.

Consider now three translations in non-coplanar directions $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$; see Figure 1. Forming the triple products as in (2.35), we find for the left-hand side

$$\begin{aligned}
&\Big(\mathcal{A}_1(\mathbf{r};\mathbf{a}_1)\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2)\Big)\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2;\mathbf{a}_3)\\
&= \exp -ie\Phi(ABC)\mathcal{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2)\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2;\mathbf{a}_3)\\
&= \exp -ie\Big(\Phi(ABC)+\Phi(ACD)\Big)\mathcal{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)
\end{aligned} \tag{3.15a}$$

while the right-hand side becomes

$$\begin{aligned}
&e^{i\omega_3(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)}\mathcal{A}_1(\mathbf{r};\mathbf{a}_1)\Big(\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2)\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2;\mathbf{a}_3)\Big)\\
&= e^{i\omega_3(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)}\mathcal{A}_1(\mathbf{r};\mathbf{a}_1)\exp ie\Phi(BCD)\mathcal{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2+\mathbf{a}_3)\\
&= e^{i\omega_3(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)}\exp ie\Big(\Phi(BCD)+\Phi(ABD)\Big)\mathcal{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)
\end{aligned} \tag{3.15b}$$

Each flux is pointing outwards and passes through the triangle specified by the three letters; see Figure 1. Comparison of the two equations (3.15) shows that the 3-cocycle is $-e$ times the total flux emerging from the tetrahedron formed from the three vectors $\mathbf{a}_i$, with one vertex at $\mathbf{r}$. Hence, it is $-4\pi eg$ when the monopole is enclosed and zero otherwise. Shrinking the three vectors to produce the infinitesimal cocycle leads to the violation of the Jacobi identify (3.9) and gives rise to the delta function in (3.10).

The 3-cocycle is trivial in that it equals, as in (2.40), to a sum of terms, each of which is the flux through the appropriate triangle. Nevertheless, if we wish to represent translations by gauge invariant operators, we must remain with the trivial 3-cocycle. Of course, removing it returns the representation to a trivial one in terms of **p**.

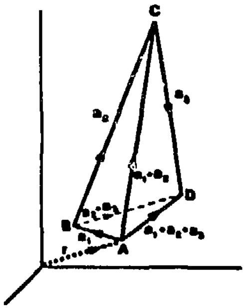

Figure 1: Tetrahedron at point **r** defined by three translations $\mathbf{a}_i$. The 3-cocycle is proportional to the flux out of the tetrahedron.

Finally, we observe that Dirac's quantization restores associativity of *finite* translations since then the 3-cocycle becomes $-2\pi n$ or zero, and has no effect in the exponential of (2.35), (2.41) and (3.15). Notwithstanding, the associativity of *finite* translations, the infinitesimal cocycle remains as an obstruction to the Jacobi identiy because *infinitesimal* generators do not associate. The argument may be reversed: By demanding that ultimately translations *must* be associative, we *derive* Dirac's quantization of eg. This, of course, also insures that a globally defined vector bundle exists.

IV. CHERN-SIMONS IN FOUR-DIMENSIONAL YANG-MILLS THEORY

Much of the topological structure of four-dimensional Yang-Mills theory and of Quantum Chromodynamics [QCD] has been understood with the help of the Chern-Simons term. The following lists some applications.[7]

A. Chern – Simons and the U(1) Anomaly

In QCD with massless quarks, the $U(1)$ axial vector current j_5^μ is not conserved, rather it satisfies $\partial_\mu j_5^\mu = 2P = 2\partial_\alpha \Omega_0^\alpha$. As a consequence, the conserved chiral charge Q_5 includes a contribution additional to the fermionic chiral charge $Q_5^F \equiv \int d^3x j_5^0$, which is given by the integrated Chern-Simons: $Q_5 = Q_5^F - 2\int d^3x \Omega_0 = Q_5^F - 2\omega_0(A)$. Although Q_5 is time independent, it is not gauge invariant as a consequence of (2.5). Thus, chiral symmetry cannot be physically realized, and this

resolves the "$U(1)$ problem". Note no reference to instantons is made in the above exact result, although they play a role in detailed, semi-classical analysis of the problem.[7]

B. Chern – Simons and the Vacuum Angle

It is known that physical states in a Yang-Mills theory are not gauge invariant; rather, they change under homotopically non-trivial gauge transformations by a phase. In a Schrödinger representation, the states are wave functionals of the spatial components of the vector potential, $\Psi(\mathbf{A})$, and the gauge transformaton is implemented by a unitary operator $U(g)$: $U(g)\Psi(\mathbf{A}) \equiv \Psi(\mathbf{A}^g) = e^{-i\theta n(g)}\Psi(\mathbf{A})$. Frequently, it is convenient to deal with gauge invariant states $\Psi'(\mathbf{A})$, and these may be obtained from $\Psi(\mathbf{A})$ by multiplying the latter by an **A**-dependent phase constructed from the integrated Chern-Simons density. $\Psi'(\mathbf{A}) = e^{i\theta\omega_0(\mathbf{A})}\Psi(\mathbf{A})$. It follows from (2.5) that $\Psi'(\mathbf{A})$ is gauge invariant: $U(g)\Psi'(\mathbf{A}) \equiv \Psi'(\mathbf{A}^g) = \Psi'(\mathbf{A})$. However, in quantum theory, a phase change of a wave function(al) corresponds to a canonical transformation, which in general induces a change in the Lagrangian by a total derivative. To find the new Lagrangian, we consider the [functional] Schrödinger equation for $\Psi(\mathbf{A}) = e^{i\theta\omega_0(\mathbf{A})}\Psi'(\mathbf{A})$

$$i\frac{\partial}{\partial t}e^{-i\theta\omega_0(\mathbf{A})}\Psi'(\mathbf{A}) = \int d^3x\left[-\frac{1}{2}\,\frac{\delta^2}{\delta A_a^i(\mathbf{x})\delta A_a^i(\mathbf{x})} + \frac{1}{2}B_a^i(\mathbf{x})B_a^i(\mathbf{x})\right]e^{-i\theta\omega_0(\mathbf{A})}\Psi'(\mathbf{A})$$
$$B_a^i \equiv -\frac{1}{2}\epsilon^{ijk}F_a^{jk} \tag{4.1}$$

and deduce that the modified Hamiltonian is, as a consequence of (2.11).

$$H' = \int d^3x\left[\frac{1}{2}\left(\Pi_a^i(\mathbf{x}) - \frac{\theta}{8\pi^2}B_a^i(\mathbf{x})\right)^2 + \frac{1}{2}B_a^i(\mathbf{x})B_a^i(\mathbf{x})\right] \tag{4.2}$$

This leads to a Lagrangian density which includes the total-derivative Chern-Pontragin density P.

$$\mathcal{L}' = \frac{1}{2}\,tr\,F^{\mu\nu}F_{\mu\nu} + \theta P \tag{4.3}$$

This is how the vacuum angle arises in the Yang-Mills Lagrangian.[7]

Note that the kinetic term in the Hamiltonian density (4.2) may also be written as $\frac{1}{2}\mathbf{V}_a^2$, where the [field] velocity $\dot{\mathbf{A}}_a \equiv \mathbf{V}_a$ differs from the canonical momentum by a further term, here $\frac{\theta}{8\pi^2}\mathbf{B}_a$. This is analogous to a point particle in an *external* field described by a vector potential. Thus, we may say that the Hamiltonian (4.2) describes a gauge potential $\mathbf{A}_a$, in an extenal $U(1)$ [functional] gauge field, whose [functional] connection is $\frac{\theta}{8\pi^2}\mathbf{B}_a$. The [functional] curvature is determined by the commutator of the velocities; compare (3.8). Here this commutator vanishes; the curvature vanishes; the connection is a "pure gauge", as is to be expected since $\frac{\theta}{8\pi^2}B_a^i = \frac{\delta}{\delta A_a^i}\theta\omega_0(\mathbf{A})$. We shall meet this situation again, but with non-vanishing curvature.

C. Chern – Simons in a Solution to the Yang – Mills Schrödinger Equation

The functional $\Psi(\mathbf{A}) = e^{\pm 8\pi^2 W(\mathbf{A})}$ is a static [zero-energy] solution to (4.1).[7] However, Ψ is not normalizable [by functional integration of over $\mathbf{A}_a$]; also, it does not transform under non-tivial gauge transformations as a physical state. I consider this a tremendous teaser of modern gauge theory: we are dealing with a highly non-trivial functional differential equation and we can find an explicit solution,which is unacceptable. Nevertheless, I continue to hope that we may yet learn something about Yang-Mills theory from this solution.

V. CHERN-SIMONS IN GAUGE-FIELD DYNAMICS

The Chern-Simons density naturally is an odd-dimensional quantity, and the variation of its integral is gauge covariant; see (2.11). Therefore, in a gauge theory in three [or properly generalized in any odd-number] dimensions, we may use the 0-cocycle $\omega_0(A)$ to supplement the conventional Yang-Mills action $I_{YM} = \int d^3x \frac{1}{2}\, tr\, F^{\mu\nu}F_{\mu\nu}$.[7]

$$I = I_{YM} + 2\pi m\, \omega_0(A) \tag{5.1}$$

The resultant field equations are gauge covariant.

$$D_\mu F^{\mu\nu} + \frac{m}{8\pi}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{5.2}$$

The proportionally constant m has dimensions of mass, and an analysis of the Abelian $U(1)$ case [electrodynamics in three-space-time] shows that the "photon" is indeed massive with spin $\frac{m}{|m|} = \pm 1$. [In two-space, spin is a (pseudo-)scalar.] This gives lie to the frequent but erroneous assertion that "gauge invariance prohibits a massive photon". The model is called a *topologically massive* gauge theory.

When quantizing the above non-Abelian theory, one must confront the fact that the total action I is not gauge invariant owing to (2.5), even though the equations of motion are gauge covariant. Since quantum mechanics makes use of the phase exponential of the action, e^{iI}, we must insist that the latter be gauge invariant and this requirement enforces a quantization condition on m: it must be an integer [in units that $\hbar$ and the dimensionful three-dimensional coupling constant is unity], so that the action changes only by an integer multiple of 2π. This field theoretic generalization of Dirac's monopole quantization condition arises because the action is multivalued.[7]

In a Hamiltonian formulation, the unitary operator $U(g)$ which implements gauge transformations does not merely gauge transform the argument of the wave functional, but also multiplies it by a phase, which is just the 1-cocycle ω_1, introduced in connection with (2.13).

$$U(g)\Psi(\mathbf{A}) = e^{-i2\pi m\omega_1(\mathbf{A};g)}\Psi(\mathbf{A}^g) \tag{5.3}$$

Gauss's law requires that physical states be unaffected by the action of U.

$$U(g)\Psi(\mathbf{A}) = \Psi(\mathbf{A}) \tag{5.4}$$

[There is no vacuum angle in three space-time dimensions.] Hence, in this theory physical states are not gauge invariant, rather

$$\Psi(\mathbf{A}^g) = e^{i2\pi m\omega_1(\mathbf{A};g)}\Psi(\mathbf{A}) \tag{5.5}$$

Iterating the gauge transformation shows that $\omega_1(\mathbf{A};g)$ must satisfy the 1-cocycle condition; as of course it does.

The occurence of the 1-cocycle does not come as a surprise in view of our general theory. The Lagrange density changes under gauge transformations according to (2.15).

$$\begin{aligned}\mathcal{L} = \frac{1}{2}tr\, F^{\mu\nu}F_{\mu\nu} + 2\pi m\Omega_0(A) &\to \frac{1}{2}tr\, F^{\mu\nu}F_{\mu\nu} + 2\pi m\Omega_0(A^g)\\ &= \frac{1}{2}tr\, F^{\mu\nu}F_{\mu\nu} + 2\pi m\Omega_0(A) + 2\pi m\partial_\alpha\Omega_1^\alpha(A;g)\end{aligned} \tag{5.6}$$

Consequently, the Lagrangian $L \equiv \int d^2x\, \mathcal{L}$ changes as

$$\begin{aligned}L \to L + 2\pi m\int d^2x\partial_\alpha\Omega_1^\alpha(A;g) &= L + \frac{d}{dt}2\pi m\int d^2x\Omega_1(\mathbf{A};g)\\ &= L + \frac{d}{dt}2\pi m\omega_1(\mathbf{A};g)\end{aligned} \tag{5.7}$$

where spatial surface terms have been dropped. Therefore, not unexpectedly, $-2\pi m\omega_1(\mathbf{A};g)$ occurs as a 1-cocycle. We shall see below that $\omega_1(\mathbf{A};g)$ is trivial, $\omega_1(\mathbf{A};g) = \alpha_0(\mathbf{A}^g) - \alpha_0(\mathbf{A})$, where $\alpha_0(\mathbf{A})$ is spatially non-local. Hence, the cocycle can be removed, and a non-local, gauge invariant Lagrangian can be given for our topologically massive gauge theory.

The Hamiltonian is

$$\begin{aligned}H &= \int d^2x\left\{\frac{1}{2}\left(\Pi_a^i(\mathbf{x}) - \frac{m}{8\pi}\epsilon^{ij}A_a^j(\mathbf{x})\right)^2 + \frac{1}{2}B_a^2\right\}\\ B_a &= -\frac{1}{2}\epsilon^{ij}F_a^{ij}\end{aligned} \tag{5.8}$$

and once again the field velocity in the kinetic term $V_a^i \equiv \Pi_a^i - \frac{m}{8\pi}\epsilon^{ij}A_a^j$ describes a gauge field in an external $U(1)$ functional connection $\frac{m}{8\pi}\epsilon^{ij}A_a^j(\mathbf{x})$, this time leading to a non-vanishing curvature,

$$i\left[V_a^i(\mathbf{x}),\, V_b^j(\mathbf{y})\right] = \frac{m}{4\pi}\delta_{ab}\,\epsilon^{ij}\delta(\mathbf{x}-\mathbf{y}) \tag{5.9}$$

which, being independent of the dynamical variable, may be called a constant [functional] magnetic field. One wonders whether there are other forms of non-Abelian field dynamics that appear as field motion in an external $U(1)$ functional connection.

One may also couple fermions to the three-dimensional gauge theory. It has been shown that massless fermions in the fundamental representation of the gauge group induce a Chern-Simons term, even if none is present in the "bare" Lagrangian. This gives rise to an anomalous violation of reflection symmetries [P and T].[7]

A physical application of the above ideas is to QCD at high temperature, since the high-temperature limit of a four-dimensional theory is governed by an effective three-dimensional Lagrangian. Indeed, it has been shown that fermionic CP[$= P$ and T] violating effects at high temperature induce the Chern-Simons action.[8]

Another application of the above, in an Abelian version, is used for a description of the Hall effect. The dynamics of charged particles in that physical situation confines them to a plane, owing to the presence of an external, homogenous magnetic field. When also a homogeneous electric field is applied perpendicularly to the magnetic field, the charged particles induce a planar effective electromagnetic action of the Chern-Simons form: $I_{\text{induced}} = 2\pi\sigma \int d^3x \frac{1}{32\pi^2}\, \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma$. [The magnetic field is in the (missing) z direction.] Here, σ is not constrained by topological considerations, since an Abelian gauge theory is under discussion. The induced covaiant current $j^\mu_{\text{induced}} = -\delta I_{\text{induced}}/\delta A_\mu = -\frac{\sigma}{4\pi}\, \epsilon^{\mu\alpha\beta} F_{\alpha\beta}$, leads to a planar current, $j^i_{\text{induced}} = \frac{\sigma}{2\pi}\, \epsilon^{ij} E^j$, which is perpendicular to the electric field E^i. This is precisely the situation in the Hall effect. The physical meaning of the parameter σ, and dynamical mechanisms that lead to its quantization are subjects of current research.[9]

VI. CHERN-SIMONS AND ANOMALOUS GAUGE THEORIES

A four-dimensional gauge theory with a single multiplet ψ of chiral fermions in the fundamental representation is in general inconsistent owing to the anomaly in the gauge current. The theory is governed by $\mathcal{L} = \frac{1}{2}\, tr\, F^{\mu\nu} F_{\mu\nu} + i\bar{\psi}(\partial + A)\psi$ and the field equation $D_\mu F^{\mu\nu} = J^\nu$ implies $0 = D_\nu D_\mu F^{\mu\nu} = D_\nu J^\nu$, but is contradicted by the anomaly

$$(D_\nu J^\nu)_a = \pm \frac{i}{24\pi^2}\, tr\, \partial_\mu\, \epsilon^{\mu\alpha\beta\gamma}\, tr\, T^a (A_\alpha \partial_\beta A_\gamma + \frac{1}{2} A_\alpha A_\beta A_\gamma) \neq 0 \quad . \qquad (6.1)$$

[The sign depends on the chirality.] An alternative/equivalent symptom of the problem is that the chiral fermion determinant $D(A) \equiv \det(\partial + A)$ is not gauge invariant.[7]

Although the model is inconsistent, one can go a certain way to give a consistent mathematical description of its anomalies.[8] This makes use of the Chern-Simons density and its descendants. However, the dimensional ladder begins with the Chern-Pontryagin density in six dimensions, as is given in (2.18). Let me remind that the Chern-Pontryagin density P is a 6-form in six dimensions; the Chern-Simons density Ω_0 is a 5-form in five dimensions; Ω_1, a 4-form, lives in four dimensions; while the 3-form Ω_2 is three dimensional. From the latter two, one may construct a 1-cocycle and a 2-cocycle by integration: $\omega_1(A;g) = \int d^4x\, \Omega_1(A;g)$, $\omega_2(A;g_1,g_2) = \int d^3x\, \Omega_2(A;g_1,g_2)$. The role that these cocycles play in the [inconsistent] gauge theory is the following.

A. 1 – Cocycle

Since the fermion deteminant is not gauge invariant, one may inquire how it transforms under a finite gauge transformation. The answer is

$$D(A^g) = e^{i2\pi\omega_1(A;g)} D(A) \quad , \qquad (6.2)$$

with a negative exponent for the opposite chirality. Infinitesimally, this reproduces the anomaly since according to (2.19b) $2\pi\omega_1(A, I+\theta) \approx \int d^4x \frac{1}{24\pi^2}\, tr\, \theta d(AdA + \frac{1}{3}A^3)$; compare (6.1). Moreover, it is easy to show that $\omega_1(A;h)$ satisfies $\omega_1(A^g; g^{-1}h) = \omega_1(A;h) - \omega_1(A;g)$. Hence, $e^{-i2\pi\omega_1(A;h)}$ transforms under g the same way as $D(A)$, and $-2\pi\omega_1(A;h)$ is recognized as the Wess-Zumino action.[3, 10]

Thus, we see that the haphazard-looking anomalous divergence (6.1) has a sensible mathematical origin. It is the infinitesimal gauge variation of the Chern-Simons 5-form; while the finite gauge change of the Chern-Simons 5-form appears as a 1-cocycle in the finite gauge change of the fermionic determinant. Of course, we recognize that the connection between these two statements arises from the fact that the covariant divergence of a gauge current is directly a measure of the infinitesimal gauge variation of the fermionic effective action [$\equiv$ logarithm of the fermionic determinant] since a current is the variation of an action with respect to the gauge potential, which under an infinitesimal gauge transformation changes by a covariant derivative. Finally, it is seen that the 1-cocycle condition (2.23), taken in infinitesimal form, is just the Wess-Zumino consistency condition.[3, 10]

It is very elegant that one can unify the form of the anomaly, the structure of the Wess-Zumino term, and the Wess-Zumino consistency condition into one mathematical structure: the Chern-Simons term, its gauge variation and the 1-cocycle. However, we physicists must not be blinded by mathematical dazzle. In particular, contrary to some assertions in the literature,[10 11] it is not true that *all* chiral anomalies need satisfy the Wess-Zumino condition, *i.e.*, be infinitesimal 1-cocycles. The point is the following. In a consistent gauge theory – in one for which all gauge currents are covariantly conserved – there may be anomalies in fermionic currents that are classically conserved not because they are gauge currents, but for some other reason, *e.g.*, because of a Noether symmetry. These anomalies are *not* related to gauge variations of a fermionic determinant, and do not take the form (6.1). Indeed, the gauge non-covariance of (6.1) indicates that gauge invariance is lost in an anomalous theory. But in a consistent gauge theory, gauge invariance is maintained, and all physically interesting quantities, including anomalously non-conversed symmetry currents, must be gauge invariant, and so must be their [anomalous] divergences.

Examples may be drawn from QCD – a consistent gauge theory – with massless quarks. Noether's theorem would indicate that an axial vector singlet current j^μ_5 is conserved as a consequence of the apparent chiral symmetry. However, the axial anomaly gives the current an anomalous divergence.

$$\partial_\mu j_5^\mu = -\frac{1}{8\pi^2} tr\, {}^*F^{\mu\nu}F_{\mu\nu} \qquad (6.3)$$

Although the right-hand side is a total divergence, see (2.1), it is not merely (6.1) with T^a replaced by I; (6.3) is gauge invariant while (6.1) is not. In the massless QCD model, there are also non-singlet axial vector currents, j_{5a}^μ, which classically are covariantly conserved. But, quantum mechanically, they too suffer covariant anomalous divergences.

$$(D_\mu j_5^\mu)_a = -\frac{i}{8\pi^2} tr\, T^{a*}F^{\mu\nu}F_{\mu\nu} \qquad (6.4)$$

Eq (6.4), unlike (6.1), is gauge covariant, the right-hand is not a total divergence. The Wess-Zumino consistency condition is not satisfied, but in no sense is (6.4) "wrong", as has been occassionally stated in the literature.

Even though the gauge covariant anomalies are not given by the infinitesimal gauge variation of the Chern-Simons term, *i.e.*, they are not infinitesimal 1-cocycles, they are determined by the *arbitary* variation of the five-dimensional Chern-Simons 5-form; see (2.19). [To obtain the normalization in (6.4), we multiply (2.19) by 2π - cochains enter with this additional factor as seen in (6.2); furthermore, we multiply by 2 because the currents in (6.3) and (6.4) refer to massless Dirac fermions, which are composed of a pair of chiral fermions.]

Since the following identity may be verified,

$$\begin{aligned}\frac{3}{2}\,{}^*F^{\mu\nu}F_{\mu\nu} &= \partial_\mu\, \epsilon^{\mu\alpha\beta\gamma}(A_\alpha\partial_\beta A_\gamma + \frac{1}{2}A_\alpha A_\beta A_\gamma) \\ &\quad + D_\mu\, \epsilon^{\mu\alpha\beta\gamma}(\partial_\alpha A_\beta A_\gamma + A_\alpha\partial_\beta A_\gamma + \frac{3}{2}A_\alpha A_\beta A_\gamma)\end{aligned} \qquad (6.5)$$

it follows that a gauge current J_a^μ which satisfies the anomaly equation (6.1) is mathematically related a current J_a^μ with anomaly of the form (6.4) by the formula[12]

$$\begin{aligned}J_a^\mu &= J_a^\mu \pm \frac{i}{24\pi^2}\, tr\, T^a\epsilon^{\mu\alpha\beta\gamma}\left(\partial_\alpha A_\beta A_\gamma + A_\alpha\partial_\beta A_\gamma + \frac{3}{2}A_\alpha A_\beta A_\gamma\right) \\ (D_\mu J^\mu) &= \pm\frac{i}{16\pi^2}\, tr\, T^a *F^{\mu\nu}F_{\mu\nu}\end{aligned} \qquad (6.6)$$

[The current j^μ is composed of both J^μ's, one for each chirality.] We emphasize that this is a mathematical relation, not a physical one, since the two types of currents arise in different physical contexts: j_a^μ is a symmetry current in a consistent gauge theory; J_a^μ is a gauge current in anomalous gauge theory.

The Feynman graphs which arise n the evaluation of matrix elements for the two currents can coincide. What is different is the regularization procedure that is required to evaluate them. The graphs do not have a unique value; their ambiguity, *confined to local polynomials*, is not fixed mathematically, but physically.

A similar story may be told for two-dimensional gauge theories. When the theory is consistent, for example massless QCD, the classically conserved axial currents [singlet and non-singlet] possess an anomaly that is gauge covariant, but does not satisfy the Wess-Zumino consistency condition.

$$\partial_\mu j_5^\mu = \frac{i}{2\pi}\, tr\, \epsilon^{\mu\nu}F_{\mu\nu} \qquad (6.7)$$

$$(D_\mu j_5^\mu)_a = \frac{1}{2\pi}\, tr\, T^a\epsilon^{\mu\nu}F_{\mu\nu} \qquad (6.8)$$

The expression in (6.8) is related to the arbitrary variation of the three-dimensional Chern-Simons 3-form in the dimensional ladder descending from the four-dimensional Chern-Pontryagin 4-form; see (2.11) and (2.16a). In the inconsistent theory, the gauge current's anomaly is

$$(D_\mu J^\mu)_a = \frac{1}{4\pi}\, tr\, T^a(g^{\mu\nu} \pm \epsilon^{\mu\nu})\partial_\mu A_\nu\ . \qquad (6.9)$$

The first term, involving $g^{\mu\nu}$ is not even a form, yet it is present to insure that fermions with one chirality couple only to $\frac{1}{2}(g^{\mu\nu}+\epsilon^{\mu\nu})A_\nu$ which is non-vanishing only in the light-cone component A^- and those with the other to $\frac{1}{2}(g^{\mu\nu}-\epsilon^{\mu\nu})A_\nu$ which is non-vanishing only for A^+. [One finds that this is what the interaction $\bar\psi(1\pm i\gamma_5)A\psi$ entails, owing to the two-dimensional formula $i\gamma^m u\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu u$. The gauge variation of the local polynomial $\frac{1}{2}\int d^2x\, tr\, A^\mu A_\mu$ is $tr\, T^a\partial_\mu A^\mu$, hence this contribution to the anomaly is trivial since it could be removed by modifying the definition of the fermionic determinant. Nevertheless, such a modification cannot be made as long as one wishes to preserve the algebraic fact that chiral fermions couple only to one component - either to A^+ or to A^-, depending on chirality. The second term in the anomalous divergence (6.9) is a form, and is related to the gauge variation of the three-dimensional Chern-Simons 3-form; see (2.12a), (2.13) and (2.16b). Once again mathematical relationships between the gauge covariant symmetry currents j_{5a}^μ and the gauge source currents J_a^μ may be given.

$$\begin{aligned}J_{\pm a}^\mu &= J_a^\mu - \frac{1}{4\pi} tr\, T^a(g^{\mu\nu} \mp \epsilon^{\mu\nu})A_\nu \\ (D_\mu J_5^\mu)_a &= \pm\frac{1}{4\pi}\, tr\, T^a\epsilon^{\mu\nu}F_{\mu\nu}\end{aligned} \qquad (6.10)$$

The two dimensional model puts into evidence yet another interesting fact. In the consistent gauge theory with massless Dirac fermions of both chiralities, the total current formally is the sum of two individual currents each constructed with spinors of the respective chirality. However, from (6.9) we see that owing to the trivial $tr\, T^a\partial_\mu A^\mu$ term, even the sum of the two currents is not conserved. To obtain a conserved current one must add the further term proportional to A_μ^a to the current. Since the current is the variation with respect to A_μ^a of the logarithm of the fermion determinant, we conclude that gauge invariance forces the determinant for massless Dirac fermions to be composed of three factors. Two factors correspond to fermions of definite chirality. However, there is a third factor, dictated by gauge invariance, involving $\exp i \int tr\, A^{\mu} A_{\mu}$. Thus, contrary to assertions in the literature, the Euclidean space determinant of chiral fermions does not possess the same real part as that of Dirac femions; nor, as in seen in (6.9), is the anomaly in Euclidean space always imaginary [proportional to the anti-symmetric tensor].[13]

Finally, let us observe that according to (6.2), the 1-cocycle may be written as

$$e^{i2\pi\omega_1(A;g)} = D(A^g)/D(A) \tag{6.11}$$

which shows that ω_1 is trivial, since it can be expressed as [compare (2.24)]:

$$\omega_1(A;g) = -\frac{i}{2\pi}\left[\ln D(A^g) - \ln D(A)\right] \ . \tag{6.12}$$

Of course, $\ln D(A)$ is a non-local functional of A; hence, the triviality noted in (6.12) cannot be used to remove the anomaly. [One is free to redefine the fermion determinant by the integrated exponential of local polynomials in A.] However, one may use (6.12), continued to Euclidean space, to remove the 1-cocycle in the realization of gauge invariance on wave functions in even-dimensional space where a Hamiltonian description for a topologically massive Lagrangian in odd-dimensional space-time is given, as in (5.3). We emphasize that in this application the fermion determinant is being used as formal mathematical entity, satisfying (6.12), hence useful for the removal of the 1-cocycle, as in (2.25).[14] No statement is being made concerning physical fermions in topologically massive gauge theories.

B. 2 – Cocycle

When a current J^{μ} possess an anomalous divergence and is present in the Hamiltonian owing to a gauge coupling, its components must satisfy anomalous commutation relations, so that $i[H,\ J^0]$ reproduce the anomaly in the divergence of J^{μ}.[15] It has now been shown that these anomalous commutations are related to the 2-cocycle introduced in (2.16) and (2.18).

When the composition law for operators implementing a representation is projective as in (2.30)

$$U(g_1)U(g_2) = e^{i2\pi\omega_2(A;g_1,g_2)}U(g_{12}) \tag{6.13}$$

and an infinitesimal description is given

$$\begin{aligned} U(I+\theta) &= I + \int d^3x\theta^a G_a(\mathbf{x}) + \dots \\ i2\pi\omega_2(A;I+\theta_1,I_2)\theta_2) &= \frac{1}{2}\int d^3x d^3y \theta_1^a(\mathbf{x})\theta_2^b(\mathbf{y})S_{ab}(A;\mathbf{x},\mathbf{y}) \end{aligned} \tag{6.14}$$

where $G_a(\mathbf{x})$ is the infinitesimal gauge transformation generator, then the infinitesimal composition law departs from the Lie algebra – S provides an extension

$$[G_a(\mathbf{x}),\ G_b(\mathbf{y})] = f_{abc}\ G_c(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) + S_{ab}(A;\mathbf{x},\mathbf{y}) \ . \tag{6.15}$$

The generator consists of two contributions $\left(D_i\frac{\delta}{\delta A_i}\right)_a$ which implements the infinitesimal gauge transformation on the gauge field variables, and J_a^0 which does that job on chiral femions. Evidently, the extension in (6.14) comprises both the anomaly in the $[J_a^0,\ J_b^0]$ commutator and the anomalous response to gauge transformations of J_a^0 [naively and formally J_a^0 is gauge invariant].

$S_{ab}(A;\mathbf{x},\mathbf{y})$ may be determined from the mathematical descent equations. One finds in three dimensions [Hamiltonian formulation of a chiral gauge theory in four-dimensional space-time] [see (2.18)]

$$S_{ab}(A;\mathbf{x},\mathbf{y}) = \frac{i}{24\pi^2}\, tr\, T^a\{T^b, T^c\}\ \epsilon^{ijk}\partial_i A_c^j \partial_k \delta(\mathbf{x}-\mathbf{y}) \tag{6.16}$$

In one dimension [Hamiltonian formulation of a chiral gauge theory in two-dimensional space-time] the answer is [see (2.16)]

$$S_{ab}(A;x,y) = \frac{i}{4\pi}\delta_{ab}\ \delta'(x-y) \tag{6.17}$$

On the other hand, the commutators may also be directly computed and, of course, the results should agree with the mathematical formulation, apart from trivial [infinitesimal] 2-cocycles. That they do in the Abelian case can be checked by comparison with the old calculations, while the non-Abelian case has now also been analyzed.[15]

It should be noticed that the Abelian 2-cocycles are trivial. In three dimensions we have

$$\begin{aligned} \omega_2(A;\theta_1,\theta_2) &= \int A d\theta_1 d\theta_2 = \Delta\alpha_1 \\ \alpha_1(A;\theta) &= -\int A dA\theta \end{aligned} \tag{6.18}$$

while in one dimension

$$\begin{aligned} \omega_2(\theta_1,\theta_2) &= \int (d\theta_1\theta_2 - \theta_1 d\theta_2) = \Delta\alpha_1 \\ \alpha_1(A;\theta) &= \frac{1}{2}\int A\theta \end{aligned} \tag{6.19}$$

However, the non-Abelian 2-cocycles are non-trivial.

The inconsistency of the theory is now apparent from (6.13): one cannot require states to be gauge invariant, even up to a phase, when the representation composition law is projective. Therefore, gauge invariance is lost.[16]

C. 3 – Cocycle

There is no known 3-cocycle associated with chiral anomalies – no failures of relevant Jacobi identities have been found. However, a violation of the Jacobi identity appears in the quark model. When the Schwinger term in the commutator between time and space components of a current is a c-number, the Jacobi identity for triple commutators of spatial current components must fail.[17] Since deep-inelastic scattering data indicates that the Schwinger term is indeed a c-number,[18] consistent with quark-model calculations,[19] the Jacobi identity for spatial current components should fail in the quark model, and this has been verified in perturbative calculations.[20] The quark-model algebra of time and space components of vector and axial vector currents closes on $U(6) \times U(6)$[21], and the above remarks indicate that a 3-cocycle occurs. However, a well-defined mathematical formulation is problematical, since the Schwinger term very likely is quadradically divergent.[22]

VII. CONCLUSION

The Chern-Simons structure has proven itself to be an unexpected and valuable addition to our mathematical tools in physics. I have here discussed some of its applications; there are others in higher dimensions which have been found by those working on various Kaluza-Klein programs, especially when embellished by a supersymmetry. Doubtlessly, other applications will come to light in the future.

Let me indicate one possible direction of research for higher-dimensional gauge theories. In four dimensions, the energy momentum tensor may be written as

$$\theta^{\mu\nu} = tr(F^{\mu\alpha} F^{\nu}_{\alpha} + {}^{*}F^{\mu\alpha *} F^{\nu}_{\alpha}) \qquad (7.1)$$

and the energy density is

$$\theta^{00} = \frac{1}{2}(E^i_a E^i_a + B^i_a B^i_a) \qquad (7.2)$$

The "potential" term $B^i_a B^i_a$ is expressible as the square of the variation of the Chern-Simons 0-cocycle by virtue of (2.11). Conventional, higher dimensional Yang-Mills theory possess the same energy density as in (7.2), since the Lagraugian in any number of dimensions is taken as in four. However, suppose we generalize the potential term in (7.2) as it is expressed with the help of the Chern-Simons 0-cocycle. Thus, in five-space [relevant to a gauge theory in six-dimensional space-time] $\frac{\delta\omega_5(A)}{\delta A^i_a} \propto \epsilon^{ijklm} tr T^a F_{jk} F_{lm}$ [compare (2.19a)] and the square of this would replace $B^i_a B^i_a$ in (7.2). Correspondingly, (7.1) would be generalized by retaining that expression, but defining the dual by

$${}^{*}F^{\mu\nu}_a \propto \epsilon^{\mu\nu\alpha\beta\gamma\delta}\, tr\, T^a F_{\alpha\beta} F_{\gamma\delta} \qquad (7.3)$$

This provides new higher-dimensional gauge theories, whose structure should surely be unraveled.

In conclusion, one may ask why structures in physical dimensions – the action in four space-time and the Hamiltonian in three-space – should be described by mathematical objects that descend from the six-dimensional Chern-Pontryagin density and the five-dimensional Chern-Simons density. While such questions often

have no answer, in the present instance the reason can be given.[23] What is involved is a topological obstruction to defining gauge avariantly is the chiral fermion determinant. To expose this obstruction, one embeds the gauge potential into a two-parameter homotopy family and the obstruction manifests itself in the vanishing of the determinant as the parameters are varied. This vanishing corresponds to zero eigenvalues of the Dirac equation in six dimensions [2 homotopy + 4 coordinate] and these are counted by the six-dimensional Atiyah-Singer index theorem, involving the Chern-Pontryagin 6-form.

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ERRATA

Page 3

Line 3: Spelling correction "their"

Page 4

2nd paragraph, line 3 from bottom: Replace comma by period after "III". Replace remainder of text by: "The latter are devoted to explaining mathematical preliminaries and exemplifying them in simple quantum mechanical settings."

Page 6

Last ~~line~~: ~~Delete~~: "With the index α suppressed, we have" and also delete eq. (2.9). ~~Replace by~~. "No simple expression is available for Ω_2^α. However, for gauge transformations near the identity, $g = e^\theta = I + \theta + \dots$ we have [with the index α supressed]

$$\Omega_2(A; I + \theta_1 + \dots,\ I + \theta_2 + \dots) = -\frac{1}{16\pi^2}\ tr \left[\theta_1 \frac{d}{dx}\theta_2 - \theta_2 \frac{d}{dx}\theta_1\right] + \dots \qquad (2.9)\text{ "}$$

Page 8

Delete last formula in eq. (2.15) and replace by:

$$\text{" }\Omega_2(A; I + \theta_1 + \dots,\ I + \theta_2 + \dots) = -\frac{1}{16\pi^2}\ tr \left[\theta_1 d\theta_2 - \theta_2 d\theta_1\right] + \dots \text{ "}$$

Delete equation (2.16c) entirely.

Page 9

Delete last formula in eq. (2.18) and replace by:

$$\text{" }\Omega_2(A; I + \theta_1 + \dots, I + \theta_2 + \dots) = \frac{i}{12(2\pi)^3}\ tr\ A(d\theta_1 d\theta_2 - d\theta_2 d\theta_1) + \dots \text{ "}$$

Line above (2.19a): Replace "the" by "other"; replace "needed" by "used".

Delete eq. (2.19c).

Page 13

Line above (2.38): Spelling correction "correspondingly".

Page 19

In eq. (3.9): Replace formula by

$$" \left[v^1, [v^2, v^3]\right] + \left[v^2, [v^3, v^1]\right] + \left[v^3 [v^1, v^2]\right] = e\nabla \cdot \mathbf{B} \qquad (3.9) "$$

Line below (3.10): Spelling correction "occurence".

Page 21

First paragraph next-to-last line: Figure caption should read: "Tetrahedron at point $\mathbf{r}$ defined by three translations $\mathbf{a}_i$. The 3-cocycle is proportional to the flux out of the tetrahedron."

Page 22

At end of Section III, before starting Section IV, insert:

"Let us also understand that the 3-cocycle arising from aribtrary magnetic sources may be put into evidence in an algebraic, gauge invariant manner, without introducing a Hilbert space, linear operators, and vector potentials as was done above.

A magnetic field, $\mathbf{B}$, regardless whether it is sourceless, does no work on a charged (e) particle at $\mathbf{r}$ which moves in $\mathbf{B}$. The particle's energy is only kinetic; consequently, the Hamiltonian does not see the magnetic field: $H = \frac{1}{2}v^2$, $\mathbf{v} \equiv \dot{\mathbf{r}}$. The magnetic field is present in the Lorentz force law, $\dot{\mathbf{v}} = e\mathbf{v} \times \mathbf{B}$, which is regained upon commutation with H when the commutator algebra (3.8) is postulated. [Commutators with $\mathbf{r}$ are conventional.]

Translations of $\mathbf{r}$ by $\mathbf{a}$ are implemented by $U(\mathbf{a}) \equiv e^{i\mathbf{a}\cdot\mathbf{v}}$, since $U(\mathbf{a})\mathbf{r}\, U^{-1}(\mathbf{a}) = \mathbf{r} + \mathbf{a}$. However, these quantities do not represent the Abelian translation group faithfully, since (3.8) implies

$$U(\mathbf{a}_1)U(\mathbf{a}_2) = e^{ie\Phi}U(\mathbf{a}_1 + \mathbf{a}_2) \qquad (3.16)$$

Moreover, by considering the triple product $U(\mathbf{a}_1)U(\mathbf{a}_2)U(\mathbf{a}_2)$, associated in the two different ways, one finds a 3-cocycle, as before, which is the flux emanating from the tetrahedron of Fig. 1.

When $\nabla \cdot \mathbf{B} = 0$, no flux emanates from a closed surface; the cocycle vanishes; associativity is regained and $\mathbf{v}$ may be realized by the linear operator $\mathbf{v} = -i\nabla - e\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$. When there are sources, $\nabla \cdot \mathbf{B} \neq 0$, the flux is non-zero, but associativity will prevail if ω_3 is 2π times an integer, since then $e^{i\omega_3} = 1$. This requirement forces: (1) $\nabla \cdot \mathbf{B}$ to consist of delta functions so that the total flux not vary continuously when the $\mathbf{a}_i$'s change, *i.e.*, the sources must be monopoles; (2) since a monopole os strength g produces the cocycle $-4\pi eg$, eg must satisfy the Dirac quantization condition. In this way, removal of the 3-cocycle, which is necessary for conventional quantum mechanics with associative operators on Hilbert space, limits magnetic sources to quantized Dirac monopoles. Other magnetic sources lead to a non-associative algebra.[5]"

Page 27

Delete last sentence: "Indeed . . . action.[8]"

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End of first paragraph: Change reference [9] to [8]. At end of Section V, before Section VI, add new paragraph:

"In a third application, the Chern-Simons term has been used to model the electomagnetic properties of graphite.[9]"

In eq. (6.1) left-hand side: Change ν to μ in subscript and superscript.

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Eq. (6.6) and lines above and below: It should be understood that the left-hand member of both equalities and in lines above and below (four places) the symbol J^μ is script capital "jay" ($\mathcal{J}$). Also, in the script captial "jay"'s, insert subscript $\pm$ below superscript μ, to left of subscript a, in all four places.

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Eq. (6.10): In first formula "$+\frac{1}{4\pi}\ldots$" is replaced by "$-\frac{1}{4\pi}\ldots$". In second formula, "$\mp\frac{1}{4\pi}\ldots$" is replaced by "$\pm\frac{1}{4\pi}\ldots$". Also, in both formulas the left-hand J^μ_a is a script capital "jay", and so is the one two lines below (three places). Insert in the script capital "jay" 's subscript $\pm$ below the superscript μ, to left of subscript a, in all three places.

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Fourth line, bracketed phrase: Delete opening of parenthesis.

References Update

3. Delete Mickelsson citation both to *Lett. Math. Phys.* and to *Comm. Math. Phys.* Insert after Faddeev and Shatashvili citation, after "(1984)": "[*Theor. Math. Phys.* **60**, 770 (1984]; J. Mickelsson, *Comm. Math. Phys.* **97**, 361 (1985)".

5. Jackiw reference: *Phys. Rev. Lett.* 54, 159 (1985). *Phys. Lett.* (in press), *Phys. Rev. Lett.* (in press). Grossman reference: *Phys. Lett.* 152B, 93 (1985); Hou reference: *Chinese Phys. Lett.* (in press); Wu and Zee reference: *Phys. Lett.* 152B, 98 (1985). Add after Wu - Zee reference: "In a non-associative algebra, where the three-fold Jacobi identity fails, one may impose a four-fold identity, the so-called Malcev identity, which requires that $\nabla \cdot B$ be constant. When this fails, one can impose a five-fold identity, *etc.* For details see B. Grossman (preprint); M. Günyadin and B. Zumino (preprint)".

8. Delete entire reference.

9. Renumber this as 8. Insert new reference 9: "G. Semenoff, *Phys. Rev. Lett.* 53, 2449 (1984)."

11. Change Z. Zee to A. Zee; delete "R. Stora (preprint)". Bardeen and Zumino reference: *Nucl. Phys.* B244, 277 (1984), add after "modified": "The variety of possible anomaly equations is surveyed by K. Fujikawa, *Phys. Rev. D* 31, 341 (1985)."

12. Change last . (period) to ; (semicolon) add "and (preprint)".

15. Next-to-last line, change J. Frenkel to I. Frenkel.

16. Jackiw and Rajaraman reference: *Phys. Rev. Lett.* 54, 1219 (1985); add: "and R. Rajaraman (preprint)".

24. Add: "J. Lott, *Comm. Math. Phys.* 97, 371 (1985)."