A geometric view on topologically massive gauge theories

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Abstract

The topologically massive gauge theory of Deser, Jackiw and Templeton is understood from Souriau's Principle of General Covariance. The non-gauge invariant mass term corresponds to a non-trivial class in the first cohomology group of configuration space, generated by the Chern-Simons secondary characteristic class. Quantization requires this class to be integral.

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Souriau's Principle of General Covariance [1] provides us with an insight into the geometric structure of field theories. For simplicity, we formulate it for a non-Abelian gauge theory. Remember, that a Yang-Mills (YM) field A is a connection form on a principal G - bundle P over a connected Riemannian manifold M, where G, the gauge group, is a compact and connected Lie group. A gauge transformation is a fibre-preserving automorphism of P which leaves an arbitrarily chosen basepoint fixed. The gauge transformations form an infinite-dimensional "Lie" group denoted by $\mathcal{G}$. Its Lie algebra consists of G - invariant vertical vector fields which vanish at the basepoint [2].

$\mathcal{G}$ acts freely on $\mathcal{A}$, the linear space of connections (P,G). The requirements of gauge invariance is expressed by saying that the true configuration space $\mathcal{C}$ - Souriau's "hyperespace" - consists of YM fields modulo gauge transformations, $\mathcal{C} = \mathcal{A}/\mathcal{G}$. $\mathcal{A}$ is a principal $\mathcal{G}$ - bundle over $\mathcal{C}$. The Principle of General Covariance says now that the theory should be described by a <u>closed 1-form</u> on $\mathcal{C}$.

To get a better understanding, consider the local picture: Choose a gauge - a family of sections $s_\alpha : V_\alpha \to P$, where the V_α's form an open covering of M - and consider the YM potentials $A_\alpha = s_\alpha^* A$. A gauge transformation $\gamma \in \mathcal{G}$ provides us with a new gauge $s_\beta(x) = \gamma(s_\alpha(x))$. In $V_\alpha \cap V_\beta$ this is also obtained as $s_\beta(x) = s_\alpha(x)\gamma_{\alpha\beta}(x)$. The action of γ on $\mathcal{A}$ is given then locally as $(\gamma \cdot A)^\alpha = s_\alpha^*(\gamma^* A) = (\gamma \cdot s_\alpha)^* A = s_\beta^* A = A_\beta = \mathrm{Ad}\gamma_{\alpha\beta}^{-1} A_\alpha + \gamma_{\alpha\beta}^{-1}\, d\gamma_{\alpha\beta}$. Assume we have a Lagrangian $\mathcal{L}$. The classical action,

$$S_\alpha(A) = \int_M \mathcal{L}(A_\alpha), \qquad (1)$$

is then a real valued function on $\mathcal{A}$, and its exterior derivative δS_α – the physicists' first variation – yields the equations of motion

$$\delta S_\alpha = 0 \qquad (2)$$

The action function (1) is however highly ambiguous: adding a surface term df(A) to $\mathcal{L}$ the action changes by

$$\int_M df(A) = \int_{\partial M} f(A). \tag{3}$$

The usual finite-action requirements imply however that the fields be constant (or vanish) on the boundary of M. Consequently, the action is changed merely by a constant and the equations of motion remain thus invariant.

Next, we have the freedom in choosing the gauge. The physical requirement is now that this have the same effect as adding a surface term to $\mathcal{L}$. Thus, a gauge transformation should change the action by a constant at most, leading to identical field equations. In other words, the first variation δS should be independent of the choice of gauge. An infinitesimal gauge transformation X (an element of the Lie algebra of $\mathcal{G}$) acts on A according to $X_A = L_X A = F(X,\cdot) + D(A(X))$. Having a gauge-invariant theory means thus that

$$\delta S_A(X_A) = 0 \quad \forall A \in \mathcal{A}. \tag{4}$$

Now we are home: (4) is just the condition for the exact 1-form δS on $\mathcal{A}$ to project to a closed 1-form σ on $\mathcal{C}$: $\delta S = \pi^* \sigma$, where π is the projection $\pi : \mathcal{A} \to \mathcal{C}$.

The standard choice for the Lagrangian is $\mathcal{L}_0 = \frac{1}{4}tr(F_A * F)$, which is strictly gauge-invariant. So the action (1) projects to a function on $\mathcal{C}$ and the 1-form σ is thus exact. More generally, a Lagrangian is strictly gauge-invariant if the associated σ is exact. The problem we address ourselves is to know of the geometric framework outlined above allows for a more subtle behaviour: is it possible to get a physically admissible theory with non-gauge invariant action? This would manifest itself in having closed, but not exact 1-forms on $\mathcal{C}$. In other words, we are interested in knowing whether the cohomology group $H^1(\mathcal{C}; R)$ is trivial or not. For additional information on cohomology theory see, for example, [3,4]. Assume hence we are given a closed 1-form σ on C. Its pullback $\pi^*\sigma$ to A is exact there, because A is contractible. Hence $\pi^*\sigma = \delta S$ for a function S on A. Choose arbitrarily a configuration $c \in C$ and denote by S^c the restriction of S to the fibre $A_c = \pi^{-1}(c)$. Identifying A_c with the structure group $\mathcal{G}$, S^c can also be viewed as a function on $\mathcal{G}$. A vector field on A which is tangent to the fibre A_c has the form X_A, $A \in A_c$, associated to an infinitesimal gauge transformation. Hence $\delta S^c_A(X_A) = \pi^*\sigma(X_A) = \sigma(\pi_* X_A) = 0$: S^c is a closed function on $\mathcal{G}$ and hence we get a cohomology class $[S^c] \in H^0(\mathcal{G}, R)$. This class is furthermore seen to be independent of the choice of S and the configuration c (as long as $\delta S = \sigma$). We conclude that

$$H^1(C, R) \simeq H^0(\mathcal{G}, R). \tag{5}$$

Observe that a closed function on $\mathcal{G}$ is locally constant, i.e. constant on the path components on $\mathcal{G}$. In fact, $H^0(\mathcal{G}, R) \simeq H^0(\mathcal{G}, \mathbb{Z}) \otimes \mathbb{R}$, and $H^0(\mathcal{G}, \mathbb{Z})$ is isomorphic to $\pi_0(\mathcal{G})$. Therefore, to get a non-trivial situation, G must have more than one path-component.

In what follows, we describe a situation where this indeed happens. We consider a 3-dimensional gauge theory over the compactified 3-space $M = S^3$. All Chern classes vanish and in fact any G-bundle P over S^3 is trivial. Choosing a global section, we write $P \simeq S^3 \times G$. A gauge transformation is given now by a function $\gamma : S^3 \to G$. γ and γ' lie in the same path component of $\mathcal{G}$ if and only if they define the same class in $\pi_3(G)$, proving that

$$\pi_0(\mathcal{G}) \simeq \pi_3(G). \tag{6}$$

But, for any compact and connected Lie group G (except SO(4)) $\pi_3(G)$ is known to be $\mathbb{Z}$. This is Abelian, so $H^1(C, R) \simeq H^0(\mathcal{G}, R) \simeq \pi_3(G) \otimes \mathbb{R} \simeq \mathbb{R}$. We conclude that we have an <u>infinite number</u> of inequivalent, gauge-non-invariant, but still physically admissible models.

Now we proceed to identify these models. More precisely, we show that $H^0(\mathcal{G}, R)$ is generated by the Chern-Simons secondary class [4].

It follows from the general theory that, for an arbitrary connection A on a G-bundle P over S^3, the (first) Chern-Simons class is given by the 3-form,

$$\Omega = \tfrac{1}{2}\, \mathrm{tr}(A \wedge F - \tfrac{1}{3} A \wedge A \wedge A) \tag{7}$$

whose pull-back to S^3 by a local section s_α is

$$\Omega_\alpha = \epsilon_{\mu\nu\lambda}\, \mathrm{tr}(A^\mu_\alpha F^{\nu\lambda}_\alpha - \tfrac{2}{3} A^\mu_\alpha A^\nu_\alpha A^\lambda_\alpha) . d^3x \tag{8}$$

Ω is closed, $d\Omega = 0$ and defines thus a class in $H^3(P, R)$, which is actually independent of the choice of A [4]. This allows us to define a function on $\mathcal{G}$, the group of gauge transformations. Indeed, having fixed a global section of P, a $\gamma \in \mathcal{G}$ is identified with a new section of P which we still denote by γ, $\gamma(x) = (x, \gamma(x))$. So $\gamma^*\Omega$ is a 3-form on S^3, and we can define

$$\omega(\gamma) = \int_{S^3} \gamma^* \Omega =$$

$$= \frac{1}{4} \int_{S^3} \epsilon_{\mu\nu\lambda}\, \mathrm{tr}(A^\mu_\gamma F^{\nu\lambda}_\gamma - \tfrac{2}{3} A^\mu_\gamma A^\nu_\gamma A^\lambda_\gamma)\, d^3x \tag{9}$$

where the subscript refers to γ viewed as a section of P. If γ_0 and γ_1 belong now to the same path component of $\mathcal{G}$, then we have a smooth path $\gamma(t)$ in $\mathcal{G}$ such that $\gamma(0) = \gamma_0$, $\gamma(1) = \gamma_1$. Stokes' theorem tells us however that

$$\omega(\gamma(1)) - \omega(\gamma(0)) = \int_{S^3} d(\gamma_t^* \Omega) = \int_{S^3} \gamma_t^*\, d\Omega = 0$$

because Ω is closed. This proves that $\omega(\gamma)$ is constant on the path components of $\mathcal{G}$. Thus $\delta\omega = 0$, and ω defines a class in $H^0(\mathcal{G}, \mathbb{R})$. Evaluating (9) gives

$$\omega(\gamma) = 4\pi^2 n, \quad [\gamma] \simeq n \in \mathbb{Z} \tag{10}$$

We conclude that (9) generates $H^0(\mathcal{G}, R)$, as stated above.

These results admit striking physical applications. As a matter of fact, the topologically massive gauge theory of Deser, Jackiw and Templeton [5] uses the Lagrangian

$$\frac{1}{2g^2} \, tr\left(F_{\mu\nu} F^{\mu\nu}\right) - \frac{m}{2g^2} \, \epsilon_{\mu\nu\lambda} \, tr\left(A^\mu F^{\nu\lambda} - \frac{2}{3} A^\mu A^\nu A^\lambda\right), \tag{11}$$

which differs from the standard $\mathcal{L}_0$ just by a multiple of the Chern-Simons term. They show then that the excitations are now massive, their mass is in fact m.

We end this paper by some remarks on quantization. Let us return to the general theory and consider, just like in finite-dimensional quantum mechanics [6], the "Feynmann factor"

$$\mathcal{F}(A) = \exp\left[iS(A)\right] \tag{12}$$

whose importance is seen, for example, from Feynman's path integral approach to quantization. Remember however, that Feynman's approach involves integration in configuration space. To have a well-behaved quantum system, (12) has project hence to a function $\mathcal{F}(c)$ on $\mathcal{C}$. But if A and A' project to the same configuration c, then $A' = \gamma \cdot A$ for some $\gamma \in \mathcal{G}$. On the other hand, $S(\gamma \cdot A) = r + S(A)$, where r is the real number labelling the class of S^c in $H^0(\mathcal{G}, \mathbb{R}) \simeq R$. So (12) projects to a function on $\mathcal{C}$ if and only if $\frac{r}{2\pi}$ is an integer. Using the isomorphism (5) this means that

$$\frac{1}{2\pi}[\sigma] \in H^1(C,\mathbb{Z}) \subset H^1(C,\mathbb{R}), \qquad (13)$$

the 1-form σ must define an integral (rather than merely real) cohomology class. Alternatively, observe that $\pi^*\sigma$ is exact, $\pi^*\sigma = dS$, and hence

$$S(A) = S(A_0) + \int_{A(t)} \pi^*\sigma = S(A_0) + \int_{\pi(A(t))} \sigma$$

where A_0 is an arbitrary basepoint in $\mathcal{A}$ and $A(t)$ a path in $\mathcal{A}$ which joins A_0 to A. But $\pi(A(t))$ is already a path in C, so (12) requires that

$$\exp\,[i\oint\sigma] = 1 \qquad (14)$$

for any loop in C, which is again (13). Conversely, if the integrality condition (13) holds, the Feynman factor (12) projects to a function on C we still denote by the same symbol F. The integer $n = [\sigma/2\pi]$ is then recovered as the winding number

$$n = \frac{1}{2\pi i}\oint\frac{\delta\mathcal{F}}{\mathcal{F}}\ . \qquad (15)$$

For the Deser-Jackiw-Templeton Langrangian (11) this gives, using (10),

$$\frac{2m}{g^2}\,\omega(\gamma) = 8\pi^2\frac{m}{g^2}\cdot n = 2\pi \times \text{integer},$$

where n labels the class of γ in $\pi_0(\mathcal{G}) \simeq \mathbb{Z}$. Mass is hence quantized,

$$m/g^2 = (1/4\pi) \times \text{integer}, \qquad (16)$$

cf. [5]. The theory can also be extended to gravity [1,5].

Remark: After this paper was completed, there appeared an article by Alvarez [7] containing similar ideas.

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