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SOME PROPERTIES OF THE FUNCTIONS SATYSFYING BELL'S INEQUALITIES IN RELATION TO QUANTUM MECHANICS.

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Contribution à la Conférence "Microphysical Reality and Quantum Formalism"
Urbino, Italie, Sept-Oct. 85

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Abstract :

A detailed comparison of Bell's inequalities (B.I.) and quantum mechanics (Q.M.) in an E.P.R.B. situation is given. It is first shown that Q.M. violates the original (3 directions) or generalized (4 directions) B.I. almost everywhere. The properties of functions satisfying the original B.I. are then derived and compared to Q.M. predictions. Finally, the behaviour of functions which satisfy B.I. and attempt to fit Q.M. is described. Altogether, an incompatibility is shown to be stronger than that resulting from just the usual examination.

Résumé :

Une comparaison détaillée des inégalités de Bell (I.B.) et de la mécanique quantique (M.Q.) dans une situation E.P.R.B. est developpée. Il est d'abord montré que la M.Q. viole les I.B. originales (3 directions) ou généralisées (4 directions) presque partout. Les propriétés des fonctions satisfaisant les I.B. originales sont ensuite établies et comparées à celles des prédictions de la M.Q. Finalement, on décrit le comportement de fonctions qui satisfont aux I.B. et tentent d'approcher la M.Q. Une incompatibilité radicale est ainsi mise en évidence.

In this contribution, a report will be made on what could be called a technical study of Bell's inequalities (B.I.) in relation to quantum mechanics (Q.M.) in the case of an E.P.R.B. (B for Bohm [1]) situation. B.I. have been derived to test some of the models (those models which are local and with added parameters in the common past) which are candidates for complementing Q.M. Only ideal cases will be treated : no sophisticated detection is allowed... nor additional hypothesis needed. Though well known, the basic scheme of the experiment is reminded as follows (fig. 1).

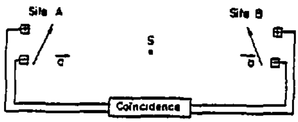


Fig. 1

The basic result of the experiment is the correlation rate defined with the usual notation as

$$(1)\quad P(\vec{a}, \vec{b}) = (\vec{a}_+\vec{b}_+) + (\vec{a}_-\vec{b}_-) - (\vec{a}_+\vec{b}_-) - (\vec{a}_-\vec{b}_+)$$

Two different B.I. will be considered

i) The original [2] 1964 B.I. which involves 3 directions and 3 polarizer settings

$$\begin{matrix} \text{site A} \\ \text{site B} \end{matrix} \quad \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{c} \end{pmatrix} \begin{pmatrix} \vec{b} \\ \vec{c} \end{pmatrix}$$

and from which 3 (double) B.I. of the type

$$(2)\quad |P(\vec{a},\vec{b}) - P(\vec{a},\vec{c})| \leq 1 + P(\vec{b},\vec{c})$$

can be written

ii) The generalized (see ref. [3]) B.I. which involves 4 directions and allows the choice of 4 + 2 = 6 polarizer settings

$$\begin{matrix} \text{site A} \\ \text{site B} \end{matrix} \quad \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{b}' \end{pmatrix} \begin{pmatrix} \vec{a}' \\ \vec{b} \end{pmatrix} \begin{pmatrix} \vec{a}' \\ \vec{b}' \end{pmatrix} \begin{pmatrix} \vec{a} \\ \vec{a}' \end{pmatrix} \begin{pmatrix} \vec{b} \\ \vec{b}' \end{pmatrix}$$

and from which 12 (double) generalized (G) B.I. of the type

(3) $$-2 \leq P(\vec{a},\vec{b}) + P(\vec{a},\vec{b}') + P(\vec{a}',\vec{b}) - P(\vec{a}',\vec{b}') \leq +2$$

can be written (12 as 3 (choices of 4 settings) times 4 (places for the minus sign)).

B.I. are used for correlated photons or spin-1/2 particles. In all cases it is assumed that the correlation rate $P(\vec{a},\vec{b})$ depends only on the unorientated angle between the vectors $\vec{a}$ and $\vec{b}$

(4) $$P(\vec{a},\vec{b}) = f(\theta_{\vec{a},\vec{b}}).$$

For spin-1/2 particles decaying from an $S = 0$ state, one has

(5) $$f(o) = -f(\pi) = -1$$

although the G.B.I. have been derived not to rely on this condition. On the other hand the condition

(6) $$f(\theta) = -f(\pi-\theta)$$

relies only the very definition (equ. 1) of $P(\vec{a},\vec{b})$ since the measurement $\vec{a}_+$ is always equivalent to that of $(-\vec{a})_-$. It follows that

(7) $$f(\pi/2) = -f(\pi/2) = 0$$ and

(8) $$f(o) = -f(\pi)$$ less restrictive than (5)

As well known, the Q.M. prediction for f is

(9) $$f_{Q.M.}(\theta) = -\cos\theta$$

which, of course, is in agreement with the conditions (4) to (8).

For photons resulting from a ($J = 0 \rightarrow J = 1 \rightarrow J = 0$) cascade, different but equivalent statements could be derived. Let us just recall the Q.M. prediction (labelled * for a photon case)

(10) $f^{*}_{QM}(\theta) = \cos 2\theta$

Indeed, most of the results can be established equivalently for photons or spin-1/2 particles because B.I. are basically concerned with the "shape" of the function f , irrespective of its phase or period. These results will be presented as follows : the direct comparison will describe how B.I. apply to Q.M. ; the reverse comparison will deal with the properties of the functions f_B, which satisfy B.I., as compared to that of f_{QM} , and finally the obstacles encountered when a fit of f_{QM} by a f_B is attempted will be examined.

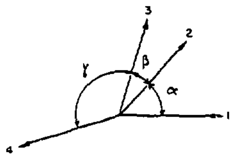
DIRECT COMPARISON : THE SYSTEMATIC VIOLATION OF BELL'S INEQUALITIES BY QUANTUM MECHANICS

It is well known that the Q.M. predictions can be in conflict with the B.I. but (in an EPRB situation) the desagreement is more systematic than sometimes claimed. Using the different inequalities which can be written for a given choice of 3 or 4 directions, it can be shown that for coplanar settings, Bell Inequalities are violated almost everywhere by Quantum Mechanics.

For the original B.I. involving the choice of three directions, equ. 2 leads to (coplanar situation) :

(11) $|f(\alpha) - f(\beta)| \leq f(\alpha-\beta) - f(o)$

and the systematic violation can easily be established. But for the generalized B.I., the statement has to be made more precise. Having chosen a set of three angles α, β, γ which define the relative setting of 4 directions (fig. 2)



six correlations can be measured and, almost everywhere, at least one of the 12 double G.B.I. that can be written is violated (of course, since only 4 correlations are used in one G.B.I., it is only those 4 that are included in the violated G.B.I. which need to be measured and, conversely, for a given setting of 4 directions (i.e. a given set of angles α, β, γ) 4 correlations randomly chosen may not lead to a violation.

The demonstration is tedious and is not reproduced here but the figure below can be used to indicate, for any given set of angles α, β, γ, which of the 12 G.B.I. is violated (or if several are, which produces the largest difference with the limit).

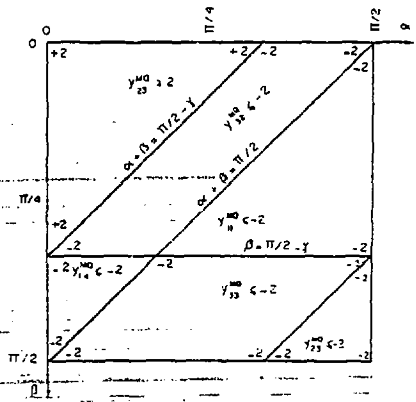
The G.B.I. are labelled by a double index. The first one indicates the choice of the 4 correlations (among 6) :

index 1	for	α	$\alpha+\beta$	$\beta+\gamma$	γ
index 2	for	α	$\alpha+\beta+\gamma$	β	γ
index 3	for	$\alpha+\beta$	$\alpha+\beta+\gamma$	β	$\beta+\gamma$

The second index indicates the place of the minus sign : 1 if in front of the last term... 4 if in front of the first term.

In each delimited zone, the true inequality, which is written, is in conflict with the corresponding G.B.I. ($-2 \leq y_{ij} \leq +2$).

The given scales and directions of the inequalities correspond to the case of photons (fig. 3).



The use of the above figure is restricted to angles α, β, γ between 0 and π/2. One of them however, say γ, may be larger than π/2 (but smaller than π). The use of identities like

$$(12) \qquad y_{11}^{MQ}\ (\alpha, \beta, \gamma - \pi/2) = y_{12}^{MQ}\ (\alpha, \beta, \gamma)$$

allows one to treat such cases.

A similar figure could be drawn for the case of spin-1/2 particles.

For the latter, $\vec{a}$, $\vec{b}$, $\vec{c}$ may not be within the same plane. This leads to the possibility of a choice of sets of directions - the measure of which is finite - where the B.I. are satisfied. But it does not change the preceding conclusions as regards the systematic violation of the B.I. by the one-variable function f_{QM} (θ). On the other hand, the knowledge of two values f(α) and f(β) leads to a more drastic constraint on f, because the B.I. must apply to the whole interval α - β to α + β instead of only its ends.

THE INVERSE COMPARISON : SOME PROPERTIES OF FUNCTIONS SATISFYING BELL'S INEQUALITIES.

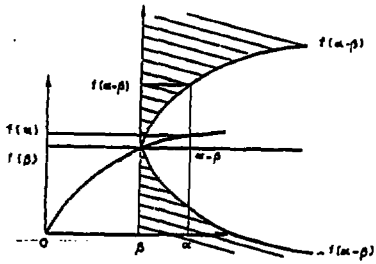
In this part of our contribution, the study will be restricted to the initial B.I. (3 directions) applied to the spin-1/2 case. No demonstrations will be given here (however, see ref. 4). Some of the results presented here and some complementary ones were already established by F. Selleri in ref.[5], it is regretted that this publication was ignored at the time of the oral contribution.

The notation f_B will be used for a function which satisfies B.I., equ. (11), and the symmetry and limit conditions (5), (6) and (7). Equ. (11) can be split into

$$(13) \qquad f_B(\alpha) - f_B(\beta) \leq f_B\ (\alpha - \beta) - f_B(o)$$

$$(14) \qquad f_B(\alpha) - f_B(\beta) \geq - \left[f_B(\alpha - \beta) - f_B(o)\right]$$

The following figure (fig. 4) provides a graphic representation of both these conditions and, because of the general behaviour imposed by the



values $f(o) = -1$; $f(\pi/2) = 0$; $f(\pi) = 1$, the condition (13) appears as a stronger constraint and will be studied more extensively.

Derivative

~~Equ. (13) leads to~~

(15) $f'_B(o) \geq f'_B(\theta)$ hence

(16) $f'_B(o) \geq \langle f'_B(\theta) \rangle = 2/\pi$

whereas $f'_{QM}(o) = 0$

This difference was already noted by Bell. The linear function

(17) $f_{B\ell}(\theta) = -1 + (2/\pi)\theta$

satisfies (15) and (16) with the sign equal on the whole interval 0, π

Integral

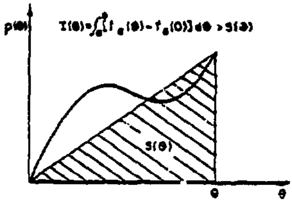
Defining $\rho(\theta) = f(\theta) - f(o)$ and

$$I(\Theta) = \int_o^\Theta \rho(\theta)\, d\theta \qquad 0 < \Theta < \pi$$

one finds

(18) $I_B(\theta) \geq \rho_B(\theta) \cdot \theta/2 = S(\theta)$

which is illustrated in the following figure (fig. 5).



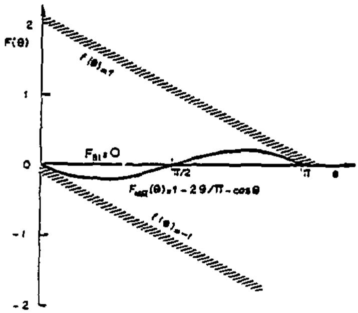
Setting $\theta = \pi/2$ leads to

(19) $I_B(\pi/2) - I_{QM}(\pi/2) \geq (4 - \pi)/4$

and again, $f_{B\ell}(\theta)$ satisfies (18) and (19) with the sign equal. $f_{B\ell}$ appears as a limit linear functions as regards the conditions on both the derivative and the integral. It will be convenient to relate it all the functions by defining

(20) $F(\theta) = f(\theta) - f_{B\ell}(\theta)$

The first part of B.I., (13), is unchanged but the advantages of this transformation are i) the sum of two F_B's is a F_B (which was not true for f_B) because $F_B(o) = F_B(\pi/2) = F_B(\pi) = 0$ ii) the relative situation below/under becomes positive/negative (the crossing points become roots). It is the properties of the F_B's which are described below, the following figure (fig. 6) describing F_{QM}.



Values and intervals

(21) If $F_B(\alpha) \geq 0$ then $F_B(\pi - \alpha) \leq 0$ and $F_B(\alpha/2^n) \geq 0$

n being an integer $1 \leq n$

(22) If $F_B(\alpha) \geq 0$ and $F_B(\beta) \leq 0$ then $F_B(\alpha - \beta) \geq 0$

(23) If $F_B(\alpha) \leq 0$ then $F_B(2\alpha) \leq 0$ and $F_B(n\alpha) \leq 0$ and $F_B(\pi - n\alpha) \geq 0$ $(n\alpha < \pi)$

(24) If $F_B(\alpha) \leq 0$ and $F_B(\beta) \leq 0$ then $F_B(\alpha + \beta) \leq 0$

It results from (23) that

(25) $F_B(\alpha) \geq 0$ if $\alpha = \pi - n\alpha$ or $\alpha = \pi/k$,

k being any integer larger than 1 (see ref. [5]).

The last result is important as regards the comparison with Q.M. since

(26) $F_{QM}(\theta) < 0$ for $0 < \theta < \pi/2$

N is the number of roots of F_B between 0 and $\pi/2$ (both ends excluded) and α_1 is the position of the first one, hence

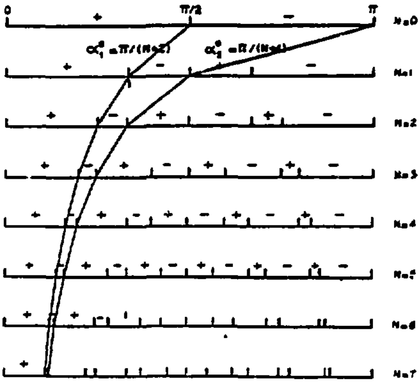
(27) $\alpha_1 \geq \pi/(N + 2)$ (and $f(\theta) > 0$ for $0 < \theta < \alpha_1$))

TRIAL TO FIT QUANTUM MECHANICS BY A FUNCTION SATISFYING BELL'S INEQUALITIES

If it is sure that no F_B can coïncide with $F_{QM}(\theta)$ on the whole interval $(0, \pi)$ (or equivalently f_B with f_{QM}), the questions remain how close it can be or on which interval they can coincide.

Starting from the limit linear function $F_{B\ell} \equiv 0$, an improvement will be attempted by allowing F_B to cross the θ axis (see (26)). Not only α_1 has to be small to get a possible agreement for small angles but the sum of the measure of the intervals where F is negative has to be as large as possible.

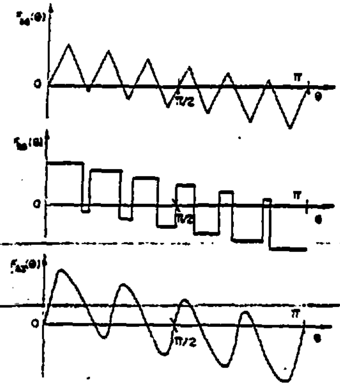
Calling this sum Θ_-, it can be shown that (N is given) setting α_1 to its limit value $\alpha_1^* = \pi/(N + 2)$, Θ_- is maximized for a limit sequence of N roots obtained by the procedure indicated in the following figure (fig. 7) for the first values of N. The resulting Θ_- is then :



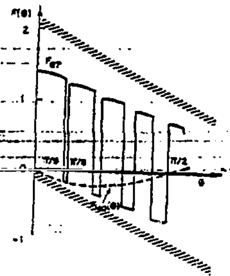
$$\Theta_- / \pi/2 = \begin{cases} \text{odd N} & (N + 3)/4\ (N + 2) \text{ which varies from 0.33 to 0.25} \\ \text{even N} & N/4\ (N + 1) \text{ which varies from 0.167 to 0.25} \end{cases}$$

Hence, the increase of N in view of an agreement for small values of θ does not allow an agreement on a large interval of θ.

Of course the knowledge of the roots is not sufficiant and several inequalities conditionning the amplitudes of F on its different segments can be established. And, again, by using the inequalities at their limit (when they reach equality) this procedure enables us to construct a (two-fold) limit F_B function from its knowledge on one of its segment between two consecutive roots. This is examplifed on the following figure where three F_B's (they are labeled with N) are constructed from three different an arbitrary initial segment. (fig. 8)

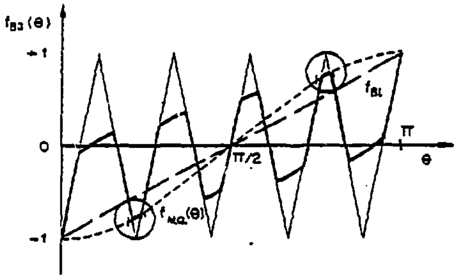


Such a limit F_B function can be used to try and obtain a fit to F_{QM}, as is done below for N = 7 (fig. 9).



This F_{B-7} is chosen to coincide with F_{MQ} on the first possible negative segment (between $\pi/9$ and $\pi/8$). Everywhere else it is as close to F_{MQ} as possible ! A somewhat equivalent result is given by F. Selleri in ref.[5])

But the whole (rel. 11) B.I. has to be satisfied. The study of the changes that this supplementary (rel. 14) condition introduces requires us to come back from $F(\theta)$ to the initial $f(\theta)$. Of course the result is a reduction of the interval on which f_B and f_{QM} can coincide, as illustrated below for the case N = 3 (fig. 10).



Thus, starting from the limit linear function $f_{B\ell}$ which never coïncides with f_{QM} but which is never too distant from it. The introduction of roots allows it to coïncide with f_{QM} on a finite θ interval but the resulting overall behaviour is less satisfying and becomes worse when N increases. Moreover, when N is increased to make possible an agreement at smaller angles, the interval of agreement (if chosen to be the first one) reduces and tends to zero as N tends to infinity.

Finally, F_{B_∞} becomes a strongly oscillating function which occupies the whole available ($\theta \times f(\theta)$) area while its average tends to zero on the whole range from 0 to π.

<u>CONCLUSION</u>

Altogether, it has been found that the - already known - uncompatibility between Q.M. and B.I. encountered in an EPRB - type experiment is stronger and more systematic than usually asserted. The present results should modify the choice of angles for an experimental test of B.I. if such an experiment is to be decided and it should also change the evaluation of its statistical degree of confidence (probably an increase from its usual evaluation).

It would be interesting to see how generalized B.I. as those presented in this meeting by A. GARUCCIO would lead to the same systematic violation and how they would reduce the range of available functions.

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