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COULOMB CORRECTIONS IN THE LOW – ENERGY SCATTERING

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Abstract

Renormalization of the coefficients of the "effective range expansion" is considered for the short-range Coulomb problem. It is shown that the coefficient at k^{2n} in the ℓ-th partial wave contains a logarithmic singularity $\sim \ln r_0/a_B$ if $n \geq \ell$. This singularity is universal, i.e. is independent of details of strong interaction. The exactly solvable model of the Coulomb plus short range potential is considered. Exact solutions are compared with approximations frequently used in the theory of hadronic atoms.

I. The calculation of Coulomb corrections to low-energy scattering parameters (scattering length a_ℓ , effective range r_ℓ etc.) is important for applications and has numerous literature - see, e.g.[1,2]. The corresponding formulae for the s-scattering length have been known long ago[3, 4], they were obtained for states with $\ell \neq 0$ in ref.[5] and for the effective range Coulomb correction in ref.[6].

We will consider here the Coulomb renormalization of an arbitrary term in the "effective range" expansion and will show that singularity at $r_0/a_B \to 0$ is independent of details of strong potential V_s and is explicitly calculable.

We consider also some exactly solvable models with a short-range plus Coulomb potential . This allows us to illustrate general formulae for Coulomb corrections and to determine the limits of applicability regions of approximate equations frequently used in the theory of lightest hadronic systems,[6-14] e.g. $pp, \bar{p}p, K^-p$, $\Sigma^- p$, $K^- {}^4He$ etc. The results of numerical calculations of low-energy parameters for potentials used in nuclear physics are also presented.

2. Consider the "effective range function"[4] $K_\ell^{(s)}(k^2)$ in the strong potential V_s

$$K_\ell^{(s)} \equiv k^{2\ell+1} ctg\, \delta_\ell^{(s)}(k) = \sum_{n=0}^{\infty} \beta_{\ell n}^{(s)} k^{2n} \qquad (1)$$

(in particular, $\beta_{\ell 0} = -1/a_\ell^{(s)}$, $\beta_{\ell 1} = \frac{1}{2} r_\ell^{(s)}$). It can be easily seen that the dimension of the coefficients in eq.(1) is 1) $[\beta_{\ell n}] = L^{2(n-\ell)-1}$. At $\xi \neq 0$ (i.e. in the presence of Coulomb interaction) the wave function of continuous spectrum has the form [2)]

$$\varphi_\ell(k,r) = r^{-\ell} v_\ell + d_\ell K_\ell^{(cs)} r^{\ell+1} u_\ell\,, \quad r > r_0\,, \qquad (2)$$

where $d_\ell = [(2\ell+1)!!(2\ell-1)!!]^{-1}$, $K_\ell^{(cs)}(k^2)$ is a meromorphic function of k^2 [2,15] which is an analog of function (1) in the presence of the Coulomb potential, $u_\ell = u_\ell(k^2, r)$ and $v_\ell = v_\ell(k^2, r)$ are connected with regular and irregular (at zero) Coulomb functions F_ℓ and G_ℓ [16], see Appendix A.

Using the wave function matching method in the region $r_0 \ll r \ll a_B$ (employed recently [17] to calculate the spectra of weakly-bound states in external fields), we extract in $\varphi_\ell(k,r)$ the term $\backsim \ln \xi r$, take into account the Coulomb potential in the internal region $0 < r \ll a_B$ with the help of iterations in the Schrödinger equation and arrive at the relations [3)]

$$K_\ell^{(cs)}(k^2) = K_\ell^{(s)}(k^2) + 2\xi \prod_{m=1}^{\ell}\left(k^2 + \frac{\xi^2}{m^2}\right) \ln|\xi| r_0 + \ldots \qquad (3)$$

$$\beta_{\ell n}^{(cs)} = \beta_{\ell n}^{(s)} + b_{\ell n}\, \xi^{2(\ell-n)+1} \ln|\xi| r_0 + \ldots \qquad (4)$$

Here $b_{\ell n}$ are the numerical coefficients defined from the identity

$$\prod_{m=1}^{\ell}\left(1 + \frac{x}{m^2}\right) = \frac{1}{2}\sum_{n=0}^{\ell} b_{\ell n}\, x^{\ell-n}$$

So, $b_{\ell 0} = 2/(\ell!)^2$, $b_{\ell 1} = \frac{1}{3}\ell(\ell+\frac{1}{2})(\ell+1)\, b_{\ell 0}\,, \ldots$

$$b_{\ell,\ell-1} = 2\sum_{m=1}^{\ell}\frac{1}{m^2} \qquad \text{and} \qquad b_{\ell\ell} = 2, \text{ see Table I.}$$

Since the radius of nuclear forces enters (4) under the logarithm its exact value (for the case of $r_0 \ll a_s$) is inessential. The singular at $\xi r_0 \to 0$ term is independent of the strong potential form. Besides, eq.(4) contains also the power corrections $\sim \xi r_0$, $(\xi r_0)^2$ etc., the coefficients at which depend on a specific model of $V_s(r)$ and are not calculable in general form.

3. Special cases. According to eq.(3), the Coulomb renormalization of the s-scattering length has the form

$$\frac{1}{a_{cs}} = \frac{1}{a_s}\left[1 + 2\left(b_1\rho + b_2\rho^2 + \dots\right)\right] - \\ - 2\xi\left[\ln|\rho| + c_0 + c_1\rho + c_2\rho^2 + \dots\right] \tag{5}$$

Here $a_s \equiv a_0^{(s)}$, $r_s \equiv r_0^{(s)}$, $\rho = \xi r_s$, $c_0 = 2C + \ln(2r_c/r_s)$, r_c and r_s are the Coulomb and effective ranges of the system (the formulae for calculating r_c and r_s and dimensionless coefficients b_1, c_1 are given in ref.[7]). We consider the case when there is a level (real, virtual or quasistationary) with the low binding energy. In this case the difference between a_s and a_{cs} is especially large.

If we neglect the corrections $\sim r_0/a_0$ in eq.(5) (i.e. put formally $b_n = c_n = 0$ at $n \geq 1$), then we arrive at the well known Schwinger formula[3]. In refs.[5,7] the formulae for calculating the coefficients b_1 and c_1 were obtained. As will be shown below, see Sect.6, the taking into account of these terms essentially expands the applicability region of the Schwinger formula.

For the states with $\ell \neq 0$ the main Coulomb correction to the inverse scattering length was found in ref.[5]. Taking into account eq.(4) we get

$$1/a_\ell^{(cs)} - 1/a_\ell^{(s)} =$$

$$= 2\xi\left[\frac{(2\ell)!}{2^\ell \ell!}\right]^2 \int_0^\infty \chi_\ell^2(r)\frac{dr}{r} + \dots - \frac{2}{(\ell!)^2}\xi^{2\ell+1}\ln|\xi| r_0 + \dots \qquad (6)$$

Here $\chi_\ell(r)$ is the wave function of the bound ℓ-level at a point of its emergence in the potential V_s normalized by condition: $\lim_{r\to\infty} r^\ell \chi_\ell(r) = 1$.

Let us now turn to dimensionless variables, putting

$$V_s(r) = -\frac{g}{2r_0^2}v(r/r_0) \quad , \qquad (7)$$

where function $v(x)$ gives the form of the potential and g is the dimensionless coupling constant. Introducing the parameter $\tilde{a}_\ell$ which is equal to the scattering length on the hard sphere of radius r_0 (see eq.(13) below) we re-write eq.(6) as

$$1/a_\ell^{(cs)} - 1/a_\ell^{(s)} =$$

$$= \frac{1}{\tilde{a}_\ell}\left\{d_\ell \xi r_0 + \dots - \frac{(2\xi r_0)^{2\ell+1}}{(2\ell+1)!(2\ell)!}\ln|\xi| r_0 + \dots\right\} \qquad (6')$$

where $\ell \geq 1$ and $d_\ell = \frac{2}{2\ell+1}\int_0^\infty \chi_\ell^2(x)\frac{dx}{x}$, $x = \frac{r}{r_0}$ (for $\ell = 1$ see Table 2). A relative change of the scattering length due to Coulomb interaction can not small: $|(a_\ell^{(cs)} - a_\ell^{(s)}/a_\ell^{(s)}| \sim$ $\sim d_\ell \xi r_0 a_\ell^{(s)}/\tilde{a}_\ell$. Note that only one term in eqs.(6),(6') has logarithmic singularity at $\xi r_0 \to 0$. Because of the factor $(2\xi r_0)^{2\ell+1}$ its contribution rapidly falls down with the angular momentum ℓ increase.

At n=1 eq.(4) defines the singular term in the renormalization of the effective range

$$r_\ell^{(cs)} - r_\ell^{(s)} = \frac{2}{3}\,\frac{\ell(\ell+1)(2\ell+1)}{(\ell!)^2}\,\xi^{2\ell-1}\,\ln(|\xi| r_0) + \cdots \qquad (8)$$

Finally, consider the coefficient at $k^{2\ell}$ in expansion (1):

$$\beta_{\ell\ell}^{(cs)} - \beta_{\ell\ell}^{(s)} = 2\xi\left[\ln(|\xi| r_0) + s_\ell + O(\xi r_0)\right] \qquad (9)$$

(the constant s_ℓ depends on a specific model of $V_s(r)$, for details see ref.[6]). The logarithmic term in eq.(9) is the main Coulomb correction to $\beta_{\ell\ell}^{(s)}$. Thus, Coulomb corrections to $1/a_\ell^{(s)}$ are especially large in the s-wave, corrections to the effective range - for the p-wave, to the form coefficient[4] - for the d-wave, etc.

At $0 \le n < \ell$ the Coulomb renormalization of $\beta_{\ell n}^{(s)}$ contains $\ln(a_B/r_0) \gg 1$, but this term is no more the main 4).

If now $n > \ell$, then the Coulomb renormalization of the coefficient $\beta_{\ell n}^{(s)}$ does not contain logarithms at all and is analytical in the parameter ξr_0.

4. Exactly solvable model. Putting $v(x) = \delta(1-x)$ we obtain the delta-function interaction on the sphere $r = r_0$. If $\xi = 0$, then for this model

$$ctg\,\delta_\ell^{(s)}(k) = \frac{2}{\pi g\, J_\mu^2(kr_0)} + \frac{N_\mu(kr_0)}{J_\mu(kr_0)} \quad , \tag{10}$$

$$K_\ell^{(s)}(k^2) = \frac{1}{\tilde{a}_\ell}\left\{ \frac{2\ell+1}{g}\left[\Lambda_\mu(kr_0)\right]^{-2} - \frac{\Lambda_{-\mu}(kr_0)}{\Lambda_\mu(kr_0)} \right\} \tag{11}$$

where $\mu = \ell + 1/2$, J_μ and N_μ are the Bessel and Neumann functions, $\Lambda_\mu(z) = \Gamma(\mu+1)(2/z)^\mu J_\mu(z)$. At $k \to 0$ we find the scattering length and the effective range

$$a_\ell^{(s)} = \tilde{a}_\ell \left(1 - \frac{g_\ell}{g}\right)^{-1}, \quad r_\ell^{(s)} = \tilde{r}_\ell \left(1 - \frac{2\ell-1}{g}\right). \tag{12}$$

Here $g_\ell = 2\ell+1$, $\tilde{a}_\ell$ and $\tilde{r}_\ell$ are corresponding values for the hard sphere of the radius r_0 :

$$\tilde{a}_\ell = \alpha_\ell\, r_0^{2\ell+1} \quad , \quad \tilde{r}_\ell = -\alpha_\ell'\, r_0^{1-2\ell} \,, \tag{13}$$

α_ℓ is defined in eq.(2) and $\alpha_\ell' = \dfrac{2(2\ell+1)}{(2\ell-1)(2\ell+3)\alpha_\ell}$

The bound ℓ -state appears at $g = g_\ell$ where $a_\ell^{(s)} = \infty$ and the effective range is equal to

$$r_\ell^{(s)} = \rho_\ell\, r_0^{1-2\ell} \quad . \tag{14}$$

$\rho_0 = 4/3$, $\rho_\ell = -2\alpha_\ell'/(2\ell+1) = -4\,(2\ell+3)^{-1}(2\ell+\ell)!!(2\ell-3)!!$ at $\ell \geqslant 1$ and $(-1)!! \equiv 1$. For states with $\ell \neq 0$ the effective range at a point of appearance of the bound ℓ -

state is negative. This holds also in the general case, i.e. for arbitrary potential $V_s(r)$, see ref.[18].

If $f \neq 0$, the formulae become to be more complicated:

$$\operatorname{ctg} \delta_\ell^{(cs)}(k) = \frac{k r_0}{g F_\ell^2(kr_0,\eta)} - \frac{G_\ell(kr_0,\eta)}{F_\ell(kr_0,\eta)}, \qquad (10')$$

$$K_\ell^{(cs)}(k^2) = \frac{1}{\tilde{a}_\ell}\left\{\frac{2\ell+1}{g[u_\ell(k^2,r_0)]^2} - \frac{v_\ell(k^2,r_0)}{u_\ell(k^2,r_0)}\right\} \qquad (11')$$

the functions u_ℓ, v_ℓ are defined in Appendix A. Making use of their expansions[5] at $k^2 \to 0$ we find the Coulomb-modified scattering length and the effective range [5])

$$a_\ell^{(cs)} = \tilde{a}_\ell \left[\frac{\xi_\ell}{\eta_\ell} - \frac{2\ell+1}{g\,\eta_\ell^2}\right]^{-1},$$

$$(15)$$

$$r_\ell^{(cs)} = \tilde{r}_\ell \left[\frac{2\ell-1}{2\ell+1}\cdot\frac{\xi_\ell\,\tilde{\eta}_\ell}{2\eta_\ell^2} + \frac{2\ell+3}{2\ell+1}\,\frac{\tilde{\xi}_\ell}{2\eta_\ell} - \frac{(2\ell-1)\,\tilde{\eta}_\ell}{g\,\eta_\ell^3}\right].$$

The functions $\xi_\ell(x)$, $\eta_\ell(x)$ etc. which enter these expressions and whose argument is $x = 2fr_0$ are expressed through the Bessel functions (see Appendix B). Note that the functions $\xi_\ell, \ldots$ have a different analytical form depending on the sign of Coulomb interaction. For instance,

$$\frac{1}{a_l^{(cs)}} = \frac{2|\xi|^{2l+1}}{(l!)^2}\begin{cases} -\pi\,\dfrac{N_{2l+1}(y)}{J_{2l+1}(y)} - \dfrac{1}{g\,J^2_{2l+1}(y)}, & \xi > 0 \\[2ex] 2\,\dfrac{K_{2l+1}(y)}{I_{2l+1}(y)} - \dfrac{1}{g\,I^2_{2l+1}(y)}, & \xi < 0 \end{cases} \qquad (15')$$

where $y = (8|\xi|/r_0)^{1/2}$. Hence at $\xi \to 0$ we get $\tilde{a}_l / a_l^{(cs)} = 1 - g_l/g$ in accordance with eq. (12).

We put further $g = g_l = 2l+1$ since the Coulomb effect on the scattering length is the most significant if there is a loosely bound state in the strong potential. Here

$$a_l^{(cs)} = \tilde{a}_l\, \eta_l^2 / (\xi_l \eta_l - 1)\ ,$$

$$r_l^{(cs)}/r_l^{(s)} = \frac{1}{4}\left[(2l+3)\frac{\tilde{\xi}_l}{\eta_l} + (2l-1)\frac{(\xi_l \eta_l - 2)\tilde{\eta}_l}{\eta_l^3}\right].$$

Expanding these expressions at $x \to 0$ we arrive for the case of s-wave to eq.(5) where [6] $r_s = 4r_0/3$,

$$b_1 = 3/4\ ,\quad b_2 = 21/32\ ,\ \ldots$$

$$c_0 = 2C + \ln\frac{3}{2} - \frac{1}{2} \approx 1.060,\quad c_1 = \frac{7}{4}\ ,\ c_2 = \frac{57}{64} \tag{16}$$

At $\ell \neq 0$ we have

$$\tilde{a}_\ell / a_\ell^{(es)} = \frac{1}{\ell(\ell+1)}\xi r_0 + \frac{8\ell+14}{(\ell+1)^2(2\ell-1)(2\ell+3)}(\xi r_0)^2 + \ldots$$
$$\ldots - \frac{1}{(2\ell+1)!(2\ell)!}(2\xi r_0)^{2\ell+1}\ln(|\xi| r_0) + \ldots \tag{17}$$

Analogously, for the effective ranges

$$r_\ell^{(cs)}/r_\ell^{(s)} = 1 - \frac{\ell(\ell+1)(2\ell-1)(2\ell+3)}{3\left[(2\ell)!\right]^2}(2\xi r_0)^{2\ell-1}\ln(|\xi| r_0) +$$
$$+ \tilde{c}_1\, \xi r_0 + \tilde{c}_2\,(\xi r_0)^2 + \ldots \tag{18}$$

Here $\tilde{c}_k$ are numerical coefficients,

$$\tilde{c}_1 = \begin{cases} \dfrac{(2\ell-1)(3\ell+7)}{2(\ell^2-1)(\ell+2)}\ , & \ell \neq 1 \\[2ex] \dfrac{5}{3}\left(\dfrac{29}{16} - 2C - \ln 2\right), & \ell = 1 \end{cases}$$

C = 0.5772 ... is the Euler constant. In particular, $\tilde{c}_1$ = 7/4, $\tilde{c}_2$ = 19/9 for s-wave (at $g = g_0 = 1$); $\tilde{c}_1$ = 0.0318, $\tilde{c}_2$ = = -857/630 for p-wave (at $g = g_1 = 3$). The Coulomb renormalization of the effective range is especially large in p-wave. In this case eq.(18) can be re-written as[6]

$$r_1^{(cs)}/r_1^{(s)} = 1 + \delta\left(\ln|\delta^{-1}| + k_1\right) + k_2\delta^2 + \dots , \tag{19}$$

where $\delta = -4\xi/r_1^{(s)}$ is a dimensionless parameter ($\delta \sim \xi r_0 \ll 1$ and $\delta > 0$ for the Coulomb attraction). In this model $\delta = \frac{5}{3}\xi r_0$, $k_1 = \frac{3}{5}\tilde{C}_1 + \ln\frac{5}{3} \approx 0.530$ and $k_2 = -857/1750 \approx -0.490$. Since the coefficients b_n, c_n and k_n are of order of unity, then provided $r_0 \ll a_B$ it suffices to confine a few first terms in expansions (5),(18) and (19).

With explicit analytical expressions for phase shifts at hand, one can readily obtain an equation for the discrete spectrum. Taking into account eq.(10') we find from the condition $ctg\,\delta_\ell^{(cs)}(i\lambda) = i$

$$\frac{\Gamma(\ell+1-\nu)}{(2\ell+1)!\,z} M_{\nu,\ell+1/2}(z)\, W_{\nu,\ell+1/2}(z) = g^{-1}, \tag{20}$$

where $\nu = \xi/\lambda$; $z = 2\lambda r_0$ and M and W are the Whittaker functions[16]. At $\xi \to 0$ this equation takes the form 7)

$$I_{\ell+1/2}(\lambda r_0)\, K_{\ell+1/2}(\lambda r_0) = g^{-1} \tag{21}$$

and defines the discrete spectrum in δ-potential[20,22]. Eqs. (20),(21) can be easily analysed numerically.[10]

5. The Yamaguchi potential. As is well known the Schrödinger equation permits exact solution for the case of the separable Yamaguchi potential[23]. The values of $a_\ell^{(cs)}$ and $r_\ell^{(cs)}$ for $\ell = 0$ were found in ref.[24] and for arbitrary ℓ in ref.[26]. For this unique potential one can find all terms 8) in eq.(5):

$$c_0 = C + \ln\left[4g/(g+2g_0)\right].$$

$$c_n = \frac{2}{n} b_n, \quad b_n = \frac{2^{2n-1}}{n!\,(1+2g_0/g)^n}, \quad n \geq 1. \tag{22}$$

Referring for calculation details to Appendix C, let us also give the following expansions:

$$z_{cs}/z_s = 1 + \sum_{k=1}^{\infty} \tilde{c}_k (\xi z_s)^k, \quad \ell = 0, \tag{23}$$

where $z_s = (1+2g_0/g)\beta^{-1}$, $\tilde{c}_1 = 4g(g+5g_0)/3(g+2g_0)^2$ and at $g=g_0=1$ and arbitrary n $\tilde{c}_n = 2^{2n}[13-(n-2)^2]/(n+1)!\,3^{n+2}$,

$$1/a_\ell^{(cs)} = \frac{2}{(\ell!)^2} \xi^{2\ell+1} \left\{ \sum_{k=1}^{2\ell} c_k \delta^{-k} - \ln|\delta| + O(1) \right\}, \quad \ell \geq 1 \tag{24}$$

$$z_\ell^{(cs)}/z_\ell^{(s)} = 1 + \frac{(2\ell-1)(11\ell+13)}{6(\ell-1)(2\ell+3)} \delta + \ldots -$$

$$- \frac{8}{3} \frac{\ell(\ell+1)(4\ell^2-1)}{(2\ell)!\,(2\ell+3)} \delta^{2\ell-1} \ln|\delta| + \ldots, \quad \ell \neq 1 \tag{25}$$

In eqs.(24),(25) $g = g_\ell = 1$, $\delta = 4\xi/\beta$ (note that β^{-1} defines the strong interaction radius), $c_{2\ell} = (2\ell-1)!$, $c_{2\ell-1} = \left[2\ell/(2\ell-1)\right] c_{2\ell}$, ... For p-wave eq.(19) takes place where

$$z_1^{(s)} = -\frac{5}{8}\beta, \quad k_1 = \frac{3}{4} - C + \ln\frac{8}{3} = 0.739, \quad k_2 = -\frac{155}{128}.$$

The examples considered enable one to make the following conclusions. The singular terms in expansions (17),(18) and (24), (25) completely correspond to formula (4) which is valid for an arbitrary short-range potential, both local or nonlocal . Here the coefficient at the Coulomb logarithm $\ln(a_s/r_0)$ is independent of the model of V_s. With ℓ increasing the relative value of Coulomb corrections decreases.

6. Compare now exact formulae for the model of δ-potential with approximate equations used in the theory of hadronic atoms [7-14].

Fig.1 shows the dependence of r_0/a_{cs} on the coupling constant, i.e. on the depth of strong potential, at several values of the ratio $\xi r_0 = \pm r_0/a_B$ ($\xi > 0$ corresponds to Coulomb attraction and $\xi < 0$ to Coulomb repulsion). Solid curves correspond to exact calculation according to eq.(15), dashed curves are obtained from eq.(5) at $a_s = r_0 g/(g-1)$ and at given in eq.(16) values of the constants b_1 , c_0 and c_1 (the terms $\lesssim (\xi r_0)^2$ were neglected). In what follows we use eq.(5) where three terms $\sim c_0$, c_1 , b_1 are remained. Such an approximation has been already used[7] when extracting the strong scattering length a_s from experimental data[27] on the $\bar{p}p$-atom (note that nowadays the experiment gives a smaller value of the $1s$-level shift[28]).

Fig.1 shows that the given above approximation has a high accuracy in the wide range of values g (but not only in the region $g \approx g_0$). The s-level in the potential V_s is not necessarily shallow (in nuclear scales), see Fig.2.

We studied also[10] the accuracy of approximation (5) as a function of the ratio r_s/a_B (but at fixed $g = g_0 = 1$, i.e. at a moment of emergence of the level). The results given in

Fig.3 show that this approximation is applicable at $|\xi| r_s \lesssim 0.3$ (recall, that $a_B = 57.6$ Fm for $\bar{p}p$-atom, $a_B = 31.0$ Fm for the $K^{-4}He$ system, so that in these cases $r_o/a_B \sim 0.03 \div 0.1$). On the other hand, if we neglect also the corrections of the order r_o/a_B, i.e. put formally $b_1 = c_1 = 0$ in eq.(5), it is applicable only at a significantly more strict condition $r_s/a_B \lesssim 0.05$ - see curve 3 and curve S in Fig.1. This illustrates the importance of taking into account the terms $\sim b_1$, c_1 in eq.(5) calculated in ref.[7].

An analogous picture takes place for the Coulomb correction to effective range in the p-wave, see Fig.4.

Figs.3 and 4 refer to the cases when Coulomb corrections contain $\ln a_B/r_o$ and are especially large. The ℓ-dependence of these corrections is clear from Fig.5. Since $1/a_\ell^{(s)} = 0$ at $g = g_\ell$ we go over from the Coulomb-modified scattering length to dimensionless quantities $\tilde{a}_\ell$ ($\tilde{a}_0 = r_0$, $\tilde{a}_1 = r_0^3/3$, $\tilde{a}_2 = r_0^5/45$ etc. (see eq.(13)). The extremely large value of Coulomb corrections to the s-scattering length and to effective range in the p-wave is explicitly seen from F ig.5.

The theory of hadronic atoms frequently uses a phenomenological approach based on the equation[7,15,29]

$$\left\{\lambda + 2\xi\left[\psi\left(1-\frac{\xi}{\lambda}\right) + \ln\frac{\lambda}{|\xi|}\right]\right\}\prod_{m=1}^{\ell}\left(\frac{\xi^2}{m^2} - \lambda^2\right) = \frac{1}{a_\ell^{(cs)}} + \frac{1}{2} r_\ell^{(cs)} \lambda^2 \quad (26)$$

which relates the shifts and widths of atomic ℓ-states with the low-energy scattering parameters. Here $\lambda = (-2E/E_c)^{1/2}$, $E_c = me^4/\hbar^2$ and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is digamma function. This equation meets various fields of physics when considering Coulomb systems with short-range forces, e.g.[7-10, 30]. In particular, eq.(26) gives a complete description of the Zeldovich phenomenon (or "atomic spectrum rearrangement") both at $\ell = 0$[7,31] and for states with $\ell \neq 0$[32].

We have studied the accuracy of eq.(26) on an example of δ-potential. Solid curves in Fig.6 are plotted from eq.(20) for $\ell = 0$, dashed curves - from approximate eq.(26) while $a_o^{(cs)}$ and $r_o^{(cs)}$ were calculated from eq.(15). It is seen from Fig.6 that the calculation error of the s-level energy from eq.(26) is not more than 10% if $\lambda r_o <$ 0.3 - i.e., the binding energy is by the order of magnitude smaller than the characteristic nuclear energy $\hbar^2/2mr_o^2$.

Fig.7 obtained from eq.(21) shows the energy dependence on the coupling constant g at ξ =0 and different ℓ . We use here the "reduced variables ε and g/g_ℓ (the energy $E = -\lambda^2/2 = \varepsilon/(2r_o^2)$.

7. Low-energy parameters for nuclear potentials. The exactly solvable models are far enough from "realistic" potentials which describe strong interactions. Therefore one should dwell on the question to which extent the results obtained are valid for other potentials V_s .

In the low-energy region where eqs.(5),(19) etc. are used, the potential enters only through the parameters c_o , c_1 , b_1 , ... Exploiting formulae from refs.[5,7] we numerically calculated these parameters for different potentials of the type (7), including for the Yukawa potential, $v(x) = e^{-x}/x$, the Hulthen potential, $v = (e^{x}-1)^{-1}$, and Gaussian, frequently used in nuclear physics.

The calculation results 9) are given in Table 2. Let us explain the notations. The Coulomb corrections to the scattering length and to the effective range are written as

$$1/a_\ell^{(cs)} - 1/a_\ell^{(s)} = f_\ell \,\xi\, \left| r_\ell^{(s)} \right|^{2\ell/2\ell-1} + \dots$$

$$r_\ell^{(cs)}/r_\ell^{(s)} = 1 + h_\ell \,\xi\, \left| r_\ell^{(s)} \right| + \dots \qquad (27)$$

where f_ℓ and h_ℓ are dimensionless coefficients depending on the form of the potential V_s . In fact, eq.(27) contains some characteristic radius $R_\ell \sim \left| r_\ell^{(s)} \right|^{1/1-2\ell}$ which is expressed via the experimental parameter $r_\ell^{(s)}$. In the case of rectangular well $r_\ell^{(s)}$ is independent of the principal quantum number n and at $\ell = 0$ coincides with the radius of the well r_o . It is natural to determine the parameter R_ℓ for an arbitrary potential using the formula [10]

$$R_\ell = \begin{cases} r_s \;, & \ell = 0 \\ \left[(2\ell+1)!!\,(2\ell-3)!! / \left| r_\ell^{(s)} \right| \right]^{1/2\ell-1} , & \ell \geq 1 \end{cases} \qquad (28)$$

(for the rectangular well $R_\ell \equiv r_o$ for all n, ℓ). The numerical calculation shows that for smooth potentials at $\ell \leq 3$ R_ℓ differs from the effective range r_s for the ground state not more than by a few per cent (see Table 3). Therefore, let us rewrite eq.(27) in a more illustrative form, introducing the parameter ξR_ℓ . As a result [11]

$$\frac{1}{a_\ell^{(cs)}} - \frac{1}{a_\ell^{(s)}} = \frac{1}{d_\ell R_\ell^{2\ell+1}} \left[\tilde{f}_\ell \,\xi R_\ell + O\left((\xi R_\ell)^2 \right) \right], \quad \ell \neq 0$$

$$r_\ell^{(cs)}/r_\ell^{(s)} = 1 + \tilde{h}_\ell \,\xi R_\ell + \dots , \quad \ell \neq 1 \qquad (29)$$

The values of the coefficients $\tilde{f}_\ell$, $\tilde{h}_\ell$ see in Table 2. This Table presents also the coefficients c_0, c_1 and b_1 for the s-wave and k_1 for the p-wave, see eq.(19). The numerical calculations permit making some conclusions.

(i) The coefficients c_0 and k_1 weakly depend on the form of the strong potential and on the number of the level [12] (the same is true for the coefficients b_1 and $\tilde{f}_\ell$ at $\ell \leq 3$). This means that calculations of the strong scattering length $a_\ell^{(s)}$ and of the effective range $r_1^{(s)}$ using eqs.(5),(19) are weakly sensitive to the used model of V_s.

(ii) On the other hand, the coefficient h_0 significantly varies. However, the corresponding Coulomb correction (29) contains a small parameter r_s/a_B.

(iii) The constants c_0, c_1 etc for the δ-potential have the same order of magnitude as for smooth potentials. Thus, in the low-energy region the δ-potential is not exceptional and that is why the sect.6 results seem to be general.

(iv) Finally, mark a curious empirical fact: for the local potentials $V_s(r)$ there is a strict correlation between the values of c_0 and of the rest coefficients.

The authors are indebted to Professor A.B.Migdal, Drs.B.M. Karnakov and A.E.Kudryavtsev for discussions and to V.I.Lisin for his help in the numerical calculations.

Appendix A

The wave function of continuous spectrum in the δ-potential model is of the form

$$\varphi_\ell(k,r) = \begin{cases} const\cdot F_\ell(kr,\eta), & 0<r<r_0, \\ G_\ell(kr,\eta) + ctg\,\delta_\ell^{(cs)} F_\ell(kr,\eta), & r>r_0, \end{cases} \qquad (A.I)$$

where η is Sommerfeld's parameter $\eta = -\xi/k = [Z_1 Z_2 e^2/\hbar]\cdot(m/2E)^{1/2}$, F_ℓ, G_ℓ are regular and irregular at $r=0$ Coul mb functions[16] which are expressed via Whittaker functions

$$F_\ell(kr,\eta) = \frac{\Gamma(\ell+1+i\eta)}{2\,[(2\ell+1)!]}\,\omega\, M_{-i\eta,\ell+1/2}(-2ikr),$$

$$F_\ell(kr,\eta) - i\,G_\ell(kr,\eta) = \omega^{-1}\, W_{-i\eta,\ell+1/2}(-2ikr), \qquad (A.2)$$

$\omega = exp\{-\pi/2\cdot[\eta - i(\ell+1)] - i\sigma_\ell\}$ and σ_ℓ - is the Coulomb phase shift, $\sigma_\ell = arg\,\Gamma(\ell+1+i\eta)$.

As is known, the δ-potential is equivalent to the boundary condition

$$\varphi_\ell'(r_0+0) - \varphi_\ell'(r_0-0) + \frac{g}{r_0}\varphi_\ell(r_0) = 0, \qquad (A.3)$$

where $\varphi_\ell(r) = r R_\ell(r)$, R_ℓ is the radial wave function. From eqs.(A.1),(A.3) for the "effective range function"[15]

$$K_\ell^{(cs)}(k^2) = \left[(2\ell+1)!!\, C_\ell / C_0\right]^2 \times$$
$$\times\ k^{2\ell+1}\left\{C_0^2 \operatorname{ctg}\delta_\ell^{(cs)} + \eta\left[\psi(1+i\eta)+\psi(1-i\eta)-\ln\eta^2\right]\right\}$$

we arrive at the formula (11') where

$$u_\ell(k^2;r) = C_\ell^{-1}(\eta)\,(kr)^{-(\ell+1)} F_\ell(kr,\eta)\ ,$$
$$v_\ell(k^2;r) = (2\ell+1)\,C_\ell(\eta)\,(kr)^\ell \times$$
$$\times\ \left[G_\ell(kr,\eta) - q(\eta) F_\ell(kr,\eta)\right]\ , \qquad (A.4)$$

$$C_\ell(\eta) = \frac{2^\ell}{(2\ell+1)!}\exp(-\pi\eta/2)\left|\Gamma(\ell+1+i\eta)\right|\ ,$$

$$q(\eta) = \frac{1}{2\pi}\left(e^{2\pi\eta} - 1\right) \times$$
$$\times\left[\psi(1+i\eta) + \psi(1-i\eta) - \ln\eta^2\right]$$

It can be shown[15] that u_ℓ and v_ℓ are entire functions of k^2. At $kz \ll 1$ the expansions[5]

$$u_\ell(k^2;z) = \eta_\ell(2\xi z) - \frac{(kz)^2}{2(2\ell+3)}\,\tilde{\eta}_\ell(2\xi z) + \dots\,,$$

(A.5)

$$v_\ell(k^2;z) = \xi_\ell(2\xi z) + \frac{(kz)^2}{2(2\ell-1)}\,\tilde{\xi}_\ell(2\xi z) + \dots$$

are valid where functions $\eta_\ell(x)$, $\xi_\ell(x)$ etc. are defined in Appendix B. At $\xi \to 0$ we have

$$u_\ell(k^2;z) \to \Lambda_{\ell+1/2}(kz)\,,$$

(A.6)

$$v_\ell(k^2;z) \to \Lambda_{-(\ell+1/2)}(kz)\,,$$

where

$$\Lambda_\mu(z) = \Gamma(\mu+1)\left(\frac{2}{z}\right)^\mu J_\mu(z) = \begin{cases} 1 - a_1 z^2 + \dots\,, & z \to 0 \\ a_2 z^{-(\mu+1/2)} \cos\left[z - (\mu+1)\frac{\pi}{2}\right], & z \to \infty \end{cases}$$

$a_1 = [4(\mu+1)]^{-1}$, $a_2 = 2^{\mu+1/2}\pi^{-1/2}\Gamma(\mu+1)$ .Hence it is seen that expression (11') transforms into (11) when the Coulomb interaction is switched off.

At small z function $u_\ell(k^2,z)$ is expanded into series only in integer powers of z while expansion of $v_\ell(k^2,z)$ contains logarithms

$$u_\ell = 1 - \frac{\xi z}{\ell+1} + \frac{2\xi^2-(\ell+1)k^2}{2(\ell+1)(2\ell+3)}z^2 + \dots \tag{A.7}$$

$$v_\ell = \begin{cases} 1 - 2\xi z\left(\ln|\xi|z + 2C + \ln 2 - 1\right) + \dots, & \ell=0 \\ 1 + \frac{\xi z}{\ell} + \frac{2\xi^2+\ell k^2}{2\ell(2\ell-1)}z^2 + \dots - \frac{(2\xi z)^{2\ell+1}}{(2\ell)!(2\ell+1)!}\ln|\xi|z + \dots, & \ell \geq 1. \end{cases} \tag{A.8}$$

The functions u_ℓ , v_ℓ are normalized by condition

$$u_\ell(k^2,0) = v_\ell(k^2,0) = 1 \tag{A.9}$$

Note that they are related to the introduced in ref.[15] functions Φ_ℓ and Θ_ℓ by the following relations

$$\Phi_\ell(k^2;z) = z^{\ell+1}u_\ell(k^2,z)$$

$$\Theta_\ell(k^2;z) = \frac{1}{(2\ell+1)}z^{-\ell}v_\ell(k^2,z) \tag{A.10}$$

Appendix B

The functions $\eta_\ell(x)$, $\xi_\ell(x)$ etc whose argument is $x=2\xi t_0$ are expressed through the Bessel functions,

$$\eta_\ell(x)=\Lambda_{2\ell+1}(2x^{1/2})=\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\,(2\ell+2)_k} \tag{B.1}$$

$$\xi_\ell(x)=\begin{cases}-\dfrac{\pi}{(2\ell)!}x^{\ell+1/2}N_{2\ell+1}(2x^{1/2}), & x>0\\[2ex] \dfrac{2}{(2\ell)!}(-x)^{\ell+1/2}K_{2\ell+1}(2\sqrt{-x}), & x<0\end{cases} \tag{B.2}$$

$$\tilde{\eta}_\ell(x)=\eta_{\ell+1}(x)-\frac{x}{6(\ell+1)(\ell+2)}\eta_{\ell+3/2}(x) \tag{B.3}$$

$$\tilde{\xi}_\ell(x)=\xi_{\ell-1}(x)-\frac{x}{6\ell(\ell-1)}\xi_{\ell-3/2}(x) \tag{B.4}$$

where, as usual, $(\alpha)_k=\alpha(\alpha+1)\dots(\alpha+k-1)$, $(\alpha)_0=1$.

Note that eq.(B.4) is inapplicable at $\ell=0$ and 1; in these cases

$$\tilde{\xi}_0(x)=\frac{2}{3x}\left(\xi_{1/2}-\eta_0\right),\ \tilde{\xi}_1=\xi_0-\frac{x}{3}\xi_{-1/2}+\frac{x}{18}\eta_1 \tag{B.5}$$

(the first two terms of expansion of these functions at $x\to 0$ were obtained in ref.[5]). Note that

$$\eta_\ell(0) = \xi_\ell(0) = \tilde{\eta}_\ell(0) = \tilde{\xi}_\ell(0) = 1 \qquad (B.6)$$

While η_ℓ and $\tilde{\eta}_\ell$ are entire functions of x, ξ_ℓ and $\tilde{\xi}_\ell$ contain the logarithms[13]

$$\xi_\ell(x) = -\frac{x^{2\ell+1}}{(2\ell)!(2\ell+1)!}\eta_\ell(x)\ln|x| + \sum_{k=0}^{\infty} b_k x^k , \qquad (B.7)$$

$$\tilde{\xi}_\ell(x) = 1 + \frac{3\ell-1}{3\ell(2\ell-2)}x + \frac{3\ell-2}{6\ell(2\ell-2)(2\ell-3)}x^2 + \dots - \qquad (B.8)$$

$$-\frac{\ell+1}{3\ell(2\ell-1)!(2\ell-2)!}x^{2\ell-1}\ln|x| + \dots , \quad \ell \geq 2$$

$$\tilde{\xi}_0(x) = 1 - x\left(\ln|x| + 2C - 1\right) + \frac{x^2}{2}\left(\ln|x| + 2C - \frac{5}{2}\right) - \qquad (B.9)$$

$$-\frac{x^3}{12}\left(\ln|x| + 2C - 10/3\right) + \dots$$

As is seen from eq.(15), the Coulomb logarithm in the renormalization of the strong scatteringl length is contained only in the term ξ_ℓ/η_ℓ :

$$\frac{\xi_\ell(x)}{\eta_\ell(x)} = -\frac{x^{2\ell+1}}{(2\ell)!(2\ell+1)!}\left(\ln|x| + 2C\right) + f_\ell(x) \qquad (B.10)$$

while the function f_ℓ has no singularity at x=0,

$$f_\ell(x) = \begin{cases} 1 + \frac{3}{2}x - \frac{7}{12}x^2 - \frac{19}{144}x^3 + \dots , & \ell = 0 \\ 1 + \frac{2\ell+1}{2\ell(\ell+1)}x + \dots , & \ell \geq 1 \end{cases}$$

Hence it immediately follows eq.(17).

Appendix C.

We present here some formulae for the separable Yamaguchi potential[23]

$$\langle\vec{p}|V_s|\vec{p}\,'\rangle = -g\,\frac{2^{4\ell+1}(\ell!)^2}{\pi(2\ell)!}\,\frac{\beta^{2\ell+3}(pp')^{\ell}}{\left[(p^2+\beta^2)(p'^2+\beta^2)\right]^{\ell+1}}\,Y_{\ell m}\left(\frac{\vec{p}}{p}\right)Y^*_{\ell m}\left(\frac{\vec{p}\,'}{p'}\right) \qquad (C.1)$$

which in q-representation corresponds to the nonlocal interaction

$$\langle\vec{r}|V_s|\vec{r}\,'\rangle = \frac{1}{(2\pi)^3}\int e^{i\vec{p}\vec{r}-i\vec{p}\,'\vec{r}\,'}\langle\vec{p}|V_s|\vec{p}\,'\rangle\, d^3p\, d^3p' =$$

$$= -g\,\frac{2^{2\ell}}{(2\ell)!}\,\beta^{2\ell+3}(rr')^{\ell-1}\exp\{-\beta(r+r')\}\,Y_{\ell m}\left(\frac{\vec{r}}{r}\right)Y^*_{\ell m}\left(\frac{\vec{r}\,'}{r'}\right) \qquad (C.2)$$

($\hbar = m = 1$, g is the dimensionless coupling constant). Unlike the δ-potential model considered in . Sect. 4, this potential explicitly depends on ℓ. There is exactly one bound state in every partial wave which emerges at $g = 1$.

For $\ell = 0$ the Schrödinger equation for the Yamaguchi plus Coulomb potential has been solved by van Haeringen[24] who has found the Coulomb modified scattering length and the effective range in analytical form. Introducing $\delta = 3\xi/\beta$, let us rewrite his formulae as

$$\frac{1}{a_{cs}} = \frac{1}{a_s}\exp\left(\tfrac{4}{3}\delta\right) + 2\xi\cdot\mathrm{Re}\,\Gamma\left(0, -\tfrac{4}{3}\delta\right), \qquad (C.3)$$

$$r_{cs} = \beta^{-1}\left[1+\frac{2\left(1-\frac{2}{3}\delta\right)}{g}\right]e^{\frac{4}{3}\delta} - \frac{1}{3\xi}\left[1-\left(1-\tfrac{4}{3}\delta\right)e^{\frac{4}{3}\delta}\right]$$

where
$$a_s = \frac{2}{\beta}\left(1 - \frac{1}{g}\right)^{-1}, \quad r_s = \frac{1}{\beta}\left(1 + \frac{2}{g}\right).$$

At $g = 1$ the strong scattering length has a pole what corresponds to appearance of a bound state (here $r_s = 3/\beta$, $\delta = \xi r_s$). From eq.(C.3), taking into account the expansion of incomplete gamma-function

$$\mathrm{Re}\,\Gamma(0,x) = -Ei(-x) = -\ln|x| - C + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n, \qquad \text{(C.4)}$$

we arrive at eqs.(22),(23). At $g \approx 1$ the numerical values of the coefficients are

$$c_0 = C + \ln\frac{4}{3} = 0.8649, \quad b_1 = \frac{1}{2}c_1 = \frac{2}{3},$$
$$b_2 = c_2 = \frac{4}{9}, \quad b_3 = \frac{16}{81}, \quad c_3 = \frac{32}{243}, \ \ldots \qquad \text{(C.5)}$$
$$\tilde{c}_1 = \frac{8}{9}, \quad \tilde{c}_2 = \frac{104}{243}, \quad \tilde{c}_3 = \frac{32}{243}, \ \ldots$$

These coefficients decrease with n increasing which results in a fast convergence of the corresponding expansions in r_0/a_s.

Recently[26], the parameters $a_\ell^{(cs)}$ and $r_\ell^{(cs)}$ for the Yamaguchi potential with arbitrary ℓ have been calculated. For instance,

$$\frac{1}{a_\ell^{(cs)}} = e^{-4\nu}\left(\frac{1}{a_\ell^{(s)}} - \frac{(2\ell)!}{2^{4\ell+1}(\ell!)^2}\beta^{2\ell+1}\right) - \qquad \text{(C.6)}$$
$$- 2\,\frac{(2\ell+1)!}{(\ell!)^2}\,\xi^{2\ell+1}\,\mathrm{Re}\,\Gamma(-2\ell-1, 4\nu)$$

$$\nu = -\frac{\xi}{\beta}\ , \quad \frac{1}{d_\ell^{(s)}} = \frac{(2\ell)!}{2^{4\ell+1}(\ell!)^2}\,\beta^{2\ell+1}\left(1-\frac{1}{\gamma}\right).$$

Making use of the identity

$$\Gamma(-n,x) = \frac{(-1)^n}{n!}\left\{\Gamma(0,x) + e^{-x}\sum_{k=1}^{n}\frac{(-1)^k (k-1)!}{x^k}\right\} \qquad (C.7)$$

and the expansion (C.4), we come to eq.(24). The coefficients C_n are defined from the relation

$$e^{z}\sum_{k=0}^{2\ell-1} k!\, z^{-(k+1)} = \sum_{n=1}^{2\ell} c_n z^{-n} + O(1)\,,\ z\to 0\,,$$

so that $c_{2\ell} = (2\ell-1)!\ ,\ c_{2\ell-1} = \frac{2\ell}{2\ell-1}\,c_{2\ell}\,,$

$c_{2\ell-2} = \frac{\ell}{2\ell-2}\,c_{2\ell}$ etc.

Formula (25) is obtained from eq.(4.13) of ref.[26] by analogous calculations.

In conclusion, mark a misprint made in ref.[26]: the term with the coefficient $8(2\ell-1)!/(\ell!)^2(4R)^{2\ell+1}$ in eq. (4.25) should be multiplied by the factor $4\xi R = -2\nu$ (here $R = \beta^{-1}$ according to the adopted in ref.[26] notations).

Footnotes

1) Thus, the scattering length and the effective range have the length dimension only for the s-wave. In the general case their dimensions are $[a_\ell] = L^{2\ell+1}$, $[r_\ell] = L^{1-2\ell}$, L being the unit of length. Furthermore, we consider the coefficient $\beta_{\ell\ell}$ at the term $k^{2\ell}$ in expansion (1). Its Dimension is $[\beta_{\ell\ell}] = L^{-1}$ at arbitrary ℓ.

2) We employ the atomic units, $\hbar = m = e = 1$ as well as the notations: ℓ - angular moment, $\xi = -z_1 z_2$ ($\xi > 0$ in the case of Coulomb attraction), $L = \hbar^2/me^2$, $a_0 = L|\xi|^{-1}$ is the Bohr radius, r_0 is the strong interaction radius. It is assumed that $r_0 << a_0$.

3) Index (cs) denotes the Coulomb- modified scattering length, the effective range $r_\ell^{(cs)}$, etc. Analogous parameters referring to strong potential V_s are denoted by index (s). A detailed derivation of formula (3) will be published elsewhere.

4) By the order of magnitude $|\beta_{\ell n}| \sim r_0^{2(n-\ell)-1}$ at $n \geq 1$, therefore in the ratio $\beta_{\ell n}^{(cs)} / \beta_{\ell n}^{(s)}$ the singular term $\sim (\xi r_0)^{2(\ell-n)+1} \ln \xi r_0$, $n \leq \ell$.
Besides, the numerical coefficient $b_{\ell n}$ decreases with the ℓ-n increase as well (see Table 1).

5) See also refs.[19,20,10]. The Coulomb problem with the δ - potential has been already considered in details in ref.[20], where the scattering length and effective range were calculated, which in fact coincide with eqs. (12),(14). However, in refs. [19,20] the logarithmic singularity is not selected in the coefficients $\beta_{\ell n}$ for any ℓ and approximate equations (5),(19) and (26) are not discussed. So our investigation properly complement the results of ref.[20]. Note also that the δ - potential was

applied to describe the $\bar{K}N$ system at low energies[21].

6) The most important (in the case $\rho z_0 \ll 1$) coefficients b_1 , c_0 , c_1 , g_1 and k_1 in expansions (5),(18) and (19) are easily calculated using formulae[5,7] which are valid for arbitrary potential V_s . Note that we exploit in eq.(5) the values of the radius z_s and of the coefficients c_0, c_1 and b_1 referring to the point of the emergence of s - level.

7) The condition $ctg\,\delta_\ell(k) = i$ defines the position of poles of the S - matrix. When going over from eq.(20) to (21) we should take into account the relations

$$M_{0,\mu}(z) = 2^{2\mu}\Gamma(\mu+1)z^{1/2}I_\mu\left(\frac{z}{2}\right), \quad W_{0,\mu}(z) = \left(\frac{z}{\pi}\right)^{1/2}K_\mu\left(\frac{z}{2}\right),$$

where $I_\mu(x)$, $K_\mu(x)$ are the modified Bessel functions. Eq. (21) can be easily obtained also from formula (10) putting $ctg\,\delta_\ell^{(s)} = i$ and taking account of identities

$$\mathcal{J}_\nu(ix) = e^{\frac{i\pi\nu}{2}} I_\nu(x),$$

$$\frac{N_\nu(ix)}{\mathcal{J}_\nu(ix)} = i - \frac{2}{\pi} e^{-i\pi\nu} \frac{K_\nu(x)}{I_\nu(x)} .$$

8) The coefficients b_1 , c_0 and c_1 can be calculated using general formulae[7] which are applicable both for local and nonlocal potentials. Note that the values of b_1 , b_2 and c_0 for the Yamaguchi potential have been calculated by Sitenko and Drobachenko[25] prior to ref.[24]. The authors are indebted to A.E. Kudryavtsev who paid our attention on the paper[25].

9) All of the constants given in Table 2 do not change at the scaling $V_s(z) \to \alpha^2 V_s(\alpha z)$, $0 < \alpha < \infty$. Therefore they depend only on the form of the potential but not on its

radius. The numerical calculation of these constants is most simple at $g = g_l$, see formulae in refs.[5,7] .

Note that local potentials are arranged in Table 2 in the course of C_0 raising.

10) For a potential of the type (7) the effective range at the point of the emergence of nl - level is equal to

$$r_l^{(s)} = \rho_{nl}\, r_0^{1-2l} ,$$

where ρ_{nl} is a numerical coefficient depending on the form of the potential. Let us give its values for the cases when analytical solution is possible. For the rectangular well

$$\rho_0 = 1 \;;\; \rho_l = -(2l+1)!!\,(2l-3)!! \,,\; l \geq 1$$

for the δ - potential[5]

$$\rho_0 = \frac{4}{3} \;;\; \rho_l = -\frac{4}{2l+3}\,(2l+1)!!\,(2l-3)!! ,\; l \geq 1 .$$

And finally, for the potential $v(x) = x^{-1}\Theta(1-x)$, $x = r/r_0$ analogous to the Yukawa potential, we have

$$\rho_{nl} = \begin{cases} \frac{4}{3}\left(1 - \frac{2}{\xi_{0,n}^2}\right), & l = 0 \\ -\frac{4}{3}(l+1)(2l-1)!!\,(2l-3)!!\left[1 + \frac{2(4l^2-1)}{\xi_{nl,n}^2}\right], & l \geq 1 \end{cases}$$

where $\xi_{\nu n}$ is the n-positive zero of the Bessel function $J_\nu(z)$. n = 0,1,2 ...... These values of $\rho_{n\ell}$ have been calculated using the formulae given in ref. [18].

11) Here we change the normalization of the coefficients f_ℓ and h_ℓ comparing with ref.[5]:

$$\tilde{f}_\ell = \frac{1}{2\ell-1}\left[(2\ell+1)!!\,(2\ell-3)!!\right]^{\frac{1}{2\ell-1}} f_\ell\,, \quad \ell \geq 1$$

$$\tilde{h}_0 = h_0\,, \quad \tilde{h}_\ell = \left[(2\ell+1)!!\,(2\ell-3)!!\right]^{-\frac{1}{2\ell-1}} h_\ell\,, \quad \ell \geq 2$$

and $(-1)!! = 1$.

12) For instance, for ns - states in the Hulthén potential c_0 = 0.865, 0.925, ... 1.065, respectively for n = 1,2, ... 10. The coefficient b_1 varies in this case from 1.000 (n=1) to 0.952 (n=10) while c_1 from 0.983 up to 1.809.

13) Here $b_k = (2\ell-k)!/k!\,(2\ell)!$ for $0 \leq k \leq 2\ell$,

$$b_{2\ell+1} = \frac{1}{(2\ell)!\,(2\ell+1)!}\left[1 + \frac{1}{2} + \ldots + \frac{1}{2\ell+1} - 2C\right],$$

and coefficients b_k with $k > 2\ell+1$ are calculated from recurrence relations:

$$b_k = \frac{(-1)^k}{k!\,(k-2\ell-1)!\,(2\ell)!}\left(2C - c_k\right),$$

$$c_{2\ell+1} = \psi(2\ell+2) - \psi(1) = 1 + \frac{1}{2} + \ldots + \frac{1}{2\ell+1}\,,$$

$$c_k = c_{k-1} + \frac{2(k-\ell)-1}{k\,(k-2\ell-1)}\,, \quad k \geq 2\ell+2 .$$

Table 1

Coefficients $b_{\ell k}$ in eq.(4)

$\ell \backslash k$	0	1	2	3	4	5
1	2	2	0	0	0	0
2	0,5	2,5	2	0	0	0
3	5,56(-2)	0,778	2,72	2	0	0
4	3,47(-3)	0,104	0,948	2,85	2	0
5	1,39(-4)	7,64(-3)	0,142	1,06	2,93	2

Note to table 1. As usual, the number in brackets show the order of magnitude: 5.56(-2) = 0.0556, 3,47(-3) = $3.47 \cdot 10^{-3}$ etc.

Table 2

$v(x)$	$l = 0$					$l = 1$			$l = 2$	
	r_s/r_o	C_o	C_1	b_1	h_o	$\tilde{f}_1$	d_1	k_1	$\tilde{f}_2$	$\tilde{h}_2$
e^{-x}/x	2.120	0.837	0.842	1.056	-0.178	1.02	0.215	-	0.517	
		0.931	1.428	0.962	-	-	-	-	-	-
$(e^{x}-1)^{-1}$	3.000	0.865	0.983	1.000	0.138	0.987	0.101	-	0.486	-
		0.924	1.453	0.905	-	-	-	-	-	-
$exp(-x)$	3.541	0.919	1.215	0.903	0.512	0.936	0.072	-	0.448	-
$(ch\,x)^{-2}$	2.000	0.946	1.323	0.863	0.693	0.906	0.223	-	0.437	-
$exp(-x^2)$	1.435	0.975	1.457	0.814	0.936	0.870	0.410	-	0.412	-
$x^{-1}\theta(1-x)$	0.872	0.985	1.500	0.793	1.043	-	-	-	-	-
$\theta(1-x)$	1.000	1.024	1.634	0.757	1.189	0.813	0.813	0.701	0.370	1.42
$\delta(1-x)$	1.333	1.060	1.750	0.750	1.312	0.780	0.500	0.856	0.352	1.35
eq.(C.1)	3.000	0.865	1.333	0.667	0.889	0.960	0.042	0.739	0.464	1.58

Note to table 2. The parameters C_o, C_1 etc. refer, as a rule, to the first level with the angular momentum l (for the Yukawa and the Hulthén potentials the parameters for the 1s and 2s levels are given - the first and the second line, correspondigly).

Table 3

$v(x)$	R_ℓ / r_s			
	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
e^{-x}/x	1.030	1.033	1.028	1.013
$(e^{x}-1)^{-1}$	1.042	1.048	1.028	0.919
$x^{-1}\,\theta(1-x)$	1.051	1.074	1.088	1.097
$\exp(-x)$	1.016	1.012	0.967	0.806
$(\mathrm{ch}\,x)^{-2}$	1.008	1.007	1.003	0.995
$\exp(-x^2)$	1.015	1.021	1.024	1.025
$\delta(1-x)$	0.938	0.904	0.882	0.867

Note to table 3. Here $r_s = r_0^{(\infty)}$ is effective range at $g = g_0$, while R_ℓ refer to the first level with the angular momentum ℓ ($g = g_\ell$).

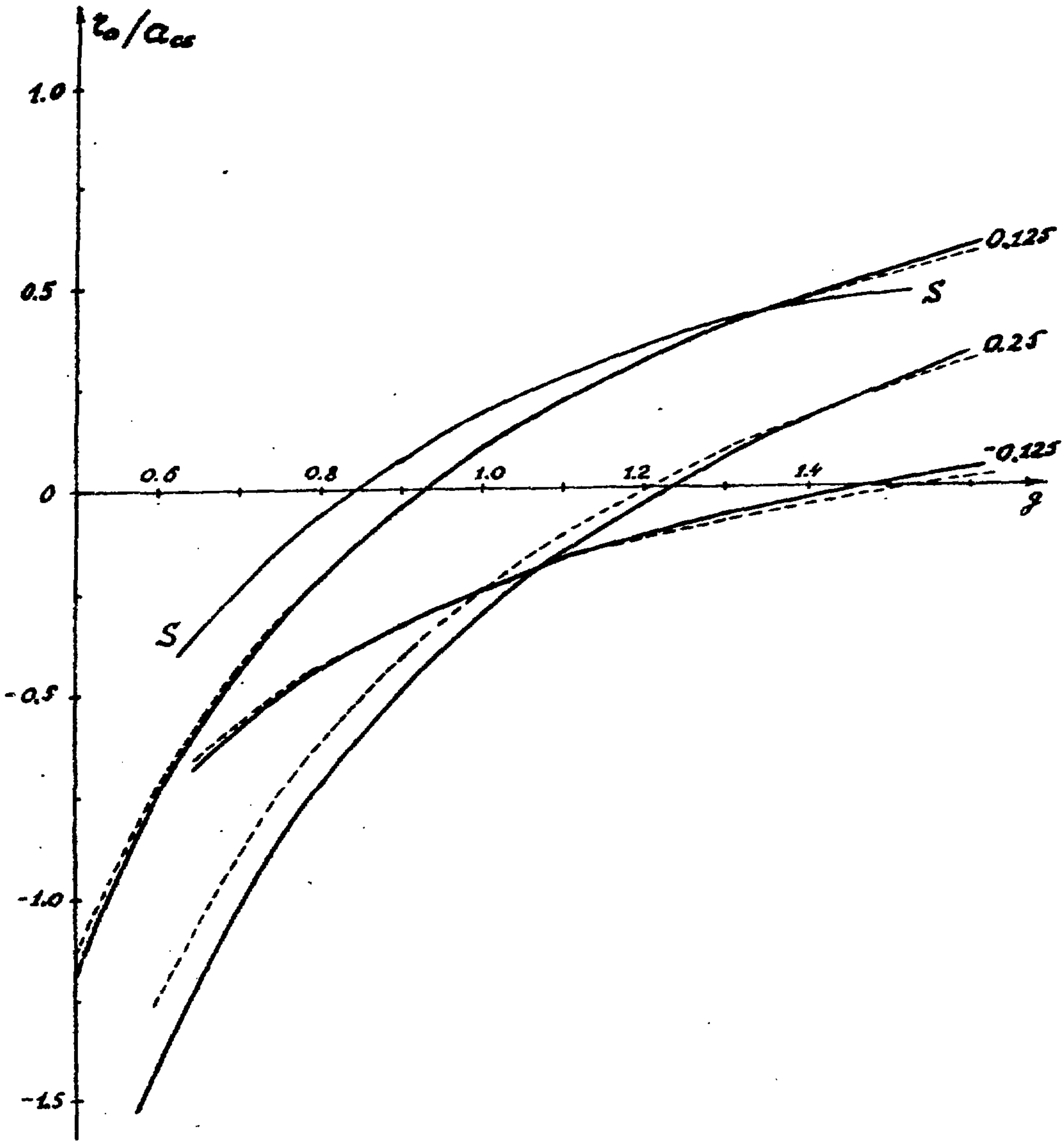

Fig.1. The inverse s-scattering length for the δ - potential. The numbers at curves give the values ξr_0. The curve S (for $\xi r_o = 0.125$) corresponds to neglecting the corrections of the order of ξr_s (i.e. $b_1 = b_2 = c_1 = c_2 = 0$ in eq.(5)).

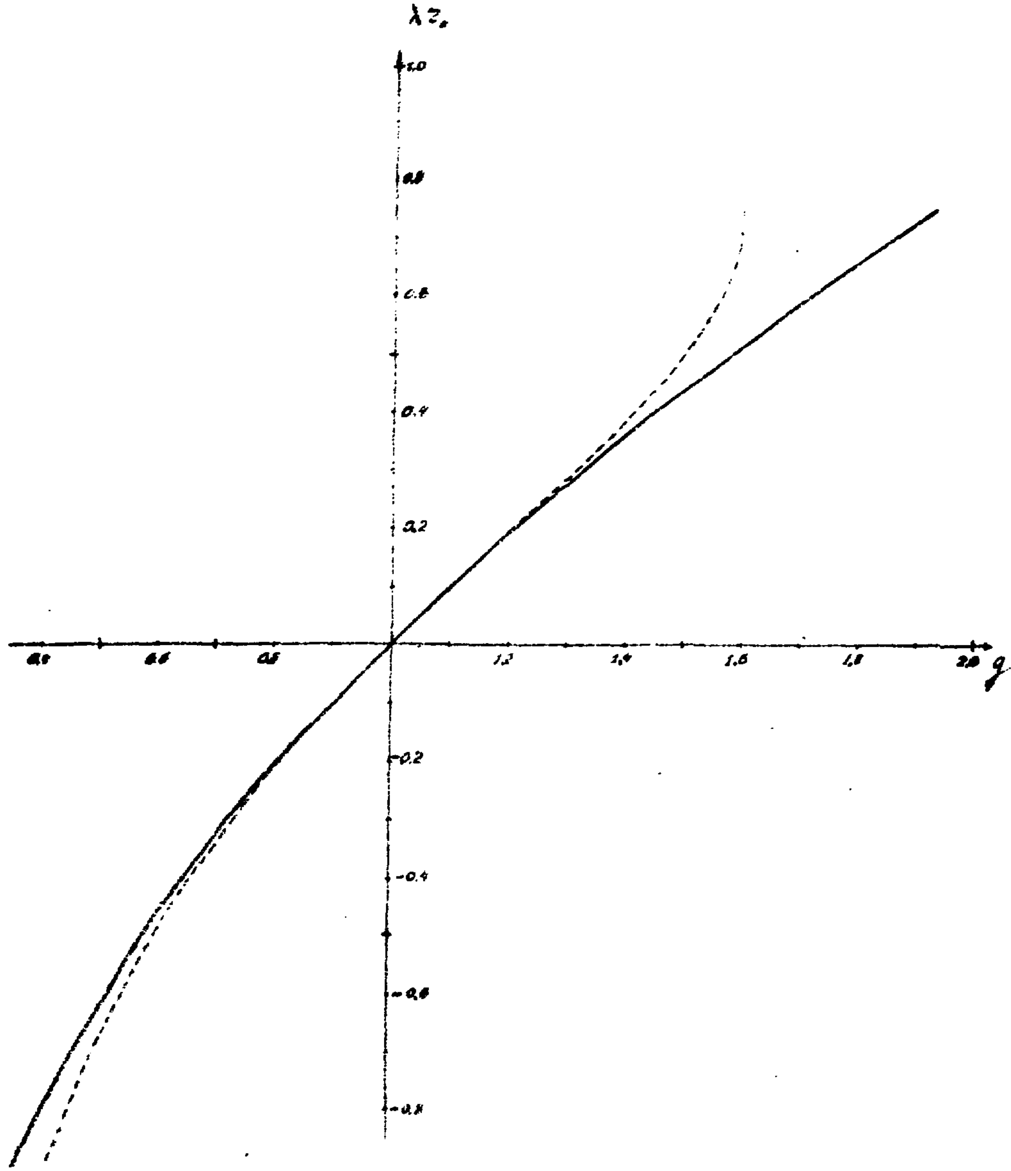

Fig.2. The position of the s-level in the δ -potential depending on the coupling constant g (the energy $E = -\lambda^2/2$). At $\lambda > 0$ the level is real, at $\lambda < 0$ -is virtual. The dashed curve is the approximation (26) at $f = 0$.

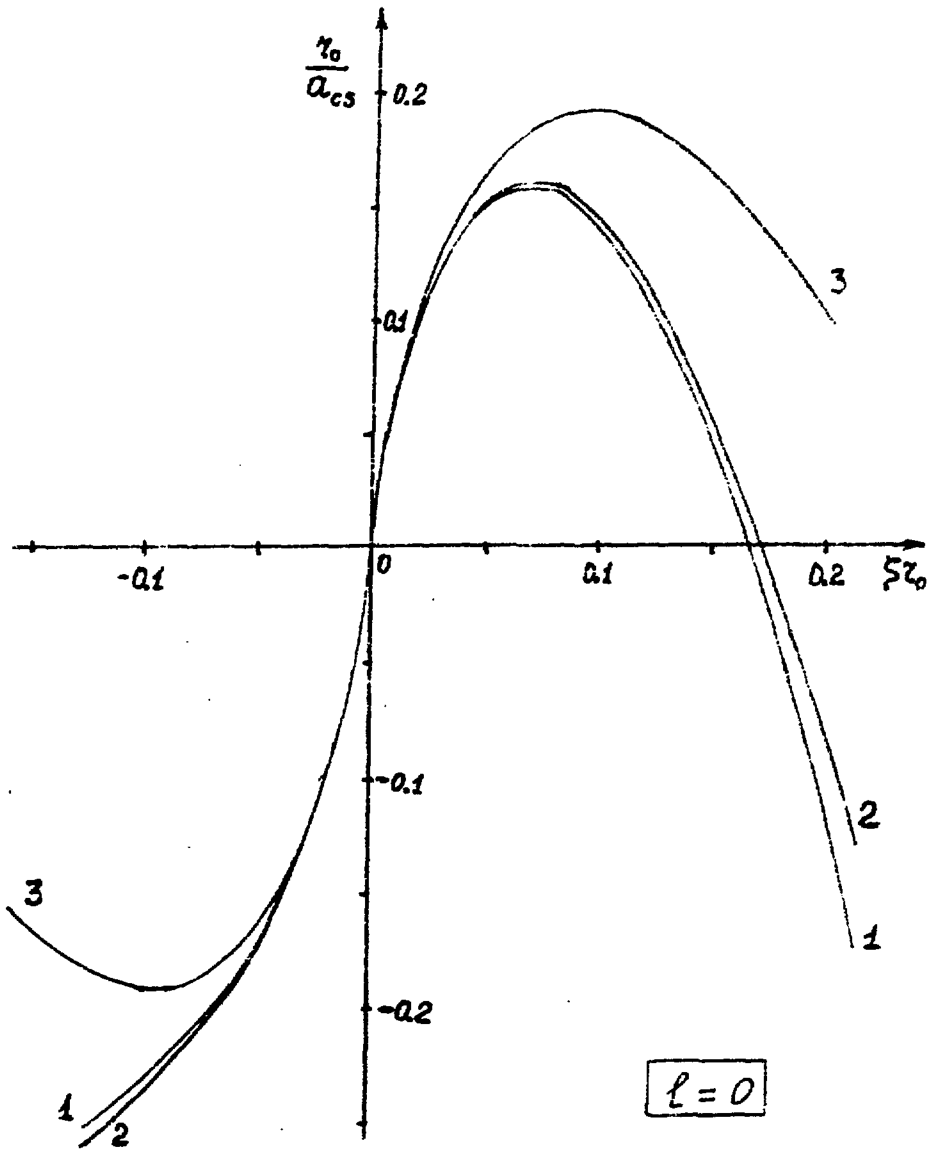

Fig.3. The ratio r_0/a_{cs} (ℓ = 0) in the δ - potential model at $g = g_0 = 1$. The curves 1 and 2 correspond to exact solution (15′) and to formula (5) with the given in (16) values of the parameters c_0 , c_1 and b_1 . The curve (3) corresponds to formula (5) in which we put $b_1 = c_1 = 0$, i.e. neglect corrections $\sim \zeta r_s$.

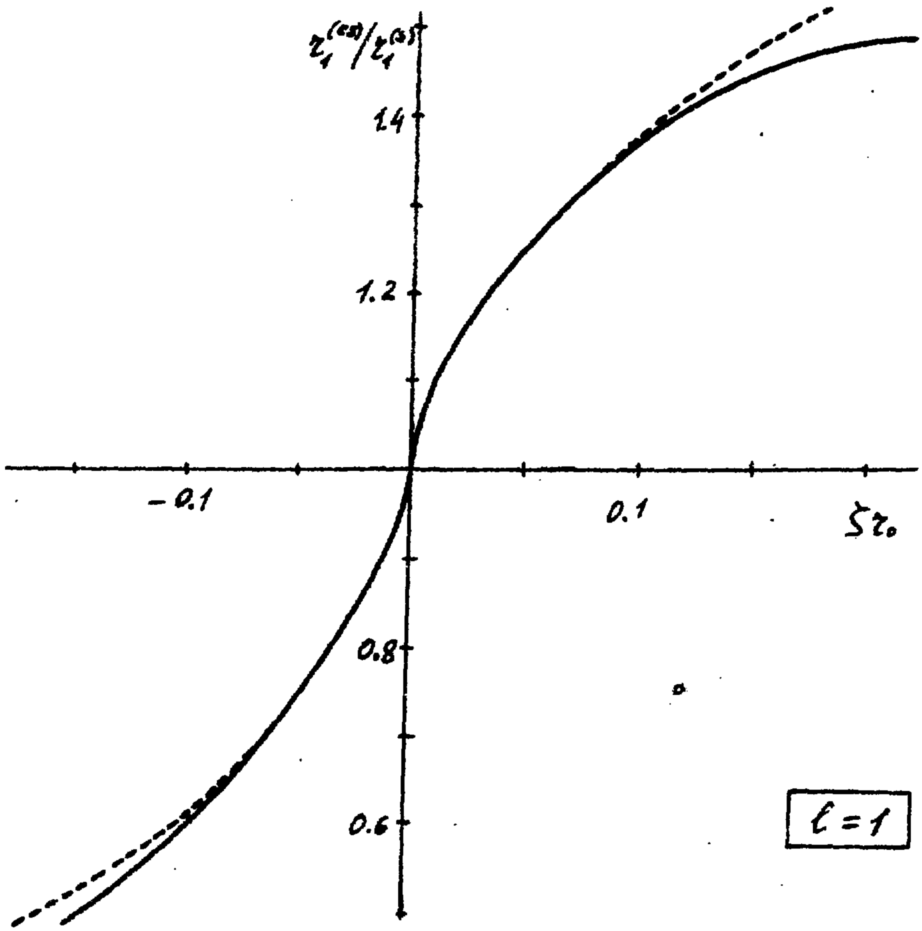

Fig.4. The ratio of effective ranges $r_1^{(cs)}/r_1^{(s)}$ for the δ-potential model ($l = 1$, $g = g_1 = 3$). The solid curve corresponds to the exact formula (15), the dashed one is plotted according to eq. (19) where the term $\sim \delta^2$ is neglected.

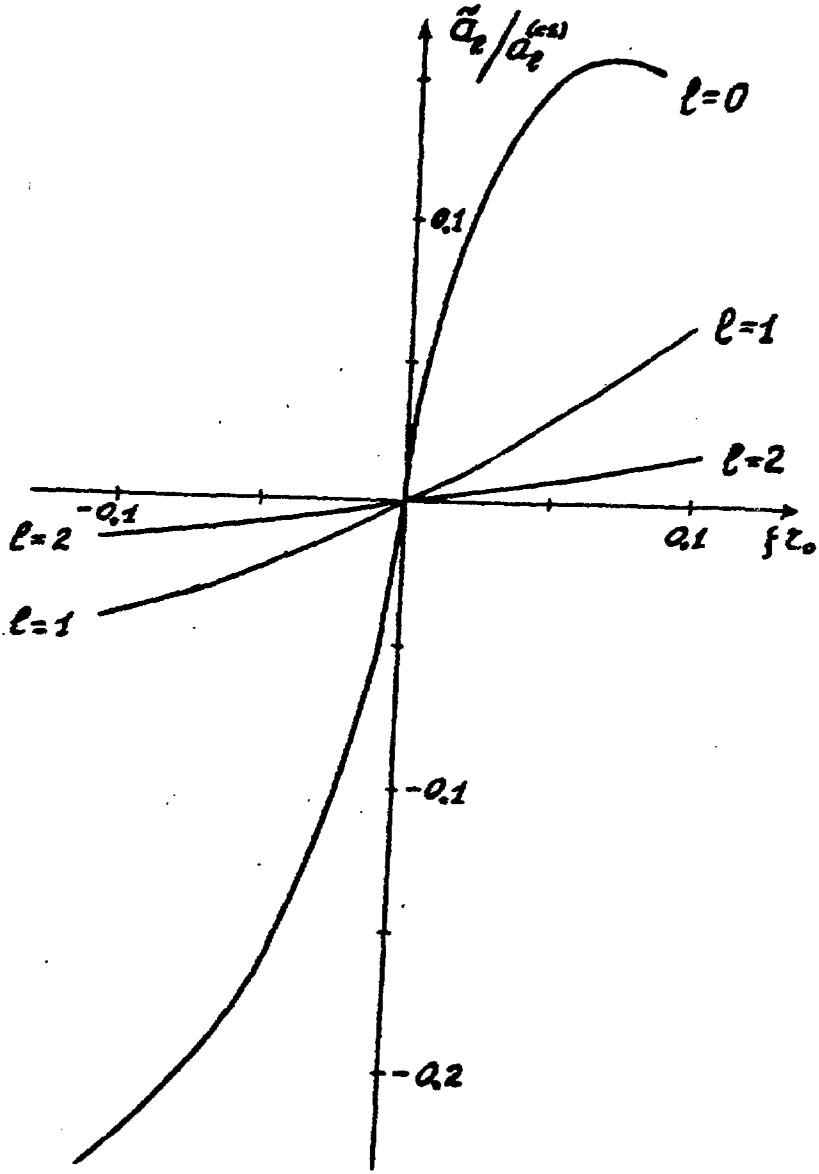

Fig.5a. The ratios $\tilde{a}_\ell / a_\ell^{(cs)}$ for the δ - potential model at $g = g_\ell = 2\ell + 1$.

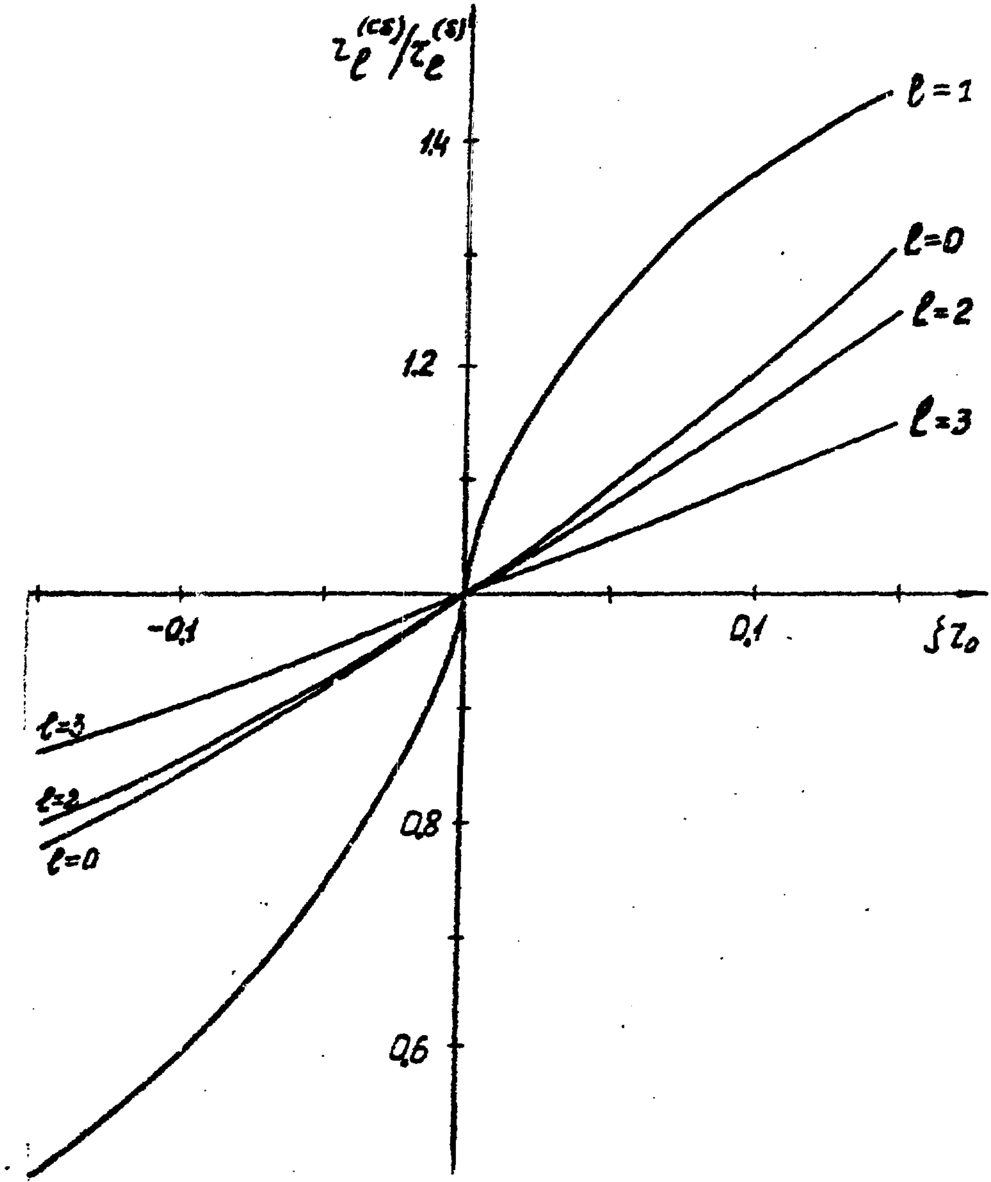

Fig.5b.The same for $\tau_\ell^{(cs)}/\tau_\ell^{(s)}$.

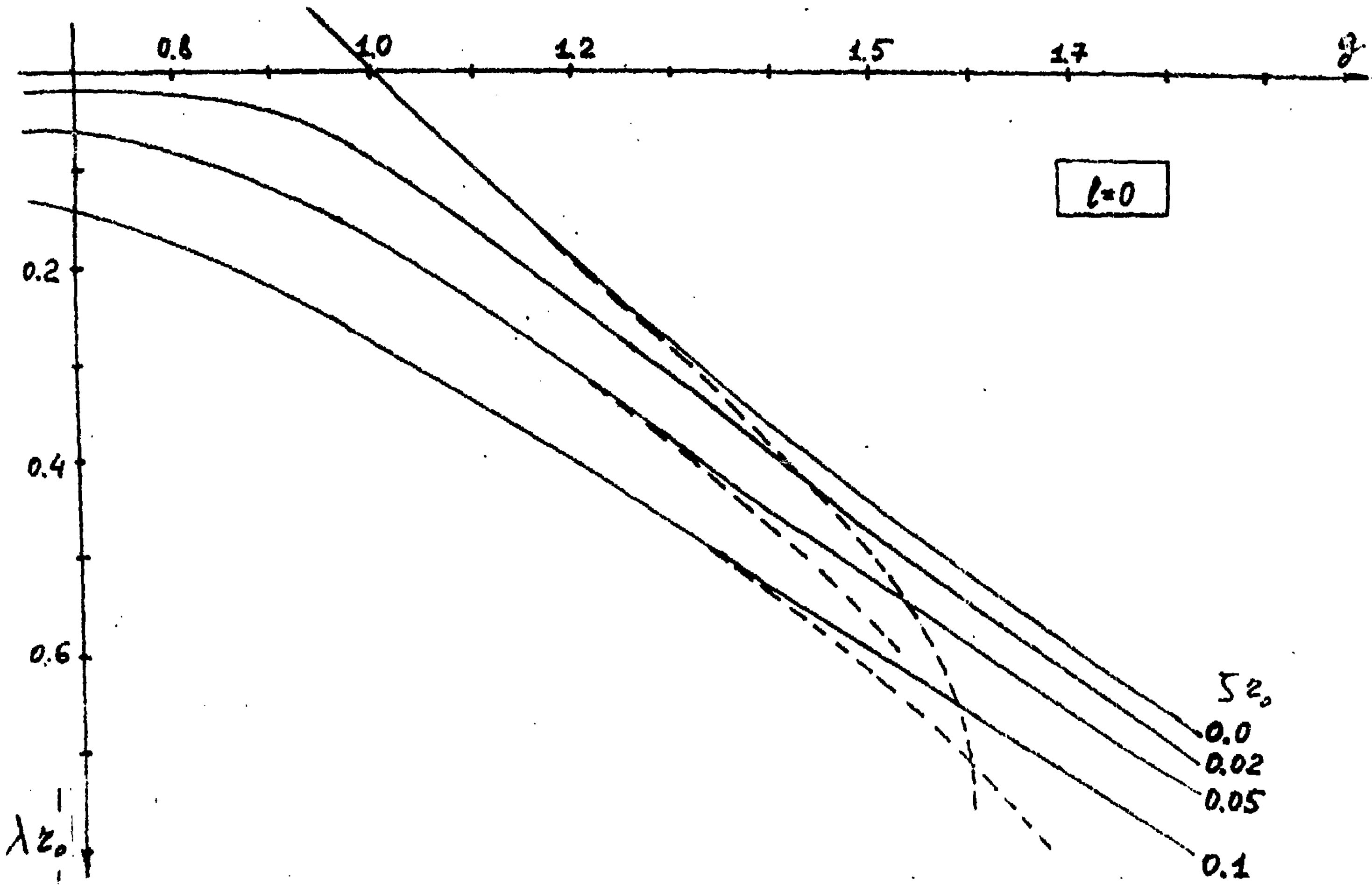

Fig.6. The dependence of λr_0 on the coupling constant g for the s-level in the δ-potential. At the curves the values $\xi r_0 = r_0/a_g$ are shown. The solid curves are plotted according to the exact eq.(20), the dashed - using the approximate eq.(26).

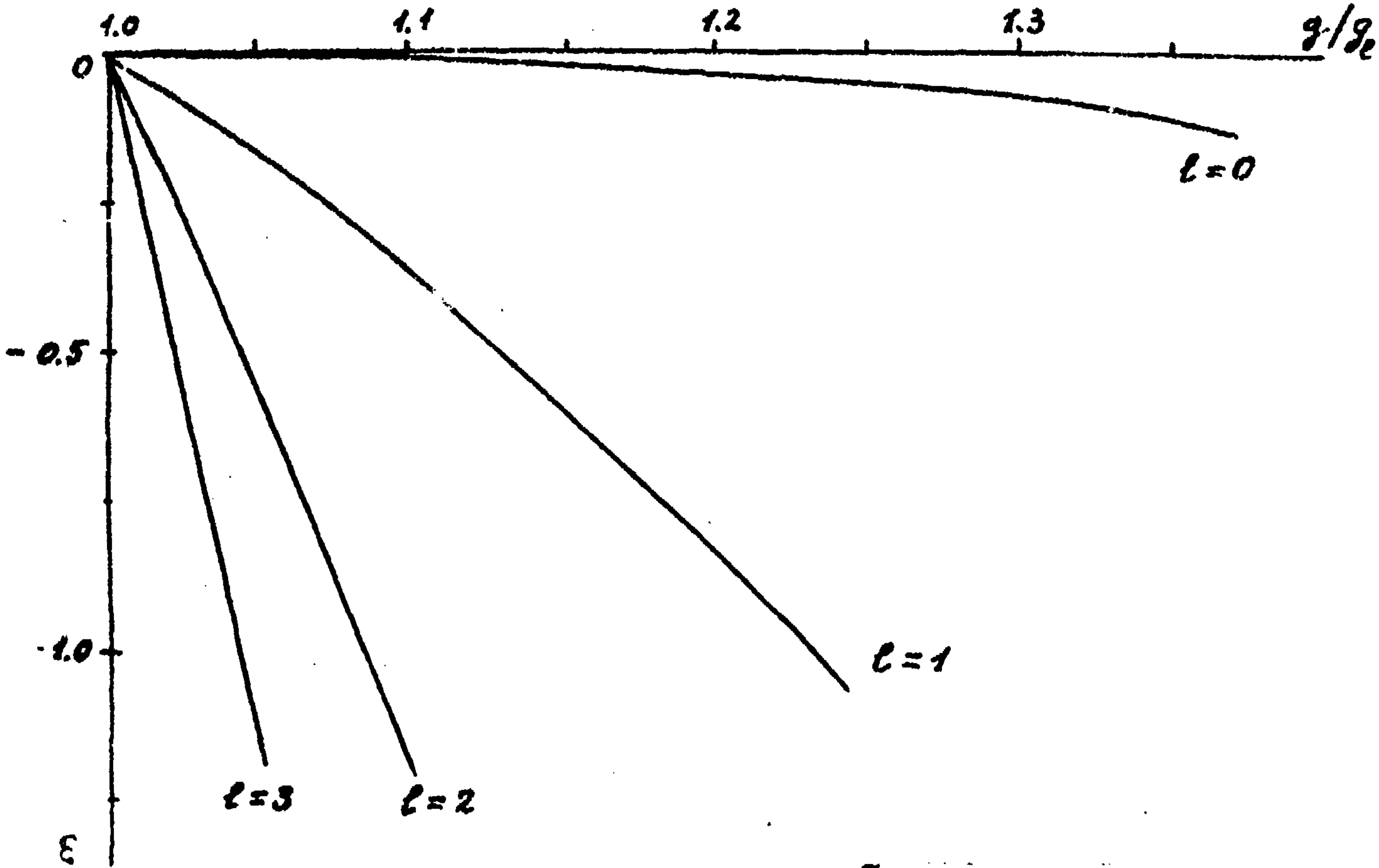

Fig.7. The bound state energies in the δ-potential model at $\xi = 0$ (the Coulomb interaction is switched off).
The value of $\varepsilon \simeq -(\lambda z_o)^2$ is plotted on the ordinate axis.

References

1. Goldberger M.L., Watson K.M. Collision Theory, Wiley, N.Y. 196-
2. Burke P.G. Potential Scattering in Atomic Physics. Plenum Press, N.Y. 1977.
3. Schwinger J.–Phys.Rev. 1950, 78, 135.
4. Jackson J.D., Blatt J.M.–Rev.Mod.Phys. 1950, 22, 77.
5. Mur V.D., Kudryavtsev A.E., Popov V.S. Preprint ITEP-180, Moscow 1980; Yad.Fiz. 1983, 37, 1417.
6. Popov V.S., Kudryavtsev A.E., Lisin V.I., Mur V.D.–Pis'ma v ZhETF, 1982, 36, 207.
7. Popov V.S., Kudryavtsev A.E., Lisin V.I., Mur V.D.–ZhETF, 1981, 80, 1271.
8. Kudryavtsev A.E., Lisin V.I., Popov V.S.–Pis'ma v ZhETF, 1981, 34, 292.
9. Kudryavtsev A.E., Mur V.D., Popov V.S.–Pis'ma v ZhETF, 1983, 37, 412; Phys.Lett. 1984, B143, 41.
10. Popov V.S., Kudryavtsev A.E., Lisin V.I., Mur V.D.–ZhETF 1985, 88, 715.
11. Kok L.P.–Phys.Rev.Lett. 1980, 45, 427.
12. Dumbrais O., Green A.M., Neskanen J.A.–Nucl.Phys. 1984, A412, 195.
13. Kerbikov B.O.–Pis'ma v ZhETF 1983, 37, 118.
14. Kerbikov B.O.–Phys.Lett. 1981, 100B, 371.
15. Lambert E.–Helv.Phys.Acta, 1969, 42, 667.
16. Abramowitz M., Stegun I.A.–Handbook of Mathematical Functions. Dover, N.Y. 1965.
17. Andreyev S.P., Karnakov B.M., Mur V.D., Polunin V.A.–ZhETF 1984, 86, 866.

18. Mur V.D., Popov V.S.– TMF, 1976, 27, 204.

19. van Haeringen H., Kok L.P.– Phys.Lett. 1981, A82, 317.

20. Kok L.P., de Maag J.W., Brouwer H.H., van Haeringen H. – Phys.Rev. 1982, C26, 2381.

21. Brower H.H., de Maag J.W., Kok L.P.– Zeits.Phys. 1984, A318, 199.

22. Galitsky V.M., Karnakov B.M., Kogan V.I. Quantum Mechanics Problems, "Nauka", M. 1981 (in Russian).

23. Yamaguchi Y.– Phys.Rev. 1954, 95, 1628.

24. van Haeringen H.– Nucl.Phys. 1975, A253, 355.

25. Sitenko A.G., Drobachenko G.V.,– Ukranian Phys.Journ. 1963, 8, 5.

26. de Maag J.W., Kok L.P., van Haeringen H.,– J.Math.Phys. 1984, 25, 684.

27. Izycki M., Backenstoss G. et al.– Z.Phys. 1980, A297, 1.

28. Ahmad S., Amsler C. et al., CERN-EP/85-05 (1985).

29. Trueman T.L.– Nucl.Phys. 1961, 26, 57.

30. Kirzhnitz D.A., Pen'kov F.M.– ZhETF 1982, 82, 657.

31. Zel'dovich Ya.B.– FTT, 1959, 1, 1637.

32. Karnakov B.M., Mur V.D., Popov V.S.– Dokl.AN SSSR 1984, 279, 345.

33. Mur V.D., Popov V.S.,– Dokl.AN SSSR, 1985, 281, No.6.

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