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THE CONSERVATION LAW OF GAUGE STRESS FIELD OF A CONTINUUM WITH DEFECTS AND ITS APPLICATION TO THE FRACTURE OF MATERIALS

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THE CONSERVATION LAW OF GAUGE STRESS FIELD OF A CONTINUUM WITH DEFECTS AND ITS APPLICATION TO THE FRACTURE OF MATERIALS *

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ABSTRACT

The conservation law of a gauge stress field of materials with defects is derived. The field equations and a criterion of the singularity (dislocation, crack, etc.) motion in a continuum with defects are obtained.

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I. INTRODUCTION

Gauge constructs in the theory of materials with defects are of current interest. The earliest paper we know was published in 1978 [1]. Almost at the same time, but one year later, there were some papers published independently [2-5]. Among them, there were some works on transformations of the field equations and the characteristic quantities of dislocations which made them invariant. But, the symmetrical theorem and the conservation law have not been discussed yet. Therefore, it is difficult to apply them to practical problems such as dislocation motion, crack propagation, etc. In this paper, the conservation law of gauge stress field of materials with defects is derived. The field equations and a criterion of the singularity (dislocation, crack, etc.) motion in a continuum with defects are obtained. It may be considered as an extension of Eshelby's energy-momentum tensor to a more general case.

II. THE GAUGE THEORY OF DISLOCATIONS AND DISCLINATIONS

As shown in Ref.5, we define the generalized spatial co-ordinates as *)

$$\hat{X} = \begin{Bmatrix} \vec{x} \\ 1 \end{Bmatrix} . \tag{1}$$

Then the generalized Cauchy strain tensors are

$$\hat{C}_{\mu\nu} = \partial_\mu \hat{X}^T \partial_\nu \hat{X} = \partial_\mu \vec{x}^T \partial_\nu \vec{x} = C_{\mu\nu} . \tag{2}$$

Therefore $L(\hat{C}_{\mu\nu}) = L(C_{\mu\nu})$, where L is the Lagrangian of classical elastic field.

Now express the matrices of $G_0 = SO(3)_0 \rhd T(3)_0$ as

$$M = \begin{pmatrix} A & \{\vec{b}\} \\ \{\vec{0}\}^T & 1 \end{pmatrix}, \qquad A \in SO(3)_0 , \quad \vec{b} \in T(3)_0 . \tag{3}$$

Under a homogeneous gauge transformation in G_0, X transforms via

$$\hat{X}' = M\hat{X}, \qquad M \in G_0 \tag{4}$$

*) The generalized spatial co-ordinates, inertial co-ordinates and displacements used in this paper take the common spatial co-ordinates, inertial co-ordinates and displacements as their first three components. The fourth components are $X_4 = 1$, $u_4 = X_4 - a_4 = 0$, $a_4 = 1$, respectively.

So that

$$\hat{C}'_{\mu\nu} = (\partial_\mu \hat{X}^T) M^T M (\partial_\nu \hat{X})$$
$$= \partial_\mu \vec{x}^T \partial_\nu \vec{x}$$
$$= C_{\mu\nu}$$
$$= \hat{C}_{\mu\nu} \tag{5}$$

i.e. $L(\hat{C}_{\mu\nu})$ is invariant under the homogeneous gauge group G_0.

Let us now consider the local gauge invariance. The infinitesimal generators of G_0 are

$$\begin{pmatrix} \gamma_i & \{\vec{0}\} \\ \{\vec{0}\}^T & 0 \end{pmatrix}, \quad i = 1,2,3; \qquad \begin{pmatrix} [0] & \{\vec{t}_j\} \\ \{\vec{0}\}^T & 0 \end{pmatrix}, \quad j = 1,2,3^{*)} \tag{6}$$

which correspond to rotation and translation operations, respectively, and in which

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{t}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{t}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{t}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7}$$

Let T_A unify the generators of G_0, $A = 1,2,\ldots,6$. T_A corresponds to the generators made by γ_i when $A = 1,2,3$ and to the ones made by $\vec{t}_j$ when $A = 4,5,6$. It is easy to verify that the matrices T_A are subject to the commutation relation

$$[T_A, T_B] = i f^{\Gamma}_{AB} T_\Gamma, \qquad A, B, \Gamma = 1, 2, \cdots 6. \tag{8}$$

Breaking of the homogeneity of the action of SO(3) $\triangleright$ T(3) is shown to give rise to a disclination-dislocation field.

According to gauge theory [6], we introduce the gauge potential functions $A^M_\mu(a)$ under a local transformation. Then we can make the replacement as follows:

*) In this paper, the following conventions are adopted: whenever small Greek letters, α, β, μ,... are used they take values from the set {1,2,3,4} ; capital Greek letters, A, B, Γ,..., take values from the set {1,2,3,4,5,6} ; and small Latin letters, i,j,k,..., from the set {1,2,3}.

$$\partial_\mu \rightarrow D_\mu(a) \equiv \partial_\mu - ig A^M_\mu(a) T^M. \tag{9}$$

The A^M_μ correspond to the W^α_μ in Ref.5 when $M = 1,2,3$ and to ϕ^i_μ when $M = 4,5,6$.

Define the intensity of the gauge field as $F^M_{\mu\nu}$,

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$$
$$F_{\mu\nu} = -ig F^M_{\mu\nu} T^M \tag{10}$$

and we have

$$F^M_{\mu\nu} = \partial_\mu A^M_\nu - \partial_\nu A^M_\mu + g f^M_{N\Gamma} A^N_\mu A^\Gamma_\nu. \tag{11}$$

Let the Lagrangian of the free gauge field be

$$L_F = -\frac{1}{4} F^M_{\mu\nu} F^{\mu\nu M} \tag{12}$$

then the total Lagrangian of the system is

$$L = L_0(\hat{C}_{\mu\nu}) + L_F \tag{13}$$

in which $L_0(\hat{C}_{\mu\nu})$ can be divided into the free elastic L_f and the interaction of elastic and gauge field L_{int}. $\hat{C}_{\mu\nu} = D_\mu \hat{X}^T D_\nu \hat{X}$ is invariant under the local gauge transformations [5], and so are L_0 and L_F.

Obviously

$$L_0(\hat{C}_{\mu\nu}) = L_0(a_\mu, X_\mu, X_{\mu,\nu}, A^M_\mu) \tag{14}$$
$$L_F = L_F(A^M_\mu, A^M_{\mu,\nu}), \tag{15}$$

and therefore,

$$L = L_0 + L_F = L(a_\mu, X_\mu, X_{\mu,\nu}, A^M_\mu, A^M_{\mu,\nu}) \tag{16}$$

is a kind of Lagrangian in a generalized continuum.

III. THE SYMMETRICAL THEOREM IN A GENERALIZED CONTINUUM

A continuous transformation group with a single parameter is given by

$$G: \begin{cases} a'_\mu = a_\mu + g_\mu \eta + O(\eta^2) \\ X'_\mu = X_\mu + p_\mu \eta + O^2(\eta^2) \\ A'^M_\mu = A^M_\mu + q^M_\mu \eta + C^3(\eta^2) \end{cases} , \tag{17}$$

where η is the parameter, $\mu = 1,2,3,4$, $M = 1,2,\ldots,6$, and

$$g_\mu = g_\mu(a_\nu, X_\nu, X_{\nu,\lambda}, A^M_\alpha, A^M_{\alpha,\nu}) = \frac{\partial a'_\mu}{\partial \eta}\Big|_{\eta=0},$$

$$p_\mu = p_\mu(a_\nu, X_\nu, X_{\nu,\lambda}, A^M_\alpha, A^M_{\alpha,\nu}) = \frac{\partial X'_\mu}{\partial \eta}\Big|_{\eta=0},$$

$$q^M_\mu = q^M_\mu(a_\nu, X_\nu, X_{\nu,\lambda}, A^M_\alpha, A^M_{\alpha,\nu}) = \frac{\partial A'^M_\mu}{\partial \eta}\Big|_{\eta=0}, \tag{18}$$

$$\delta a_\mu = g_\mu \eta, \quad \delta X_\mu = p_\mu \eta, \quad \delta A^M_\mu = q^M_\mu \eta .$$

It is obvious that we can obtain some kinds of transformation groups by assigning g_μ, p_μ and q^M_μ some special values.

Let the action functional in the generalized continuum disucssed be

$$S = \int_V L(a_\mu, X_\mu, X_{\mu,\nu}, A^M_\mu, A^M_{\mu,\nu})\, dV \tag{19}$$

we have [7] Noether's symmetrical theorem. If S is infinitesimal invariant under the gauge group G, there must be (Appendix I):

$$\frac{\delta L}{\delta X_\mu}\bar{\delta} X_\mu + \frac{\delta L}{\delta A^M_\mu}\bar{\delta} A^M_\mu = J^M_{,\mu} \tag{20}$$

in which

$$\frac{\delta L}{\delta X_\mu} = \frac{\partial L}{\partial X_\mu} - \left(\frac{\partial L}{\partial X_{\mu,\nu}}\right)_{,\nu}, \quad \frac{\delta L}{\delta A^M_\mu} = \frac{\partial L}{\partial A^M_\mu} - \left(\frac{\partial L}{\partial A^M_{\mu,\nu}}\right)_{,\nu},$$

$$\bar{\delta} X_\mu = \delta X_\mu - X_{\mu,\nu}\delta a_\nu, \quad \bar{\delta} A^M_\mu = \delta A^M_\mu - A^M_{\mu,\nu}\delta a_\nu,$$

$$J^M = \left(\frac{\partial L}{\partial X_{\nu,\mu}}\bar{\delta} X_\nu + \frac{\partial L}{\partial A^M_{\nu,\mu}}\bar{\delta} A^M_\nu + L\,\delta a^M\right) .$$

Noether's symmetrical theorem plays an important role in modern field theory. From it we can obtain the field equation of materials with defects, the dynamical conservation and the criterion of a singularity motion (a dislocation starts to move or crack starts to propagate, etc.) at its initial stage.

IV. THE FIELD EQUATIONS IN A GENERALIZED CONTINUUM

Following Noether's theroem and (17) let

$$g_\mu = 0, \quad p_\mu\big|_{\partial V} = 0, \quad q^M_\mu\big|_{\partial V} = 0 . \tag{21}$$

We have

$$\frac{\delta L}{\delta X_\mu}\delta X_\mu + \frac{\delta L}{\delta A^M_\mu}\delta A^M_\mu = -\left(\frac{\partial L}{\partial X_{\nu,\mu}}\delta X_\nu + \frac{\partial L}{\partial A^M_{\nu,\mu}}\delta A^M_\nu\right)_{,\mu} . \tag{22}$$

After integration we obtain

$$\int_V \left(\frac{\delta L}{\delta X_\mu}\delta X_\mu + \frac{\delta L}{\delta A^M_\mu}\delta A^M_\mu\right) dV = -\int_V \left(\frac{\partial L}{\partial X_{\nu,\mu}}\delta X_\nu + \frac{\partial L}{\partial A^M_{\nu,\mu}}\delta A^M_\nu\right)_{,\mu} dV . \tag{23}$$

An application of the Gauss theorem yields

$$\int_V \left(\frac{\delta L}{\delta X_\mu}\delta X_\mu + \frac{\delta L}{\delta A^M_\mu}\delta A^M_\mu\right) dV = -\oint_{\partial V} \left(\frac{\partial L}{\partial X_{\nu,\mu}}\delta X_\nu + \frac{\partial L}{\partial A^M_{\nu,\mu}}\delta A^M_\nu\right) n_\mu\, ds . \tag{24}$$

By the above hypothesis we have

$$\delta X_\nu\big|_{\partial V} = p_\nu \eta\big|_{\partial V} = 0, \quad \delta A^M_\nu = q^M_\nu \eta\big|_{\partial V} = 0, \tag{25}$$

therefore

$$\int_V \frac{\delta L}{\delta X_\mu}\delta X_\mu\, dV + \int_V \frac{\delta L}{\delta A^M_\mu}\delta A^M_\mu\, dV = 0 . \tag{26}$$

Considering the arbitrariness of V and δX_μ, δA^M_μ in V, we have

$$\frac{\delta L}{\delta X_i} = 0, \quad \frac{\delta L}{\delta A^M_\mu} = 0. \tag{27}$$

namely

$$\begin{cases} \dfrac{\partial L}{\partial X_i} - \left(\dfrac{\partial L}{\partial X_{i,\nu}}\right)_{,\nu} = 0 \\ \dfrac{\partial L}{\partial A^M_\mu} - \left(\dfrac{\partial L}{\partial A^M_{\mu,\nu}}\right)_{,\nu} = 0 \end{cases} . \tag{28}$$

Eqs.(28) are the field equations when dislocations and disclinations exist.

If we start from the homogeneous linear elastic medium, and take its Lagrangian as

$$L_0 = \tfrac{1}{2}[\lambda(e_{ij}\delta^{ij})^2 + 2\mu e_{ij} e_{ij}] - \tfrac{1}{2}\rho_0 \dot{X}_i \dot{X}_i$$

$$e_{ij} = \tfrac{1}{2}(C_{ij} - \delta_{ij}) \qquad (29)$$

then Hooke's law can be expressed as

$$e_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\,\sigma_{kk}. \qquad (30)$$

After replacement (9) we have

$$L_0(\hat{C}_{\mu\nu}) = \tfrac{1}{2}[\lambda(\hat{e}_{ij}\delta^{ij})^2 + 2\mu \hat{e}_{ij}\hat{e}_{ij}] - \tfrac{1}{2}\rho_0 \hat{C}_{44}$$

$$\hat{e}_{ij} = \frac{1}{2\mu}\hat{\sigma}_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\,\hat{\sigma}_{kk} \qquad (31)$$

Let displacements u = X - a, gauge potentials A and their derivatives be infinitesimal,and we have the linear field equations as follows (Appendix II):

$$\hat{\sigma}_{ij,j} - \rho_0[u_{i,44} - ig(T^M_{i4}A^M_{4,4} + T^M_{ik}A^M_{4,4}a^k)] = 0$$
$$-ig\hat{\sigma}_{ij}(T^M_{i4} + T^M_{ik}a^k)\delta^j_\mu + \rho_0[igT^M_{k4}u_{k,4} +$$
$$+ igT^M_{kh}u_{k,4}a^h + 2g^2A^N_4(T^M_{k4}T^N_{k4} + T^M_{k4}T^N_{kk}a^k +$$
$$+ T^M_{ik}T^N_{k4}a^k + T^M_{kk}T^N_{kh}a^ka^h)]\delta^4_\mu + A^M_{\mu,\nu\nu} - A^M_{\nu,\mu\nu} = 0 . \qquad (32)$$

In plane statics, the equations can be written as (Appendix III):

$$A^N_{1,22} - A^N_{2,12} = 0$$
$$A^N_{2,11} - A^N_{1,21} = 0$$
$$ig(\hat{\sigma}_{11}a_2 - \hat{\sigma}_{12}a_1) + A^3_{1,22} - A^3_{2,12} = 0$$
$$ig(\hat{\sigma}_{12}a_2 - \hat{\sigma}_{22}a_1) + A^3_{2,11} - A^3_{1,21} = 0$$
$$-ig\hat{\sigma}_{11} + A^4_{1,22} - A^4_{2,12} = 0$$
$$-ig\hat{\sigma}_{12} + A^4_{2,11} - A^4_{1,21} = 0$$
$$-ig\hat{\sigma}_{21} + A^5_{1,22} - A^5_{2,12} = 0$$
$$-ig\hat{\sigma}_{22} + A^5_{2,11} - A^5_{1,21} = 0 \qquad (33)$$

Eq.(33) can be used to solve problems in plane statics under the existence of defects (dislocations and disclinations).

Considering dislocations, we can obtain the physical implication of the gauge potential A. From stresses of a single straight edge dislocation [8]

$$\sigma_{11} = -D\frac{a_2(3a_1^2 + a_2^2)}{r^4} = \hat{\sigma}_{11}$$
$$\sigma_{12} = \sigma_{21} = -D\frac{a_1(a_2^2 - a_1^2)}{r^4} = \hat{\sigma}_{12} \qquad (34)$$
$$\sigma_{22} = -D\frac{a_2(a_2^2 - a_1^2)}{r^4} = \hat{\sigma}_{22}$$

in which $D = \mu b/2\pi(1-\nu)$, $r = \sqrt{a_1^2 + a_2^2}$, and $\vec{b}$ is a Burgers vector along a_1.

After proper manipulations and arrangement, we can obtain the relation between gauge potential and Burgers vector (Appendix IV):

$$b = -i\frac{2\pi(1-\nu)}{\mu g}\lim_{a_1\to 0} a_1(A^4_{2,11} - A^4_{1,21})\Big|_{a_2=0} \qquad (35)$$

Under similar circumstances, we can also obtain the relation between gauge potential and stress intensity factor of a small crack as dislocation pile ups in the material (Appendix V)

$$K = -i\frac{\pi}{gb}\left(\frac{\pi}{a}\right)^{1/2}\frac{1}{\frac{\pi}{2} - \arcsin\frac{c}{a}}\lim_{a_1\to a}(a_1 - a)(A^4_{2,11} - A^4_{1,21})\Big|_{a_2=0} . \qquad (36)$$

Here $C \in (-a,a)$, where 2a is the length of a crack.

Therefore the physical meaning of the gauge potential A is specified to be a gauge potential of dislocations.

V. THE CRITERION OF A SINGULARITY MOTION OR THE CRACK PROPAGATION IN A GAUGE POTENTIAL OF DISLOCATIONS

It is well known that Eshelby [10] had firstly given a formula for the force acting on a general elastic singularity [9,10], which is called an expression of energy momentum tensor. The J integral or Rice [11] is equal to the 2-D form of it. Now, we may extend them to a generalized case, a continuum with defects.

We define

$$a'_i = a_i + C_i\,\varepsilon \qquad (i = 1,2,3) \qquad (37)$$

as a spatial translation group in which C_i is a constant vector and ε is an infinitesimal quantity. Therefore

$$g_4 = 0, \qquad g_i = c_i, \qquad F_\mu = 0, \qquad g^M_\mu = 0$$

Substituting these into Noether's theorem (20) we obtain the conservation law (on the energy release rate of the field in certain K direction) under this group,

$$\Big(L\delta^k_l - \frac{\partial L}{\partial X_{f,l}} X_{f,k} - \frac{\partial L}{\partial A^M_{\nu,l}} A^M_{\nu,k}\Big)_{,l}$$
$$= \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot X_\nu} X_{\nu,k} + \frac{\partial L}{\partial \dot A^M_\nu} A^M_{\nu,k}\Big)$$
$$+ \Big[\frac{\partial L}{\partial X_f} - \Big(\frac{\partial L}{\partial X_{f,\nu}}\Big)_{,\nu}\Big] X_{f,k} + \Big[\frac{\partial L}{\partial A^M_\mu} - \Big(\frac{\partial L}{\partial A^M_{\mu,\nu}}\Big)_{,\nu}\Big] A^M_{\mu,k} \quad . \tag{38}$$

From the physical consideration, let X and A be extremals, then

$$\frac{\delta L}{\delta X_f} = \frac{\partial L}{\partial X_f} - \Big(\frac{\partial L}{\partial X_{f,\nu}}\Big)_{,\nu} = 0, \qquad \frac{\delta L}{\delta A^M_\mu} = \frac{\partial L}{\partial A^M_\mu} - \Big(\frac{\partial L}{\partial A^M_{\mu,\nu}}\Big)_{,\nu} = 0 \tag{39}$$

(38) becomes

$$\Big(L\delta^k_l - \frac{\partial L}{\partial X_{f,l}} X_{f,k} - \frac{\partial L}{\partial A^M_{\nu,l}} A^M_{\nu,k}\Big)_{,l} = \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot X_\nu} X_{\nu,k} + \frac{\partial L}{\partial \dot A^M_\nu} A^M_{\nu,k}\Big) \tag{40}$$

Integrate it

$$\oint_\Sigma \Big(L\delta^k_l - \frac{\partial L}{\partial X_{f,l}} X_{f,k} - \frac{\partial L}{\partial A^M_{\nu,l}} A^M_{\nu,k}\Big) n_l\, ds$$
$$= \frac{d}{dt}\int \Big(\frac{\partial L}{\partial \dot X_\nu} X_{\nu,k} + \frac{\partial L}{\partial \dot A^M_\nu} A^M_{\nu,k}\Big) dV \quad . \tag{41}$$

The right-hand side of the equality is the energy release rate or the force acting on the singularity (dislocation, crack, etc.). Here we mainly aim at the crack propagation force, it can be expressed by the left-hand side of the equality. Take a closed surface surrounding the tip as Fig.1. It consists of the free surface of the crack tip π_1, and a smooth surface π. The total driving force acting on the crack tip is

$$\oint_\Sigma \Big(L\delta^K_l - \frac{\partial L}{\partial X_{f,l}} X_{f,k} - \frac{\partial L}{\partial A^M_{\nu,l}} A^M_{\nu,k}\Big) n_l\, ds \quad . \tag{42}$$

Because the surface of the crack is free, the total driving force acting on the crack can be expressed by an integral as

$$F_k = \int_\pi \Big(L\delta^k_l - \frac{\partial L}{\partial X_{f,l}} X_{f,k} - \frac{\partial L}{\partial A^M_{\nu,l}} A^M_{\nu,k}\Big) n_l\, ds \quad . \tag{43}$$

Therefore we obtain a dynamical criterion of crack propagation in a generalized continuum with defects. Let the critical driving or energy release rate of a crack in the existence of dislocations and disclinations be F_{kc}. As

$$F_K > F_{Kc} \tag{44}$$

the crack starts to propagate. F_{kc} is a material constant. F_k can also be written as

$$F_K = \int_\pi \Big\{\Big[L_0(\hat C_{\mu\nu}) - \frac{1}{4}\big(A^M_{[\nu,\mu]} + g f^M_{N\Gamma} A^N_\mu A^\Gamma_\nu\big)\cdot\big(A^M_{[\nu,\mu]} + g f^M_{HI} A^H_\mu A^I_\nu\big)\Big]\delta^k_i - \frac{\partial L_0}{\partial \hat C_{\alpha\beta}} X_{j,k}\big(X_{j,(\alpha}\delta^i_{\beta)} - i g T^M_{j\lambda} X_\lambda A^M_{(\alpha}\delta^i_{\beta)}\big) + \big(A^M_{[\nu,i]} + \frac{1}{2} g f^M_{N\Gamma} A^N_{[i} A^\Gamma_{\nu]}\big) A^M_{\nu,k}\Big\} n_i\, ds, \tag{45}$$

where

$$A^M_{[\nu,\mu]} = A^M_{\nu,\mu} - A^M_{\mu,\nu}, \qquad X_{j,(\alpha}\delta^i_{\beta)} = X_{j,\alpha}\delta^i_\beta + X_{j,\beta}\delta^i_\alpha,$$

and other similar symbols have the same meaning.

Because no limitation is given to the particular form of $L_0(\hat C_{\mu\nu})$, the above conclusion has a generalized significance.

If we start from the homogeneous linear elastic medium, and take L_0 as (29), we obtain F_k as follows:

$$F_k = \int_\pi \Big\{\Big[\frac{1}{2}\lambda(\hat e_{ij}\delta^{ij})^2 + \mu \hat e_{ij}\delta^{ih}\delta^{jl}\hat e_{hl} - \frac{1}{2}\rho_0 \hat C_{44} - \frac{1}{4}\big(A^M_{[\nu,\mu]} + g f^M_{N\Gamma} A^N_\mu A^\Gamma_\nu\big)\big(A^M_{[\nu,\mu]} + g f^M_{HI} A^H_\mu A^I_\nu\big)\Big]\delta^k_i - \big(X_{j,(\alpha}\delta^i_{\beta)} - i g T^M_{j\lambda} X_\lambda A^M_{(\alpha}\delta^i_{\beta)}\big)\Big(\frac{1}{2}\lambda \hat e_{mn}\delta^{mn}\delta^\alpha_f\delta^\beta_f + \mu \hat e_{hl}\delta^\alpha_h\delta^\beta_l - \frac{1}{2}\rho_0\delta^\alpha_4\delta^\beta_4\Big) X_{j,k} + \big(A^M_{[\nu,i]} + \frac{1}{2} g f^M_{N\Gamma} A^N_{[i} A^\Gamma_{\nu]}\big) A^M_{\nu,k}\Big\} n_i\, ds \tag{46}$$

where each field quantity is determined by its field equations and the conditions of each particular problem. Thus after proper treatment and simplification, it is hoped that F_k can be used as a criterion of the fracture of materials with defects.

In the case of defect free materials (or pure elastic case) F_k is simplified to be

$$F_K = \int_\pi \left(L_f \delta_i^k - \frac{\partial L_f}{\partial X_{j,i}} X_{j,k} \right) n_i \, ds \quad . \tag{47}$$

It is easy to prove that F_k is a conservational integral. Rewrite it as

$$F_K = \int_\pi \left(L_f \delta_i^k - \hat{\sigma}_{ij} u_{j,k} \right) n_i \, ds \quad . \tag{48}$$

Compared with Eshelby's energy-momentum tensor [9,10] for the motion of elastic singularity

$$\begin{aligned} T_{jl} &= P_{jl} - \tfrac{1}{2} \rho \dot{u}^2 \delta_{jl} \\ &= W \delta_{jl} - \sigma_{ij} u_{i,l} - \tfrac{1}{2} \rho \dot{u}^2 \delta_{jl} \\ &= L \delta_{jl} - \sigma_{ij} u_{i,l} \, , \end{aligned} \tag{49}$$

F_k is an integral over the curved surface of T_{jl}, it expresses the force acting on the surface Σ in the direction of a.

Let $a_1 = x$, $a_2 = y$, $a_3 = z$, project F_k onto the x-y plane, we have

$$F_K = \int_\Gamma \left(W dy - T_i u_{i,x} ds \right) \tag{50}$$

$T_i = \sigma_{ik} n_k$ is the deriving force acting on Γ. It is the same as Rice's J integral [11], the extension force acting on a crack tip. Therefore, the F-integral given in (43) is the generalization of the elastic (including non-linear elastic) J-integral in the existence of dislocations and disclinations, and the criterion of fracture given in this paper is the generalization of the criterion of fracture of defect free materials.

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APPENDIX I

PROOF OF NOETHER'S THEOREM

Under the infinitesimal transformation group given in (17) the total variations of field quantities are

$$\bar{\delta} X_\mu = \delta X_\mu - X_{\mu,\nu} \delta a^\nu, \qquad \bar{\delta} A^M_\mu = \delta A^M_\mu - A^M_{\mu,\nu} \delta a^\nu,$$

so that

$$\begin{aligned} L' = L &+ \frac{\partial L}{\partial a_\mu} \delta a_\mu + \frac{\partial L}{\partial X_\mu} \bar{\delta} X_\mu - \frac{\partial L}{\partial A^M_\mu} \bar{\delta} A^M_\mu \\ &+ \frac{\partial L}{\partial X_{\mu,\nu}} \bar{\delta} X_{\mu,\nu} + \frac{\partial L}{\partial A^M_{\mu,\nu}} \bar{\delta} A^M_{\mu,\nu} \end{aligned}$$

Suppose that δa_μ, δA^μ_μ, $\delta X_{\mu,\nu}$ and $\delta A^\mu_{\mu,\nu}$ are infinitesimal. In the first order approximation there is

$$\begin{aligned} L'(1 + \partial_\mu \delta a^\mu) - L = {} & \mathrm{div}(L \delta a) + \frac{\partial L}{\partial X_\mu} \bar{\delta} X_\mu + \frac{\partial L}{\partial A^M_\mu} \bar{\delta} A^M_\mu + \frac{\partial L}{\partial X_{\mu,\nu}} \delta X_{\mu,\nu} \\ & - \frac{\partial L}{\partial X_{\mu,\nu}} X_{\mu,\nu\lambda} \delta a^\lambda + \frac{\partial L}{\partial A^M_{\mu,\nu}} \delta A^M_{\mu,\nu} - \frac{\partial L}{\partial A^M_{\mu,\nu}} A^M_{\mu,\nu\lambda} \delta a^\lambda . \end{aligned}$$

Consider that

$$\begin{aligned} \delta X_{\mu,\nu} &= - X_{\mu,\lambda} (\delta a^\lambda)_{,\nu} + (\delta X_\mu)_{,\nu} \\ \delta A^M_{\mu,\nu} &= - A^M_{\mu,\lambda} (\delta a^\lambda)_{,\nu} + (\delta A^M_\mu)_{,\nu} \end{aligned}$$

and substitute it into the last formula, we have

$$\begin{aligned} L'(1 + \partial_\mu \delta a^\mu) - L = {} & \mathrm{div}(L \delta a) + \frac{\partial L}{\partial X_\mu} \bar{\delta} X_\mu + \frac{\partial L}{\partial A^M_\mu} \bar{\delta} A^M_\mu - \frac{\partial L}{\partial X_{\mu,\nu}} X_{\mu,\nu\lambda} \delta a^\lambda \\ & - \frac{\partial L}{\partial A^M_{\mu,\nu}} A^M_{\mu,\nu\lambda} \delta a^\lambda - \frac{\partial L}{\partial X_{\mu,\nu}} X_{\mu,\lambda} (\delta a^\lambda)_{,\nu} + \frac{\partial L}{\partial X_{\mu,\nu}} (\delta X_\mu)_{,\nu} \\ & - \frac{\partial L}{\partial A^M_{\mu,\nu}} A^M_{\mu,\lambda} (\delta a^\lambda)_{,\nu} + \frac{\partial L}{\partial A^M_{\mu,\nu}} (\delta A^M_\mu)_{,\nu} \\ = {} & \mathrm{div}(L \delta a) + \frac{\partial L}{\partial X_\mu} \bar{\delta} X_\mu + \frac{\partial L}{\partial A^M_\mu} \bar{\delta} A^M_\mu - \frac{\partial L}{\partial X_{\mu,\nu}} X_{\mu,\nu\lambda} \delta a^\lambda - \frac{\partial L}{\partial A^M_{\mu,\nu}} A^M_{\mu,\nu\lambda} \delta a^\lambda \\ & - \left(\frac{\partial L}{\partial X_{\mu,\nu}} X_{\mu,\lambda} \delta a^\lambda \right)_{,\nu} + \left(\frac{\partial L}{\partial X_{\mu,\nu}} \right)_{,\nu} X_{\mu,\lambda} \delta a^\lambda + \end{aligned}$$

$$+\frac{\partial L}{\partial X_{\mu,\nu}} X_{\mu,\nu\lambda}\delta a^\lambda + \left(\frac{\partial L}{\partial X_{\mu,\nu}}\delta X_\mu\right)_{,\nu} - \left(\frac{\partial L}{\partial X_{\mu,\nu}}\right)_{,\nu}\delta X_\mu - \left(\frac{\partial L}{\partial A^M_{\mu,\nu}} A^M_{\mu,\lambda}\delta a^\lambda\right)_{,\nu}$$

$$+\left(\frac{\partial L}{\partial A^M_{\mu,\nu}}\right)_{,\nu} A^M_{\mu,\lambda}\delta a^\lambda + \frac{\partial L}{\partial A^M_{\mu,\nu}} A^M_{\mu,\nu\lambda}\delta a^\lambda + \left(\frac{\partial L}{\partial A^M_{\mu,\nu}}\delta A^M_\mu\right)_{,\nu} - \left(\frac{\partial L}{\partial A^M_{\mu,\nu}}\right)_{,\nu}\delta A^M_\mu$$

$$= \mathrm{div}(L\delta a) + \frac{\partial L}{\partial X_\mu}\bar\delta X_\mu + \frac{\partial L}{\partial A^M_\mu}\bar\delta A^M_\mu - \left(\frac{\partial L}{\partial X_{\mu,\nu}}\right)_{,\nu}\bar\delta X_\mu - \left(\frac{\partial L}{\partial A^M_{\mu,\nu}}\right)_{,\nu}\bar\delta A^M_\mu$$

$$+\left(\frac{\partial L}{\partial X_{\mu,\nu}}\bar\delta X_\mu + \frac{\partial L}{\partial A^M_{\mu,\nu}}\bar\delta A^M_\mu\right)_{,\nu}$$

$$= \frac{\delta L}{\delta X_\mu}\bar\delta X_\mu + \frac{\delta L}{\delta A^M_\mu}\bar\delta A^M_\mu - J^M_{,\mu}$$

By virtue of the infinitesimal invariance of S, namely

$$S' = S$$

or

$$\int_{V'} L' dV' = \int_V L\, dV ,$$

we have

$$\int_V [L'(1+\partial_\mu \delta a^\mu) - L]\, dV = 0$$

or

$$\int_V \left(\frac{\delta L}{\delta X_\mu}\bar\delta X_\mu + \frac{\delta L}{\delta A^M_\mu}\bar\delta A^M_\mu - J^M_{,\mu}\right) dV = 0$$

Considering the arbitrariness of V, we finally have

$$\frac{\delta L}{\delta X_\mu}\bar\delta X_\mu + \frac{\delta L}{\delta A^M_\mu}\bar\delta A^M_\mu = J^M_{,\mu}$$

APPENDIX II

DEDUCTION OF EQUATION (32)

We can rewrite (28) as

$$\frac{\partial L_0}{\partial X_i} - \left(\frac{\partial L_0}{\partial X_{i,\nu}}\right)_{,\nu} = 0$$

$$\frac{\partial L_0}{\partial A^M_\mu} + \frac{\partial L_F}{\partial A^M_\mu} - \left(\frac{\partial L_F}{\partial A^M_{\mu,\nu}}\right)_{,\nu} = 0 ,$$

and deduce the following under the first order approximation.

By virtue of (29) we have

$$\frac{\partial L_0}{\partial X_i} = \frac{\partial L_0}{\partial \hat e_{kj}}\frac{\partial \hat e_{kj}}{\partial X_i} + \frac{\partial L_0}{\partial \hat C_{44}}\frac{\partial \hat C_{44}}{\partial X_i}$$

and considering that

$$\hat C_{\mu\nu} = X_{k,\mu}X_{k,\nu} - ig\, T^M_{k\beta} A^M_{(\mu} X^k_{,\nu)} X^\beta - g^2 A^M_\mu A^N_\nu T^M_{\alpha\beta} T^N_{\alpha\gamma} X^\beta X^\gamma$$

we have

$$\frac{\partial \hat C_{44}}{\partial X_i} \approx -2ig\, T^M_{k\beta} A^M_4 \delta^\beta_i X^k_{,4}$$

$$= -2ig\, T^M_{ki} A^M_4 u^k_{,4}$$

$$\approx 0 ,$$

$$\frac{\partial \hat e_{kj}}{\partial X_i} = \frac{1}{2}\frac{\partial \hat C_{kj}}{\partial X_i} \cong -\frac{1}{2} ig\, T^M_{\ell\beta} A^M_{(k} X^\ell_{,j)} \delta^\beta_i$$

$$\approx 0$$

therefore,

$$\frac{\partial L_0}{\partial X_i} = \frac{\partial L_0}{\partial \hat e_{kj}}\frac{\partial \hat e_{kj}}{\partial X_i} = 0$$

In the same way, we have

$$\frac{\partial L_0}{\partial X_{i,\nu}} = \frac{\partial L_0}{\partial \hat e_{kj}}\frac{\partial \hat e_{kj}}{\partial X_{i,\nu}} + \frac{\partial L_0}{\partial \hat C_{44}}\frac{\partial \hat C_{kk}}{\partial X_{i,\nu}} ,$$

$$\frac{\partial \hat e_{kj}}{\partial X_{i,\nu}} = \frac{1}{2}\left[X_{\ell,j}\delta^\ell_i\delta^k_\nu + X^\ell_{,k}\delta^\ell_i\delta^j_\nu - ig\, T^M_{\ell\beta}(A^M_k\delta^\ell_i\delta^j_\nu + A^M_j\delta^\ell_i\delta^k_\nu)X^\beta\right]$$

$$\frac{\partial L_0}{\partial \hat e_{kj}}\frac{\partial \hat e_{kj}}{\partial X_{i,\nu}} = \hat\sigma_{kj}\frac{\partial \hat e_{kj}}{\partial X_{i,\nu}} \approx \hat\sigma_{kj}\delta^j_i\delta^k_\nu ,$$

$$\frac{\partial \hat{C}_{44}}{\partial X_{i,\nu}} = 2X_{k,4}\delta_k^i\delta_4^\nu - 2ig T_{\ell\beta}^M A_4^M \delta_i^\ell \delta_\nu^4 X^\beta$$

$$\approx 2u_{k,4}\delta_k^i\delta_4^\nu - 2ig(T_{i4}^M A_4^M \delta_\nu^4 + T_{ik}^M A_4^M \delta_\nu^4 a^k),$$

$$\frac{\partial L_0}{\partial \hat{C}_{44}}\frac{\partial \hat{C}_{44}}{\partial X_{i,\nu}} = -\rho_0[u_{k,4}\delta_i^k\delta_4^\nu - ig(T_{i4}^M A_4^M \delta_\nu^4 + T_{ik}^M A_4^M \delta_\nu^4 a^k)],$$

$$\frac{\partial L_0}{\partial X_{i,\nu}} = \hat{\sigma}_{kj}\delta_i^j\delta_\nu^k - \rho_0[u_{k,4}\delta_k^i\delta_4^\nu - ig(T_{i4}^M A_4^M\delta_\nu^4 + T_{ik}^M A_4^M \delta_\nu^4 a^k)]$$

$$\left(\frac{\partial L_0}{\partial X_{i,\nu}}\right)_{,\nu} = \hat{\sigma}_{kj,k}\delta_i^j - \rho_0[u_{k,44}\delta_k^i - ig(T_{i4}^M A_{4,4}^M + T_{ik}^M A_{4,4}^M a^k)],$$

then the first one of (32) is

$$\hat{\sigma}_{ij,j} - \rho_0[u_{i,44} - ig(T_{i4}^M A_{4,4}^M + T_{ik}^M A_{4,4}^M a^k)] = 0$$

For the second one of (32) there are

$$\frac{\partial L_0}{\partial A_\mu^M} = \frac{\partial L_0}{\partial \hat{e}_{ij}}\frac{\partial \hat{e}_{ij}}{\partial A_\mu^M} + \frac{\partial L_0}{\partial \hat{C}_{44}}\frac{\partial \hat{C}_{KK}}{\partial A_\mu^M},$$

$$\frac{\partial L_0}{\partial \hat{e}_{ij}}\frac{\partial \hat{e}_{ij}}{\partial A_\mu^M} = \frac{1}{2}\hat{\sigma}_{ij}\frac{\partial \hat{C}_{ij}}{\partial A_\mu^M}$$

$$\approx \frac{1}{2}\hat{\sigma}_{ij}[-igT_{k\beta}^M(\delta_\mu^i X_{,j}^k + \delta_\mu^j X_{,i}^k)X^\beta]$$

$$\approx -\frac{1}{2}ig\hat{\sigma}_{ij}(\delta_\mu^i\delta_j^k + \delta_\mu^j\delta_i^k)(T_{k4}^M + T_{kh}^M a^h)$$

$$= -ig\hat{\sigma}_{ij}(T_{i4}^M + T_{ih}^M a^h)\delta_\mu^j,$$

$$\frac{\partial \hat{C}_{KK}}{\partial A_\mu^M} = -2igT_{K\beta}^M\delta_4^\mu X_{,4}^K X^\beta - 2g^2\delta_4^\mu A_4^N T_{\alpha\beta}^M T_{\alpha\gamma}^N X^\beta X^\gamma$$

$$\approx -2igT_{k4}^M\delta_4^\mu u_{,4}^k - 2igT_{kh}^M\delta_4^\mu u_{,4}^k a^h$$

$$-4g^2\delta_4^\mu A_4^N(T_{\alpha 4}^M T_{\alpha 4}^N + T_{\alpha 4}^M T_{\alpha k}^N a^k + T_{\alpha k}^M T_{\alpha 4}^N a^k + T_{\alpha k}^M T_{\alpha h}^N a^k a^h),$$

$$\frac{\partial L_0}{\partial A_\mu^M} = -ig\hat{\sigma}_{ij}(T_{i4}^M + T_{ih}^M a^h)\delta_\mu^j + \rho_0[igT_{k4}^M u_{,4}^k\delta_4^\mu + igT_{kh}^M\delta_4^\mu u_{,4}^k a^h$$

$$+2g^2\delta_4^\mu A_4^N(T_{\alpha 4}^M T_{\alpha 4}^N + T_{\alpha 4}^M T_{\alpha k}^N a^k + T_{\alpha k}^M T_{\alpha 4}^N a^k + T_{\alpha k}^M T_{\alpha h}^N a^k a^h)]$$

$$\frac{\partial L_F}{\partial A_\mu^M} = -\frac{1}{4}\frac{\partial}{\partial A_\mu^M}(F_{\alpha\nu}^N F^{\alpha\nu N})$$

$$= -\frac{1}{2}F_{\alpha\nu}^N\frac{\partial F^{\alpha\nu N}}{\partial A_\mu^M}$$

$$= -\frac{1}{2}(A_{\nu,\alpha}^N - A_{\alpha,\nu}^N + gf_{M\Gamma}^N A_\alpha^M A_\nu^\Gamma)gf_{BE}^N(\delta_M^B\delta_\mu^\alpha A_\nu^E + \delta_\mu^E\delta_\mu^\nu A_\alpha^B)$$

$$\approx 0$$

$$\frac{\partial L_F}{\partial A_{\mu,\nu}^M} = -\frac{1}{4}\frac{\partial}{\partial A_{\mu,\nu}^M}(F_{\alpha\lambda}^N F^{\alpha\lambda N})$$

$$= -\frac{1}{2}F_{\alpha\lambda}^N(\delta_\nu^\alpha\delta_\mu^\lambda\delta_M^N - \delta_\mu^\alpha\delta_\nu^\lambda\delta_M^N)$$

$$\cong -\frac{1}{2}(A_{\lambda,\alpha}^N - A_{\alpha,\lambda}^N)(\delta_\nu^\alpha\delta_\mu^\lambda\delta_M^N - \delta_\mu^\alpha\delta_\mu^\lambda\delta_M^N)$$

$$= A_{\nu,\mu}^M - A_{\mu,\nu}^M$$

$$\left(\frac{\partial L_F}{\partial A_{\mu,\nu}^M}\right)_{,\nu} = A_{\nu,\mu\nu}^M - A_{\mu,\nu\nu}^M$$

Then the second one of (32) is

$$-ig\hat{\sigma}_{ij}(T_{i4}^M + T_{ih}^M a^h)\delta_\mu^j + \rho_0[igT_{k4}^M u_{,4}^k + igT_{kh}^M u_{,4}^k a^h$$

$$+2g^2 A_4^N(T_{\alpha 4}^M T_{\alpha 4}^N + T_{\alpha 4}^M T_{\alpha k}^N a^k + T_{\alpha k}^M T_{\alpha 4}^N a^k a^h)]\delta_\mu^4$$

$$+A_{\mu,\nu\nu}^M - A_{\nu,\mu\nu}^M = 0$$

APPENDIX III

DEDUCTION OF EQUATION (33)

In plane statics the first one of (32) is

$$\hat{\sigma}_{ij,j} = 0 \qquad i, j = 1, 2,$$

the second one is

$$-ig\,\hat{\sigma}_{ij}(T^M_{i4} + T^M_{ih}a^h) + A^M_{j,hh} - A^M_{h,jh} = 0, \qquad i, j, h = 1, 2,$$

or written as follows:

$$\hat{\sigma}_{11,1} + \hat{\sigma}_{12,2} = 0$$

$$\hat{\sigma}_{21,1} + \hat{\sigma}_{22,2} = 0$$

$$-ig\,\hat{\sigma}_{11}(T^M_{14} + T^M_{1h}a^h) - ig\,\hat{\sigma}_{21}(T^M_{24} + T^M_{2h}a^h) + A^M_{1,hh} - A^M_{h,1h} = 0$$

$$-ig\,\hat{\sigma}_{12}(T^M_{14} + T^M_{1h}a^h) - ig\,\hat{\sigma}_{22}(T^M_{24} + T^M_{2h}a^h) + A^M_{2,hh} - A^M_{h,2h} = 0$$

where h = 1,2; M = 1,2,...,6.

Considering the values of $T^M_{\alpha\beta}$ we can write the last equation as

$$A^N_{1,hh} - A^N_{h,1h} = 0$$

$$A^N_{2,hh} - A^N_{h,2h} = 0$$

$$ig(\hat{\sigma}_{11}a^2 - \hat{\sigma}_{12}a^1) + A^3_{1,hh} - A^3_{h,1h} = 0$$

$$ig(\hat{\sigma}_{12}a^2 - \hat{\sigma}_{22}a^1) + A^3_{2,hh} - A^3_{h,2h} = 0$$

$$-ig\,\hat{\sigma}_{11} + A^4_{1,hh} - A^4_{h,1h} = 0$$

$$-ig\,\hat{\sigma}_{12} + A^4_{2,hh} - A^4_{h,2h} = 0$$

$$-ig\,\hat{\sigma}_{21} + A^5_{1,hh} - A^5_{h,1h} = 0$$

$$-ig\,\hat{\sigma}_{22} + A^5_{2,hh} - A^5_{h,2h} = 0$$

where h = 1,2; N = 1,2,6, or

$$A^N_{1,22} - A^N_{2,12} = 0$$

$$A^N_{2,11} - A^N_{1,21} = 0$$

$$ig(\hat{\sigma}_{11}a^2 - \hat{\sigma}_{12}a^1) + A^3_{1,22} - A^3_{2,12} = 0$$

$$ig(\hat{\sigma}_{12}a^2 - \hat{\sigma}_{22}a^1) + A^3_{2,11} - A^3_{1,21} = 0$$

$$-ig\,\hat{\sigma}_{11} + A^4_{1,22} - A^4_{2,12} = 0$$

$$-ig\,\hat{\sigma}_{12} + A^4_{2,11} - A^4_{1,21} = 0$$

$$-ig\,\hat{\sigma}_{21} + A^5_{1,22} - A^5_{2,12} = 0$$

$$-ig\,\hat{\sigma}_{22} + A^5_{2,11} - A^5_{1,21} = 0$$

N = 1,2,6.

APPENDIX IV

PROOF OF (35)

From the sixth one of (33) that

$$-ig\hat{\sigma}_{12} + A^4_{2,11} - A^4_{1,21} = 0$$

and (34) we have

$$-ig(-D)\frac{a_1(a_2^2-a_1^2)}{r^4} + A^4_{2,11} - A^4_{1,21} = 0$$

Let $a_2 = 0$, then

$$-igD\frac{1}{a_1} + (A^4_{2,11} - A^4_{1,21})\Big|_{a_2=0} = 0$$

and

$$-igD + \lim_{a_1\to 0} a_1(A^4_{2,11} - A^4_{1,21})\Big|_{a_2=0} = 0$$

therefore we have

$$b = -i\frac{2\pi(1-\nu)}{\mu g}\lim_{a_1\to 0} a_1(A^4_{2,11} - A^4_{1,21})\Big|_{a_2=0}$$

APPENDIX V

PROOF OF (36)

The density of crack dislocations at the tip of a crack is

$$\mathcal{D} = \frac{K}{\pi A(2\pi)^{1/2}}(-s)^{-1/2}, \qquad s = a_1 - a \ll a$$

where $A = \mu b/2\pi(1-\nu)$. $\vec{b}$ is a Burgers vector, k, a stress intensity factor [9], [10], [12].

Take a linear distribution of crack dislocation density to represent a crack as

$$\mathcal{D} = \pm(n/\pi)(a^2 - a_1^2)^{-1/2}$$

or

$$\mathcal{D} = \frac{1}{\pi A}\left(\frac{a}{\pi}\right)^{1/2}K(a^2 - a_1^2)^{-1/2}$$

Therefore the total stress is $\sigma_{ij} = \sigma^A_{ij} + \sigma^D_{ij}$, where the σ^A_{ij} represent the elastic field,

$$\sigma^D_{ij} = b\int_{-a}^{a}\mathcal{D}\,\tilde{\sigma}_{ij}(a_1 - a_1', a_2, a_3)\,da_1',$$

and the $\tilde{\sigma}_{ij}$ is the stress of a single dislocation located at $(a_1,0,0)$, which Burgers vector parallels to a_1.

By virtue of the sixth one of (33) we have

$$-ig\sigma^A_{12} - igb\int_{-a}^{a}\frac{1}{\pi A}\left(\frac{a}{\pi}\right)^{1/2}K(a^2 - a_1'^2)^{-1/2}(-A)\frac{(a_1-a_1')[a_2^2-(a_1-a_1')^2]}{[(a_1-a_1')^2+a_2^2]^2}\,da_1'$$

$$+ A^4_{2,11} - A^4_{1,21} = 0$$

Let $a_2 = 0$, we have

$$-ig\sigma^A_{12}\Big|_{a_2=0} - igb\frac{1}{\pi}\left(\frac{a}{\pi}\right)^{1/2}K\int_{-a}^{a}(a^2 - a_1'^2)^{-1/2}\frac{1}{a_1 - a_1'}\,da_1'$$

$$+ (A^4_{2,11} - A^4_{1,21})\Big|_{a_2=0} = 0$$

As a result of the mean-value theorem, there is

$$\int_{-a}^{a}(a^2 - a_1'^2)^{-1/2}\frac{1}{a_1 - a_1'}\,da_1' = \frac{1}{a_1 - a}\int_{c}^{a}(a^2 - a_1'^2)^{-1/2}\,da_1'$$

$$= \frac{1}{a_1 - a} \arcsin \frac{a_1'}{a} \Big|_c^a$$

$$= \frac{1}{a_1 - a} \left(\frac{\pi}{2} - \arcsin \frac{c}{a} \right)$$

where C(-a,a). Multiply the last equation by (a,-a). We have

$$\lim_{a_1 \to a} (a_1 - a)\, \sigma_{12}^A \Big|_{a_2 = 0} = 0$$

therefore

$$-igb \frac{1}{\pi} \left(\frac{a}{\pi} \right)^{1/2} K \left(\frac{\pi}{2} - \arcsin \frac{c}{a} \right) +$$

$$+ \lim_{a_1 \to a} (a_1 - a) \left(A^4_{2,11} - A^4_{1,21} \right) \Big|_{a_2 = 0} = 0$$

and finally we have

$$K = -i \frac{\pi}{gb} \left(\frac{\pi}{a} \right)^{1/2} \frac{1}{\frac{\pi}{2} - \arcsin \frac{c}{a}} \lim_{a_1 \to a} (a_1 - a) \left(A^4_{2,11} - A^4_{1,21} \right) \Big|_{a_2 = 0}$$

REFERENCES

[1] O. Far (Ou Fa), Journal of Harbin Institute of Technology, (3-4), 1978, 196 (in Chinese).

[2] A.A. Golebiewska-Lasota and D.G.B. Edelen, Int. J. Engng. Sci., Vol.17, 1979, pp.335-339.

[3] A.A. Golebiewska-Lasota, Int. J. Engng. Sci. Vol.17, 1979, pp.329-333.

[4] Ou Fa, Acta Physica Sinica, Vol.30, No.7, July 1981, pp.968-971 (in Chinese).

[5] Aida Kadic and Dominic G.B. Edelen, A Gauge Theory of Dislocations and Disclinations (Springer Verlag 1983),pp.226-230.

[6] Ernest S. Abers and Benjamin W. Lee, Phys. Rep. 9c, No.1, 1973.

[7] N.P. Konopleva and V.N. Popov, Gauge Fields (Harwood Academic Publishers, 1981), pp.53-65.

[8] J. Friedel, Dislocations (Pergamon Press, 1964).

[9] B.A. Bilby and J.D. Eshelby, Fracture, Vol.I, Ed. H. Liebowitze (Academic Press, New York 1968), pp.126-140.

[10] J.D. Eshelby, Solid State Physics, Vol.3, 1956, pp.79-144.

[11] J.R. Rice, Appl. Mech. 35, No.2, 1968, pp.379-386.

[12] C.W. Lung, Scientia Sinica, Vol. XXIII, No. 4 (1980) 441 or ICTP, Trieste, preprint IC/79/72 (1979).

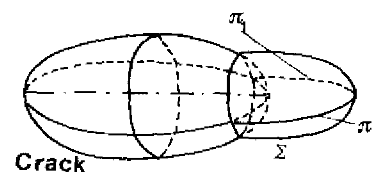

Fig.1

The closed surface surrounding the crack tip.