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ASYMPTOTICS OF THE NODAL LINES OF SOLUTIONS OF 2-DIMENSIONAL SCHRÖDINGER EQUATIONS

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1. Introduction and Previous Results

In this paper we sharpen results on nodal properties of L^2-solutions of 2-dimensional Schrödinger equations, recently obtained in collaboration with T. Hoffmann-Ostenhof and J. Swetina in [1]. We shall consider real valued $W^{2,2}$-solutions $\phi(x)$ of the Schrödinger equation

$$\begin{aligned} &(-\Delta + V - E)\phi = 0 \quad \text{for} \quad x \in \Omega_R , \\ &\Omega_R = \{x \in \mathbb{R}^2 \,|\, r \equiv |x| > R\},\ R > 0 , \end{aligned} \tag{1.1}$$

(where the Sobolev space $W^{2,2}$ is defined as in [2]). In the following it will always be assumed that

$$E < 0 \tag{1.2}$$

and that

$$\left.\begin{aligned} &V(x) \text{ is real valued and continuous in } \bar{\Omega}_R \\ &\text{and } \lim_{r\to\infty} V(x) = 0 . \end{aligned}\right\} \tag{1.3}$$

Due to these assumptions we can choose R so that

$$\inf_{x\in\Omega_R} \left(V(x) - \frac{1}{4r^2} - E\right) > 0 \tag{1.4}$$

which implies that the Dirichlet problem (1.1) with continuous boundary data is uniquely solvable (see [1]). Note also that $\phi \in C^1(\Omega_R)$ (see e.g. [2]). In the following we shall use polar coordinates $x_1 = r\cos\omega$, $x_2 = r\sin\omega$ with $r \geq R$ and $\omega \in [-\pi,\pi]$, and denote $\phi = \phi(r,\omega)$.

Under additional suitable assumptions on V the generally unbounded nodal set of ϕ, i.e. $\{x \in \Omega_R,\ \phi(x) = 0\}$ will be investigated for $r \to \infty$. Particularly it will be shown (Theorem 2.3) that for large r the nodal set of ϕ consists of non-intersecting nodal lines which look roughly speaking asymptotically either like straight lines or like branches of parabolas.

By the following example (compare also [1]) it is illustrated that already for spherically symmetric V it is in general far from trivial to determine the asymptotic behaviour of the zeros of ϕ: Let a_ℓ, $b_\ell \in \mathbb{R}$ for $0 \leq \ell \leq m$ with $m \in \mathbb{N} \cup \{0\}$ and denote by $W_{o,\ell}(r)$ for $0 \leq \ell \leq m$ the Whittaker functions (see [3]). Define

$$\phi(r,\omega) = r^{-1/2} \sum_{\ell=0}^{m} (a_\ell \sin \ell\omega + b_\ell \cos \ell\omega) W_{o,\ell}(r) .$$

Then it is easily seen that $(-\Delta + 1/4)\phi = 0$ in Ω_R and since for all ℓ $W_{o,\ell}(r) = e^{-r/2}(1 + O(r^{-1}))$ (see [3]),

$$A(\omega) \equiv \lim_{r\to\infty} \frac{\phi(r,\omega)}{r^{-1/2}\, W_{o,o}(r)} = \sum_{\ell=0}^{m} (a_\ell \sin \ell\omega + b_\ell \cos \ell\omega)$$

whereby $(-\Delta + 1/4) r^{-1/2}\, W_{o,o}(r) = 0$ in Ω_R. Obviously given any $M \geq 1$, then m, a_ℓ, b_ℓ can be chosen suitably so that A vanishes e.g. in $\omega = 0$ of order M. In Theorem 2.3 it is demonstrated how the order M of the zero of A is connected with the asymptotics of the nodes of ϕ in a cone $|\omega| < \varepsilon$ for ε small enough.

In the following we suppose (as in [1]) that

$$\left.\begin{array}{l} V(x) = V_1(r) + V_2(x) \\ \text{where } V_1 \text{ and } V_2 \text{ obey (1.3) and (1.4) separately.} \end{array}\right\} \quad (1.5)$$

The above assumptions imply (see [1] and [4]) that there exists $v \in L^2(\Omega_R)$, $v > 0$ for $r \geq R$ such that

$$(-\Delta + V_1 - E)v = 0 \quad \text{for} \quad r > R . \qquad (1.6)$$

Now define

$$u(r,\omega) = \phi(r,\omega)/v(r) \qquad (1.7)$$

and note that u and ϕ have the same zeros. The derivation of our results on the nodal lines of ϕ will be based on results on the asymptotic behaviour of u given in [1] (see also [5]). We shall summarize these relevant results in Theorem 1.1. For this and later on we need

Def. 1.1. (i) Let $I', I \subset \mathbb{R}$ denote finite open intervals and let $f: (R,\infty)\times I \to \mathbb{R}$ denoted by $f = f(r,s)$. f is called real analytic in s uniformly with respect to r, if $\forall \bar{R} > R$ f is real analytic in the variable s $\forall r \geq \bar{R}$ and if $\forall I' \subset I$ there exist $\delta, C > 0$ (not depending on r) such that $|\partial^k f(r,s)/\partial s^k| \leq C\, k!/\delta^k$ $\forall s \in I'$, $\forall r \geq \bar{R}$ and for $k \in \mathbb{N} \cup \{0\}$.

(ii) Let $g: \Omega_R \to \mathbb{R}$ and define $\forall \omega \in (-\pi,\pi)$ $\phi_{\bar{\omega}}^{-1}(\omega) = (\cos(\omega-\bar{\omega}), \sin(\omega-\bar{\omega})) \in S^1$ $\forall \bar{\omega} \in [-\pi,\pi]$. We say g is real analytic in ω uniformly with respect to r, if for all $\bar{\omega}$ $g(r\phi_{\bar{\omega}}^{-1}(\omega))$ is real analytic in ω uniformly with respect to r (as defined in (i)) with C, δ not depending on $\bar{\omega}$. In accordance with the foregoing we denote $g(r\phi_0^{-1}(\omega)) = g(r,\omega)$.

According to [1] (resp. [4]) we have

Theorem 1.1. Let $V = V_1 + V_2$ be given according to (1.3), (1.4) and (1.5). Assume that V_1 is continuously differentiable with

$$\left|\frac{dV_1}{dr}\right| \leq c\, r^{-1-\varepsilon} \quad \text{for} \quad r > R \tag{1.8}$$

for some $c, \varepsilon > 0$ and that

$$\left.\begin{array}{l}\text{for some } \alpha > \frac{1}{2},\ r^{1+\alpha}\, V_2 \quad \text{is real analytic in } \omega \\ \text{uniformly with respect to } r.\end{array}\right\} \tag{1.9}$$

Let ϕ and v be given according to (1.1) and (1.6).

(i) Then u is real analytic in ω uniformly with respect to r,

$$\lim_{r\to\infty} u(r,\omega) \equiv A(\omega)$$

exists, A is real analytic in ω and for $k \in \mathbb{N} \cup \{0\}$

$$\left|\frac{\partial^k}{\partial\omega^k}(u(r,\omega) - A(\omega))\right| \leq C_k\, r^{-a}, \quad a = \min(1,\alpha)$$

in $\Omega_{\bar{R}}$ for $\bar{R} > R$ large enough, with some $C_k < \infty$ (not depending on r). (1.10)

(ii) Let $\beta \in (0,\frac{1}{2})$ and $D_\beta = \{x \in \Omega_{R_\beta} \mid |\omega| < r^{-\beta}\}$ with R_β sufficiently large. Suppose $A(0) = 0$, then for some $M \in \mathbb{N}$ and $|\omega|$ small

$$A(\omega) = \omega^M + O(\omega^{M+1})$$

and in D_β for some $\nu, \delta > 0$

$$u(r,\omega) = (2b)^{-M} r^{-M/2} H_M(b\sqrt{r}\omega)(1+O(r^{-\nu})) + O(r^{-M/2-\delta}) \qquad (1.11)$$

where $b=(|E|/4)^{1/4}$ and H_M denotes the Hermitepolynomial of order M

$$H_M(z) = \sum_{k=0}^{[M/2]} (-1)^k \frac{M!}{k!(M-2k)!} (2z)^{M-2k}, \quad z \in \mathbb{R},$$

($[M/2]$ denoting the integer part of $M/2$).

Some immediate consequences of Theorem 1.1 on the nodes of ψ have been already noted in [1]. See Remark 2.3.

<u>Corollary 1.1</u>. Choosing $\omega = z/(b\sqrt{r})$, (1.11) implies

$$u(r,\frac{z}{b\sqrt{r}})r^{M/2} \to (2b)^{-M} H_M(z) \quad \text{for} \quad r \to \infty, \forall z \in \mathbb{R} \qquad (1.12)$$

and the convergence is uniformly in any compact interval.

In the following we denote $H_M^{(k)} = \frac{d^k}{dz^k} H_M$.

$$U_M(r,z) = u(r,\frac{z}{b\sqrt{r}})r^{M/2} \quad \text{and} \quad U_M^{(k)} = \frac{\partial^k}{\partial z^k} U_M, \qquad k \in \mathbb{N}. \qquad (1.13)$$

Note that $U_M^{(k)}$ exists since $\partial^k u/\partial\omega^k$ exists for all $k \in \mathbb{N}$ due to Theorem 1.1.

Theorem 2.1 deals with the behaviour of $U_M^{(k)}$ for $r \to \infty$ for $k \in \mathbb{N}$. In Theorem 2.2 the asymptotics of $\partial u/\partial r$ is characterized. With the help of these two theorems the main result on the nodal lines of ψ, stated in Theorem 2.3 will be obtained. In sections 3, 4, and 5 the theorems given in section 2 are proven.

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2. Statement of the Results

The first result is concerned with the asymptotic properties of U_M as defined in (1.13).

Theorem 2.1. Under the assumptions of Theorem 1.1, $U_M(r,z)$ is real analytic in z uniformly with respect to r (in the sense of Def. 1.1) whereby $z \in I$, I any finite open interval.

Furthermore for $k \in \mathbb{N} \cup \{0\}$

$$\lim_{r\to\infty} \frac{\partial^k}{\partial z^k} U_M(r,z) = (2b)^{-M} \frac{d^k}{dz^k} H_M(z)$$
$$\text{with } b = (|E|/4)^{1/4}, \text{ for } z \in \mathbb{R}, \tag{2.1}$$

and the convergence is uniformly in any compact interval.

Remark 2.1. Clearly (2.1) implies that $U_M^{(k)} \to 0$ for $r \to \infty$, $\forall z \in \mathbb{R}$ for $k \geq M+1$.

The next result gives detailed information on the asymptotics of $\partial u/\partial r$.

Theorem 2.2. Under the assumptions of Theorem 1.1, $r^{1+a} \frac{\partial u}{\partial r}$ (with $a = \min(1,\alpha)$) is real analytic in ω uniformly with respect to r.

Further let $A(0) = 0$ with

$$A(\omega) = \omega^M + d\omega^{M+1} + O(\omega^{M+2}) \quad \text{for } |\omega| \text{ small} \tag{2.2}$$

for some $d \in \mathbb{R}$ and $M \in \mathbb{N}$.

If $M = 1$, then for some $\epsilon > 0$

$$r^2 \frac{\partial u}{\partial r}(r,\omega) = \frac{d}{\sqrt{|E|}} + O(r^{-\epsilon})$$
$$\text{for all } \omega \text{ with } |\sqrt{r}\,\omega| \text{ bounded for } r \to \infty . \tag{2.3}$$

If $M = 2m$, $m \geq 1$, then

$$r^{M/2+1} \frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=z/(b\sqrt{r})} = (2b)^{-M} M(M-1)H_{M-2}(z) + o(1) \qquad (2.4)$$

for $|z|$ bounded and $r \to \infty$.

If $M = 2m+1$, $m \geq 1$, then for some $\epsilon > 0$

$$r^{M/2+1} \frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=z/(b\sqrt{r})} = (2b)^{-M} M(M-1)H_{M-2}(z) +$$

$$+ (-1)^m \frac{(M+1)!}{m!} d(2b)^{-M-1} r^{-1/2}(1+O(r^{-\epsilon})) +$$

$$+ O(r^{-1/2-a}) + |z|\, o(1) \qquad (2.5)$$

for $|z|$ bounded and $r \to \infty$.

Remark 2.2. (a) An immediate consequence of Theorem 2.2 is that $r\partial U_M/\partial r \to 0$ for $r \to \infty$ for $z \in \mathbb{R}$.
(b) Clearly (2.5) implies that (2.4) holds for M odd. However, for M odd we shall need the more detailed asymptotics of (2.5) later on.

Theorem 2.1 and 2.2 together will enable us to obtain our main result:

Theorem 2.3. Suppose the assumptions of Theorem 1.1 hold. Assume $A(0) = 0$ with

$$A(\omega) = \omega^M + d\omega^{M+1} + O(\omega^{M+2}) \quad \text{for } |\omega| \text{ small} \qquad (2.6)$$

for some $d \in \mathbb{R}$ and $M \in \mathbb{N}$. Let $z_i \in \mathbb{R}$ for $1 \leq i \leq M$ denote the zeros of the Hermite polynomial H_M, i.e. $H_M(z_i) = 0$ for $1 \leq i \leq M$.

Then for $\epsilon > 0$ sufficiently small and R_ϵ large the nodal set of ψ in $D_\epsilon \equiv \{x \in \Omega_R \mid r > R_\epsilon, |\omega| < \epsilon\}$ consists of M nodal lines (corresponding to the M zeros of H_M). They admit a representation in cartesian coordinates ($(x_1,x_2) \in \mathbb{R}^2$) denoted by $x_2 = G_i(x_1)$ for $1 \leq i \leq M$. Therefore denoting $\psi = \psi(x_1,x_2)$, $\psi(x_1,G_i(x_1)) = 0$ for $1 \leq i \leq M$. For all i, G_i is continuously differentiable and the nodal lines have the following asymptotic behaviour:

For $M \geq 2$ and $z_i \neq 0$

$$G_i(x_1) = (\frac{z_i}{b} + o(1))\sqrt{x_1} \qquad \text{for large } x_1 \tag{2.7}$$

with $b = (|E|/4)^{1/4}$. Further if $z_i > 0$ (< 0), then G_i is strictly monotonically increasing (decreasing) for large x_1.

For M odd, $H_M(0) = 0$ and without loss let $z_1 = 0$, then

$$G_1(x_1) = \frac{d}{\sqrt{|E|}} + o(1) \qquad \text{for large } x_1 \tag{2.8}$$

with d given in (2.6).

Remark 2.3. As a consequence of the results summarized in Theorem 1.1 it was noted in [1] that in D_c for each r there exist $\omega_i(r)$, $1 \leq i \leq M$ with $u(r,\omega_i(r)) = 0$. In Theorem 2.3 the case $A(0) = 0$ is considered without loss of generality, since by rotation of the coordinate system corresponding results to (2.7) and (2.8) are immediately obtained if, for instance,

$$A(\omega) = (\omega - \omega_o)^M + d(\omega - \omega_o)^{M+1} + O((\omega - \omega_o)^{M+1}) \text{ for } |\omega - \omega_o| \text{ small.}$$

Note that since A is real analytic it has only a finite number of zeros. Hence the zero set of ϕ consists of non-intersecting nodal lines characterized by the results given in Theorem 2.3.

Remark 2.4. In some sense our asymptotic results on nodes might be considered as analogs of the local results on nodes of L. Bers [6], S.Y. Cheng [7] and recently L.A. Cafarelli and A. Friedmann [8].

There are some results on generic properties of eigenfunctions of elliptic operators on compact manifolds by J. Albert [9] and K. Uhlenbeck [10]. In the appropriate setting the generic case for the nodal lines of ϕ for $r \to \infty$ should be straight lines as given in (2.8). We hope to investigate this problem in future work.

Remark 2.5. The results given in Theorem 1.1 have been generalized to the n-dimensional case in [5]. Naturally the structure of the nodal set near infinity of such a solution can show a much more complicated pattern than in two dimensions. Partial results will be given in [11].

3. Proof of Theorem 2.1

To verify the uniform real analyticity of J_M it suffices to show that given I, then for some c, $\delta > 0$,

$$|U_M^{(k)}(r,z)| \le c\,\frac{k!}{\delta^k} \qquad \forall z \in I \quad \text{and} \quad \forall r \ge \bar{R} \tag{3.1}$$

for some $\bar{R} \ge R$ large

To derive (3.1) we first show that given any compact interval $J \subset \mathbb{R}$, then the family of functions

$$F_k = \{U_M^{(k)}(r,\cdot):\ J \to \mathbb{R}, r \ge \bar{R}\} \quad (\text{with some } \bar{R} \ge R)$$

is uniformly bounded for $0 \le k \le M$.

This can be verified by making use of the following inequality: If f is an n-times differentiable function on a closed interval $J \subset \mathbb{R}$ of length $|J|$ and if $|f(x)| \le M_0$ and $|f^{(n)}(x)| \le M_n$, where $M_j = \sup_{x\in J} |f^{(j)}(x)|$, $1 \le j \le n$, then for $x \in J$ and for $0 < k < n$

$$|f^{(k)}(x)| \le c_{n,k}\, M_0^{1-k/n}\, M_n'^{\,k/n} \tag{3.2}$$

where $M_n' = \max(M_n, M_0 n!|J|^{-n})$ and $c_{n,k}$ is a constant depending only on n and k. (See e.g. [12].)

Since for every arbitrary fixed $r > R$, $U_M(r,z)$ fulfills the above conditions (due to the known properties of u) on any compact interval $J \subset \mathbb{R}$, inequality (3.2) can be applied and it remains to show that $\sup_{z\in J} |U_M(r,z)|$ and $\sup_{z\in J} |U_M^{(M)}(r,z)|$ are bounded for $r \to \infty$, which will become clear from the following: That

$$\sup_{z\in J} |U_M(r,z)| \le C(J) \quad \text{for} \quad r > R$$

is an immediate consequence of Corollary 1.1. On the other hand, since

$$U_M^{(M)} = b^{-M} \frac{\partial^M}{\partial\omega^M} u(r,\frac{z}{b\sqrt{r}})$$

we conclude by (1.10) (with $k = M$) that $\forall z \in J$

$$|U_M^{(M)}(r,z) - b^M \frac{\partial^M}{\partial\omega^M} A(\frac{z}{b\sqrt{r}})| \le C(J)\ r^{-a} \quad \text{for large } r \tag{3.4}$$

with some $C(J) < \infty$. Particularly (3.4) implies that

$$U_M^{(M)}(r,z) \to (\frac{1}{2b})^M\ H_M^{(M)}(z) \quad \text{for } r \to \infty \text{ uniformly in } J. \tag{3.5}$$

Hence it follows via inequality (3.2) that F_k is uniformly bounded for $0 \le k \le M$. For $k \ge M+1$ the uniform boundedness of F_k is easily seen from

$$U_M^{(M+j)}(r,z) = b^{-M-j}\ r^{-j/2} \frac{\partial^{M+j}}{\partial\omega^{M+j}} u(r,\frac{z}{b\sqrt{r}}) \quad \forall j \in \mathbb{N} \tag{3.6}$$

and the fact that due to Theorem 1.1 u is real analytic in ω uniformly with respect to r. But this implies further that given $I \subset \mathbb{R}$, then for some c, $\delta > 0$,

$$|U_M^{(M+j)}(r,z)| \le c\ r^{-j/2} \frac{(M+j)!}{\delta^{M+j}} \quad \forall z \in I \text{ and large } r\ . \tag{3.7}$$

(3.7) together with the uniform boundedness of F_k for $0 \le k \le M$ verifies (3.1). Furthermore (3.7) implies (2.1) for $k \ge M+1$.

So finally it remains to verify (2.1) for $0 \le k \le M$: Note first that F_k is for all $k \in \mathbb{N}$ an equicontinuous family of functions since for $z_1, z_2 \in J$ and $r \ge \bar{R}$

$$|U_M^{(k)}(r,z_1) - U_M^{(k)}(r,z_2)| \le \int_{z_1}^{z_2} |U_M^{(k+1)}(r,z)|dz \le c_{k+1}|z_2 - z_1|$$

for some $c_{k+1} < \infty$ (not depending on r) due to the uniform boundedness of F_k for all k.

To simplify notation let $g(z) = (2b)^{-M} H_M(z)$ and $g_n(z) = U_M(r_n, z)$ with $z \in J$, where $\{r_n\}$ is an arbitrary but fixed sequence with $r_n \to \infty$ for $n \to \infty$.

From Theorem 1.1 we know that $g_n \to g$ uniformly in J. Now let $k \in \{1,2,\ldots,M-1\}$, $\bar{z} \in J$ arbitrary but fixed and let a_k denote an accumulation point of the sequence $\{g_n^{(k)}(\bar{z})\}$.

Then a subsequence $\{g_{n(i)}\}$ of $\{g_n\}$ exists such that $g_{n(i)}^{(k)}(\bar{z}) \to a_k$ for $i \to \infty$. But $g_{n(i)} \to g$ for $i \to \infty$ uniformly on J and F_k is uniformly bounded and equicontinuous. Hence by Arcela-Ascoli's theorem (see e.g. [13]) it follows that a subsequence $\{g_{\bar{n}(i)}\}$ of $\{g_{n(i)}\}$ exists with $g_{\bar{n}(i)}^{(j)} \to g^{(j)}$ for $i \to \infty$ uniformly on J for $j = 0,1,2,\ldots,M$. Therefore $g^{(k)}(\bar{z}) = a_k$ and further $g_n^{(k)}(\bar{z}) \to g^{(k)}(\bar{z})$. Since $\bar{z} \in J$ was arbitrary we obtain $g_n^{(k)} \to g^{(k)}$ in J for $n \to \infty$, and the convergence is uniformly since $g_n^{(M)} \to g^{(M)}$ for $n \to \infty$ uniformly on J due to (3.5).

This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

For the proof we shall need the following

Lemma 4.1. Let V_1 and v be given according to Theorem 1.1 so that $-\tilde{v}'' + (V_1 - 1/4r^2 - E)\tilde{v} = 0$ for $r > R$, where $\tilde{v} = \sqrt{r}\, v$. Then for large r

$$\int_r^\infty \tilde{v}^2(x)dx = \frac{1}{2\sqrt{V_1 - 1/4r^2 - E}}\,(1 + O(r^{-1}))\,\tilde{v}^2(r) \,, \qquad (4.1)$$

for $\gamma > 0$

$$\tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(x)x^{-1-\gamma}\,dx = \frac{1}{2\sqrt{|E|}}\, r^{-1-\gamma}(1 + O(r^{-\epsilon})) \qquad (4.2)$$

for some $\epsilon > 0$.

Let $\gamma_i > 0$ for $1 \le i \le k$, $k \in \mathbb{N}$, denote

$$Q_i = \tilde{v}^2(y_i)/\tilde{v}^2(x_i)$$

and

$$\langle \prod_{i=1}^{k} Q_i y_i^{-1-\gamma_i} \rangle = \int_r^\infty \int_{x_1}^\infty \int_{y_1}^\infty \int_{x_2}^\infty \cdots \int_{y_{k-1}}^\infty \int_{x_k}^\infty \prod_{i=1}^{k} Q_i y_i^{-1-\gamma_i} \, dy_k dx_k \ldots dy_1 dx_1$$

then for large r

$$\langle \prod_{i=1}^{k} Q_i y_i^{-1-\gamma_i} \rangle = O(r^{-\gamma}) \quad \text{where } \gamma = \sum_{i=1}^{k} \gamma_i, \text{ specifically for}$$

$$\text{some } \varepsilon > 0 \qquad \langle \prod_{i=1}^{k} Q_i y_i^{-2} \rangle = \left(\frac{1}{2\sqrt{|E|}}\right)^k \frac{1}{k!} r^{-k}(1 + O(r^{-\varepsilon})) \ . \tag{4.3}$$

Proof of Lemma 4.1. For a proof of (4.1) see Lemma 2.5 in [1]. Applying (4.1) we obtain immediately that for some $\varepsilon > 0$

$$\tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(x) x^{-1-\gamma} \, dx \le r^{-1-\gamma} \frac{1}{2\sqrt{|E|}} (1 + O(r^{-\varepsilon})) \ .$$

To derive the lower bound we use partial integration, apply (4.1) and obtain for some c, $\varepsilon > 0$

$$\int_r^\infty \tilde{v}^2(x) x^{-1-\gamma} \, dx = r^{-1-\gamma} \int_r^\infty \tilde{v}^2(x) dx - (1+\gamma) \int_r^\infty x^{-2-\gamma} \int_x^\infty \tilde{v}^2(y) dy dx \ge$$

$$\ge r^{-1-\gamma} \frac{1 - c\, r^{-\varepsilon}}{2\sqrt{|E|}} \tilde{v}^2(r)$$

implying (4.2). Using induction (4.3) follows easily by application of (4.2). □

Now we investigate the properties of $\partial u/\partial r$ for $r \to \infty$: Noting that ϕ obeys (1.1) and v obeys (1.6) it follows that

$$-\frac{\partial^2 u}{\partial r^2} - \frac{2\tilde{v}'}{\tilde{v}} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \omega^2} + V_2 \, u = 0 \qquad \text{in } \Omega_R \ . \tag{4.4}$$

Having in mind that $\lim_{r\to\infty} u = A$ it is easily seen that u obeys the following integrodifferential equation

$$u(r,\omega) = A(\omega) + \int_r^\infty \tilde{v}^{-2}(x) \int_x^\infty \tilde{v}^2(y)\left(-y^{-2}\frac{\partial^2}{\partial\omega^2} + V_2(y,\omega)\right)u(y,\omega)\,dy\,dx \quad (4.5)$$

(see Equ. (4.2) in [1]). Therefrom

$$\frac{\partial u}{\partial r}(r,\omega) = -\tilde{v}^{-2}(r)\int_r^\infty \tilde{v}^2(y)\left(-y^{-2}\frac{\partial^2}{\partial\omega^2} + V_2(y,\omega)\right)u(y,\omega)\,dy \qquad (4.6)$$

follows. Since u, $r^{1+\alpha} V_2$ are real analytic in ω uniformly with respect to r, $\forall\varepsilon > 0$ small, there exist $C_0, C_1, \delta > 0$ such that for $|\omega| \le \pi-\varepsilon$ and large r

$$\left|\frac{\partial^j}{\partial\omega^j} V_2\, r^{1+\alpha}\right| \le C_0 \frac{j!}{\delta^j} \quad \text{and} \quad \left|\frac{\partial^{2+j}}{\partial\omega^{2+j}} u\right| \le C_1 \frac{j!}{\delta^j} \quad \text{for } j \in \mathbb{N} \cup \{0\}$$

Therefore for some $C < \infty$ (not depending on k and y)

$$\left|\frac{\partial^k}{\partial\omega^k}\left(-y^{-2}\frac{\partial^2}{\partial\omega^2}u(y,\omega) + V_2(y,\omega)u(y,\omega)\right)\right| \le$$

$$\le \left|y^{-2}\frac{\partial^{2+k}}{\partial\omega^{2+k}}u\right| + \sum_{j=0}^{k}\binom{k}{j}\left|\frac{\partial^{k-j}}{\partial\omega^{k-j}}V_2\right|\left|\frac{\partial^j}{\partial\omega^j}u\right| \le C\frac{k!}{\delta^k}y^{-1-a} \quad \text{for } k \in \mathbb{N} \cup \{0\}$$

with $a = \min(1,\alpha)$. Applying (4.2) in Lemma 4.1 we obtain for large r and some $C < \infty$

$$\tilde{v}^{-2}(r)\int_r^\infty \tilde{v}^2(y)\left|\frac{\partial^k}{\partial\omega^k}\left(-y^{-2}\frac{\partial^2 u}{\partial\omega^2} + V_2 u\right)\right|dy \le C\frac{k!}{\delta^k}r^{-1-a} \quad \text{for } k \in \mathbb{N} \cup \{0\}\,. \tag{4.7}$$

Hence we conclude from (4.6) and (4.7) that for $k \in \mathbb{N}$, $\forall|\omega| \le \pi-\varepsilon$ and large r

$$\frac{\partial^{k+1}}{\partial r\,\partial\omega^k}u(r,\omega) = -\tilde{v}^{-2}(r)\int_r^\infty \tilde{v}^2(y)\frac{\partial^k}{\partial\omega^k}\left(-y^{-2}\frac{\partial^2 u}{\partial\omega^2} + V_2 u\right)dy \qquad (4.8)$$

and

$$\left|\frac{\partial^{k+1}}{\partial r\,\partial\omega^k}u(r,\omega)\right| \le C\frac{k!}{\delta^k}r^{-1-a} \qquad (4.9)$$

Clearly an analogous estimate to (4.9) is obtained after rotation of the coordinate system by proceeding in the above manner. But this implies that $r^{1+a}\frac{\partial u}{\partial r}$ is real analytic in ω uniformly with respect to r, verifying the first part of Theorem 2.2.

To prove the second part of Theorem 2.2 we start with the case M = 1: Rewriting (4.6) we have

$$\frac{\partial u}{\partial r}(r,\omega) = -\tilde{v}^{-2}(r)\int_r^\infty \tilde{v}^2(y)(-y^{-2}\frac{d^2A}{d\omega^2}+V_2A)dy -$$

$$-\tilde{v}^{-2}(r)\int_r^\infty \tilde{v}^2(y)(-y^2\frac{\partial^2}{\partial\omega^2}+V_2)(u-A)dy\ .$$

Having in mind (1.10) of Theorem 1.1 application of Lemma 4.1 leads to

$$\frac{\partial u}{\partial r}(r,\omega) = \frac{d^2A(\omega)/d\omega^2}{2\sqrt{|E|}}\,r^{-2}(1+O(r^{-\epsilon})) + A(\omega)O(r^{-1-a}) + O(r^{-1-2a})\ .$$

Since for $\omega = O(r^{-1/2})$, $d^2A/d\omega^2 = 2d + O(r^{-1/2})$ and $A = O(r^{-1/2})$, (2.2) follows immediately from the above.

Now we have to investigate the case $M \geq 2$: Due to the real analyticity of $\partial u/\partial r$ we have

$$\frac{\partial u}{\partial r}(r,\omega) = \sum_{k=0}^{\infty}\frac{\partial^{k+1}}{\partial r\partial\omega^k}u(r,0)\frac{\omega^k}{k!} \quad \text{for small } |\omega| \text{ and large } r. \tag{4.10}$$

To derive the asymptotics of the r.h.s. of (4.10) we shall use (4.8) with $\omega = 0$ and the fact that

$$\frac{\partial^k}{\partial\omega^k}u(r,\frac{z}{b\sqrt{r}}) = b^k\, r^{(-M+k)/2}\,U_M^{(k)}(r,z) \quad \text{for } z\in\mathbb{R} \text{ and } k\in\mathbb{N}\cup\{0\} \tag{4.11}$$

Therefore we obtain

$$\sum_{k=1}^{M-1}\frac{\partial^{k+1}}{\partial r\partial\omega^k}u(r,0)\frac{\omega^k}{k!} = \sum_{k=1}^{M-1}\frac{\omega^k}{k!}\tilde{v}^{-2}(r)\int_r^\infty \tilde{v}^2(y)[y^{-1+(k-M)/2}\,b^{k+2}\,U_M^{(k+2)}(r,0) -$$

$$-\sum_{j=0}^{k}\binom{k}{j}\frac{\partial^{k-j}}{\partial\omega^{k-j}}V_2(y,0)\,U_M^{(j)}(y,0)b^j y^{(j-M)/2}]dy\ .$$

Since due to Theorem 2.1 $|U_M^{(j)}(r,0) - (2b)^{-M}H_M^{(j)}(0)| \to 0$ for $r\to\infty$ the r.h.s. of the above equation

$$= \sum_{k=1}^{M-1} \frac{\omega^k}{k!} \tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(y)[y^{-1+(k-M)/2}\, b^{k+2}((2b)^{-M} H_M^{(k+2)}(0) + o(1)) -$$

$$- \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^{k-j}}{\partial\omega^{k-j}} V_2(y,0)\, y^{(j-M)/2}\, b^j((2b)^{-M} H_M^{(j)}(0) + o(1))]dy$$

Further since due to our assumptions $\frac{\partial^j}{\partial\omega^j} V_2\, r^{1+a}$ is uniformly bounded for $r \to \infty$ for all $j \in \mathbb{N} \cup \{0\}$, application of Lemma 4.1 implies that the above

$$= \sum_{k=1}^{M-1} \frac{\omega^k}{k!} [\frac{b^{k+2}}{2\sqrt{|E|}}\, r^{-1+(k-M)/2}((2b)^{-M} H_M^{(k+2)}(0) + o(1))(1 + O(r^{-\epsilon})) +$$

$$+ O(r^{-1-a+(k-M)/2})] \, . \tag{4.12}$$

Therefore for $\omega = z/(b\sqrt{r})$ and $z \in J$ (J an arbitrary but fixed compact interval)

$$\sum_{k=1}^{M-1} \frac{\partial^{k+1}}{\partial r \partial\omega^k} u(r,0) (\frac{z}{b})^k\, r^{-k/2} =$$

$$= r^{-1-M/2} [\frac{1}{4}(2b)^{-M} \sum_{k=1}^{M-1} \frac{z^k}{k!} (H_M^{(k+2)}(0) + o(1))(1 + O(r^{-\epsilon})) + |z| O(r^{-a})] \, . \tag{4.13}$$

On the other hand by applying (4.9) it is straightforward to see that for $z \in J$

$$\sum_{k=M}^{\infty} \frac{1}{k!} |\frac{z}{b\sqrt{r}}|^k |\frac{\partial^{k+1}}{\partial r \partial\omega^k} u(r,0)| \le C\, |z|\, r^{-M/2-1-a} \tag{4.14}$$

for large r with some $C < \infty$. Combining (4.13) and (4.14) we arrive at

$$\frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=z/(b\sqrt{r})} = \frac{\partial u}{\partial r}(r,0) +$$

$$+ r^{-1-M/2} \frac{1}{4}(2b)^{-M} \sum_{k=1}^{M-1} \frac{z^k}{k!}(H_M^{(k+2)}(0) + o(1))(1 + O(r^{-\epsilon})) + |z| O(r^{-1-a-M/2}) \, . \tag{4.15}$$

Due to (4.6) and (4.11) and Theorem 2.1 it follows that

$$\frac{\partial u}{\partial r}(r,0) = \tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(y)(y^{-1-M/2}\, b^2\, U_M^{(2)}(y,0) + V_2(y,0)y^{-M/2}U_M(y,0))dy =$$

$$= (\tfrac{1}{4}(2b)^{-M}\, H_M^{(2)}(0) + o(1))r^{-1-M/2}(1+O(r^{-\varepsilon})) + O(r^{-1-a-M/2}) \ . \qquad (4.16)$$

Further note that

$$\sum_{k=0}^{M-1} \frac{z^k}{k!}\, H_M^{(k+2)}(0) = H_M^{(2)}(z) = 4M(M-1)\, H_{M-2}(z) \ . \qquad (4.17)$$

Now if $\underline{M = 2m,\ m \in \mathbb{N}}$, then (4.16), (4.17) together with (4.15) obviously verify (2.4).

Finally let $\underline{M = 2m+1,\ m \in \mathbb{N}}$: Since $H_M(0) = 0$ (4.17) together with (4.15) yield

$$\frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=z/(b\sqrt{r})} = \frac{\partial u}{\partial r}(r,0) + r^{-1-M/2}(\frac{M(M-1)}{(2b)^M}\, H_{M-2}(z) + |z|\ o(1)) \qquad (4.18)$$

for large r and $z \in J$. However (4.16) only implies that

$$\frac{\partial u}{\partial r}(r,0)\ r^{\ 1+M/2} \to 0 \qquad \text{for } r \to \infty \ .$$

Suppose we have shown

<u>Lemma 4.2</u>. For $M = 2m+1$, $m \in \mathbb{N}$,

$$r^{m+2}\, \frac{\partial u}{\partial r}(r,0) = \frac{(-1)^m}{(2b)^{M+1}}\, \frac{(M+1)!}{m!}\, d(1+O(r^{-\varepsilon})) + O(r^{-a}) \ . \qquad (4.19)$$

Then (4.19) together with (4.18) verify (2.5), finishing the proof of Theorem 2.2.

<u>Proof of Lemma 4.2</u>. For the proof we shall proceed in an analogous way as in [1] resp. [5] for working out the asymptotics of u. We shall use the following notation (compare Lemma 4.1):

$$T_i = -\, y_i^{-2}\, \frac{\partial^2}{\partial\omega^2} + V_2(y_i,\omega) \ , \qquad T = -\, y^{-2}\, \frac{\partial^2}{\partial\omega^2} + V_2(y,\omega) \ ,$$

$$Q_i = \tilde{v}^2(y_i)\ \tilde{v}^{-2}(x_i) \ ,$$

$$< \prod_{i=1}^{\ell} Q_i T_i f> = \int_r^{\infty} \int_{x_1}^{\infty} \int_{y_1}^{\infty} \int_{x_2}^{\infty} \cdots \int_{y_{\ell-1}}^{\infty} \int_{x_\ell}^{\infty} Q_1 T_1 \cdots Q_\ell T_\ell f(y_\ell,\omega) dy_\ell dx_\ell dy_{\ell-1} \cdots dy_1 dx_1$$

where f is identified with u or A. When necessary the dependence of <...> on the variable r will be denoted by $<\ldots>_r$. Using the above notation, equation (4.5) resp. (4.6) read

$$u = A + <Q_1\ T_1\ u> \tag{4.20}$$

$$\frac{\partial u}{\partial r} = - \tilde{v}^{-2}(r) \int_r^{\infty} \tilde{v}^2(y)\ T\ u(y,\omega)dy \ . \tag{4.21}$$

Iterating equation (4.20) gives

$$u = A + \sum_{\ell=1}^{M} < \prod_{i=1}^{\ell} Q_i T_i A> + < \prod_{i=1}^{M} Q_i T_i (u - A)> \ . \tag{4.22}$$

Combining (4.22) with (4.21) leads to

$$\frac{\partial u}{\partial r}(r,\omega) = - \tilde{v}^{-2}(r) \int_r^{\infty} \tilde{v}^2(y) T(A + \sum_{\ell=1}^{M} < \prod_{i=1}^{\ell} Q_i T_i A>_y + < \prod_{i=1}^{M} Q_i T_i (u-A)>_y)dy \ . \tag{4.23}$$

Now we investigate the asymptotics of the terms on the r.h.s. of (4.23): Note that due to assumption (1.9) $r^{1+a} \frac{\partial^k}{\partial\omega^k} v_2$ is uniformly bounded in r for $r \to \infty$ for all $k \in \mathbb{N} \cup \{0\}$ and that $r^a \frac{\partial^k}{\partial\omega^k}(u-A)$ is in the same sense bounded because of (1.10). Taking this into account and using Lemma 4.1 it is straightforward to show that for all ω and large enough r

$$T< \prod_{i=1}^{M} Q_i T_i (u - A)>_y = < \prod_{i=1}^{M} Q_i\ T T_1 \ldots T_M (u(y_M,\omega) - A)>_y =$$

$$= O(< \prod_{i=1}^{M-1} Q_i y_i^{-1-a}\ Q_M y_M^{-1-2a}>_y\ y^{-1-a}) = O(y^{-aM-2a-1}) = o(y^{-m-2-a})$$

and therefore

$$\tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(y)\ T < \prod_{i=1}^{M} Q_i T_i (u - A) >_y dy = o(r^{-m-2-a}) \ . \tag{4.24}$$

Next we observe in an analogous way that for $1 \le t \le m$

$$< \prod_{i=1}^{\ell} Q_i\, T T_1 \ldots T_\ell A>\Big|_{\omega=0} = < \prod_{i=1}^{\ell} Q_i y_i^{-2}>_y\ (-1)^{\ell+1}\ y^{-2}\ \frac{d^{2\ell+2}}{d\omega^{2\ell+2}} A(0) + R$$

where R is a sum of terms, each of them depending on $\frac{d^k}{d\omega^k} A(0)$ for some k with $0 \le k \le 2\ell$. Since $\frac{d^k}{d\omega^k} A(0) = 0$ for $0 \le k \le 2m$ and $1 \le \ell \le m$ the above implies that

$$T< \prod_{i=1}^{\ell} Q_i T_i A>|_{\omega=0} = \begin{cases} 0 & \text{for } 1 \le \ell \le m-1 \\ < \prod_{i=1}^{m} Q_i y_i^{-2}>_y (-1)^m y^{-2}\ \frac{d^{2m+2}}{d\omega^{2m+2}} A(0) & \text{for } \ell = m \end{cases}$$

and we conclude from the above via Lemma 4.1 that for some $\varepsilon > 0$

$$- \tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(y)\ T \sum_{\ell=1}^{m} < \prod_{i=1}^{\ell} Q_i T_i A>_y dy|_{\omega=0} =$$

$$= \frac{(-1)^m}{m!}\ \frac{1}{(4b^2)^m}\ (1 + O(r^{-\varepsilon}))\tilde{v}^{-2}(r) \int_r^\infty \tilde{v}^2(y)\ y^{-2-m}\ dy\ \frac{d^{2m+2}}{d\omega^{2m+2}} A(0) =$$

$$= \frac{(-1)^m}{m!}\ (M+1)!\ \frac{d}{(4b^2)^{m+1}}\ (1 + O(r^{-\varepsilon}))\ r^{-m-2}\ , \tag{4.25}$$

where we used $\frac{d^{2m+2}}{d\omega^{2m+2}} A(0) = d(M+1)!$.

It remains to investigate the asymptotics of $T< \prod_{i=1}^{\ell} Q_i T_i A>|_{\omega=0}$ for $m+1 \le \ell \le 2m+1$: Thereby it is not difficult to see that

$$< \prod_{i=1}^{\ell} Q_i\, T T_1 \ldots T_\ell A>|_{\omega=0} =$$

$$= < \prod_{i=1}^{\ell} Q_i y_i^{-2}>((-1)^{\ell+1}\ y^{-2}\ \frac{d^{2\ell+2}}{d\omega^{2\ell+2}} A(0) + V_2(y,0)(-1)^\ell\ \frac{d^{2\ell}}{d\omega^{2\ell}} A(0)) + R \tag{4.26}$$

where the rest R is a finite sum of terms of two types denoted by $J_{k,\ell}$ and $I_{k,\ell}$ with $1 \le k+1 \le \ell$ characterized by the following:

Let $(i_1,\dots,i_\ell)$ denote a permutation of $(1,2,\dots,\ell)$ and let $a \in \mathbb{N}$, $a_j \in \mathbb{N} \cup \{0\}$ for $k+1 \le j \le \ell$, then

$$J_{k,\ell} = \langle \prod_{i=1}^{\ell} Q_i \prod_{j=1}^{k} y_{i_j}^{-2} \prod_{j=k+1}^{\ell} \frac{\partial^{a_j}}{\partial \omega^{a_j}} V_2(y_{i_j},0)\rangle_y \, y^{-2} \frac{d^a}{d\omega^a} A(0)$$

$$\text{with } 2m+1 \le a \le 2\ell \text{ and } \sum_{j=k+1}^{\ell} a_j + a = 2k+2 \tag{4.27}$$

and

$$I_{k,\ell} = \langle \prod_{i=1}^{\ell} Q_i \prod_{j=1}^{k} y_{i_j}^{-2} \prod_{j=k+1}^{\ell} \frac{\partial^{a_j}}{\partial \omega^{a_j}} V_2(y_{i_j},0)\rangle \, V_2(y,0) \frac{d^a}{d\omega^a} A(0)$$

$$\text{with } 2m+1 \le a \le 2\ell-2, \quad \sum_{j=k+1}^{\ell} a_j + a = 2k . \tag{4.28}$$

Analogously to the foregoing considerations it follows that

$$J_{k,\ell} = O(\langle \prod_{i=1}^{\ell} Q_i \prod_{j=1}^{k} y_{i_j}^{-2} \prod_{j=k+1}^{\ell} y_{i_j}^{-1-\varepsilon}\rangle_y \, y^{-2}) = O(y^{-2-k-\varepsilon(\ell-k)}) =$$

$$= O(y^{-2-m-\varepsilon}) \tag{4.29}$$

using $k \ge m$ in the last step (according to (4.27)) and

$$I_{k,\ell} = O(\langle \prod_{i=1}^{\ell} Q_i \prod_{j=1}^{k} y_{i_j}^{-2} \prod_{j=k+1}^{\ell} y_{i_j}^{-1-\varepsilon}\rangle_y \, y^{-1-\varepsilon}) = O(y^{-k-\varepsilon(\ell-k)-1-\varepsilon}) =$$

$$= O(y^{-2-m-2\varepsilon}) \tag{4.30}$$

using $k \ge m+1$ in the last step (according to (4.28)). (4.29) and (4.30) imply that

$$R = O(y^{-2-m-\varepsilon}) .$$

Therefore and again with the help of Lemma 4.1, (4.26) yields

$$\tilde{v}^{-2}(r) \int_r^{\infty} \tilde{v}^2(y) \left\langle \prod_{i=1}^{\ell} Q_i \, T \, T_1 \cdots T_\ell A \right\rangle \Big|_{\omega=0} dy = O(r^{-2-m-a}) \tag{4.31}$$

$$\text{for } m+1 \leq \ell \leq 2m+1 = M .$$

Application of (4.31), (4.25) and (4.29) to (4.23) verifies Lemma 4.2.□

Remark 4.1. (4.19) holds also for $m = 0$ as can be seen easily by proceeding as above, but we shall not need it for the following.

5. Proof of Theorem 2.3

We first show the existence of exactly M nodal lines in D_c: Let $H_M(\bar{z}) = 0$ and choose $\delta_0 > 0$ such that (without loss)

$$\frac{d}{dz} H_M(z) > 0 \quad \text{for } z \in I_{\delta_0}(\bar{z}) \; , \quad \text{where } I_{\delta_0}(\bar{z}) = \{z \,|\, |z - \bar{z}| \leq \delta_0\} . \tag{5.1}$$

This is possible since H_M has only nondegenerate zeros. Further choose R_0 so large that

$$U_M^{(1)}(r,z) > 0 \quad \text{for } r \geq R_0 \quad \text{and} \quad z \in I_{\delta_0}(\bar{z}) \tag{5.2}$$

which is possible due to Theorem 2.1. Further by Theorem 2.1 the above implies that $\forall \delta \in (0,\delta_0)$ there is some $R_\delta \geq R_0$ such that for $r \geq R_\delta$

$$\operatorname{sgn} U_M(r, \bar{z} \pm \delta) = \operatorname{sgn} H_M(\bar{z} \pm \delta) .$$

Hence for all $r \geq R_\delta$ there exists $g(r) \in I_\delta(\bar{z})$ with $U_M(r, g(r)) = 0$ and due to (5.2) it is unique. Having in mind that $u \in C^1(\Omega_R)$ the implicit function theorem implies that g is continuously differentiable. Furthermore it follows that $g(r) \to \bar{z}$ for $r \to \infty$.

The foregoing considerations imply that for each zero z_i $(1 \leq i \leq M)$ of H_M there is at least one nodal line of u in D_c given by $\omega_i = g_i(r)/(b\sqrt{r})$, $u(r,\omega_i) = 0$ where $g_i(r) \to z_i$ for $r \to \infty$.

Now suppose there exists $r_n \to \infty$ for $n \to \infty$ and $\bar{\omega}(r_n)$, such that $\forall n$ $|\bar{\omega}(r_n)| < \varepsilon$, $\bar{\omega}(r_n) \neq \omega_i(r_n)$, for $1 \leq i \leq M$ and $u(r_n,\bar{\omega}(r_n)) = 0$. Then since $u \to A$ for $r \to \infty$ uniformly and $A(\omega) \neq 0$ for $0 < |\omega| < \varepsilon$ for ε small enough, $\bar{\omega}(r_n) \to 0$ for $n \to \infty$ follows. Together with the foregoing considerations we obtain that for some $\omega(r_n) \to 0$ for $n \to \infty$ $\frac{\partial^M u}{\partial \omega^M}(r_n,\omega(r_n)) = 0$ $\forall n$. Since $\frac{\partial^M u}{\partial \omega^M} \to \frac{\partial^M A}{\partial \omega^M}$ for $r \to \infty$ uniformly we obtain $\partial^M A(0)/\partial\omega^M = 0$ which is a contradiction to the assumption on A. Hence there are exactly M nodal lines of u in D_ε for ε small enough.

Let $g \to \bar{z}$ for $r \to \infty$ be given as before and denote $f(r) = g(r)/(b\sqrt{r})$. Then for large r

$$u(r,f(r)) = 0$$

and

$$\frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=f(r)} + \frac{\partial u}{\partial \omega}(r,f(r))\ f'(r) = 0\ . \qquad (5.3)$$

This implies further

$$f'(r) = -\frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=f(r)}\ r^{(M-1)/2}\ (b\ U_M^{(1)}(r,g(r)))^{-1}. \qquad (5.4)$$

Since due to Theorem 2.1

$$\lim_{r\to\infty} U_M^{(1)}(r,g(r)) = (2b)^{-M}\ 2M\ H_{M-1}(\bar{z}) \neq 0 \qquad (5.5)$$

and since due to Theorem 2.2 $r^{1+M/2}\frac{\partial u}{\partial r}(r,\omega)\big|_{\omega=f(r)}$ is bounded for $r \to \infty$ we obtain from (5.4) that for some $C < \infty$

$$|f'| \leq C\ r^{-3/2} \quad \text{for large } r\ . \qquad (5.6)$$

Denoting $\gamma_1(r) = r\cos f(r)$, $\gamma_2(r) = r\sin f(r)$ we conclude from (5.6) that for large r for some $c > 0$

$$\gamma_1'(r) = \cos f(1 - r\ f'\ \mathrm{tg}\ f) \geq \cos f(1 + O(r^{-1})) \geq c > 0\ . \qquad (5.7)$$

Therefore the inverse γ_1^{-1} exists, implying the representation of the nodal line in cartesian coordinates $(x_1,x_2) \in \mathbb{R}^2$ by $x_2 = G(x_1)$ with $G = \gamma_2 \circ \gamma_1^{-1}$.

Next we verify the asymptotics of the nodal lines of ϕ. We start with the simplest case:

<u>$M = 1$</u>

We use the asymptotics of $\partial u/\partial r$ given in (2.3) of Theorem 2.2, take into account (5.5) and apply these findings to (5.4). This gives

$$f'(r) = \left(-\frac{d}{\sqrt{|E|}} + o(1)\right) r^{-2}$$

and integrating from r to ∞ gives

$$f(r) = \left(\frac{d}{\sqrt{|E|}} + o(1)\right) r^{-1} .$$

Therefore

$$\gamma_2(r) = r \sin f(r) = rf + O(r^{-1/2}) = \frac{d}{\sqrt{|E|}} + o(1)$$

and $\gamma_1(r)/r = \cos f(r) \to 1$ for $r \to \infty$, implying $G(x_1) = d/\sqrt{|E|} + o(1)$ and verifying (2.8) für $M = 1$.

Next we consider the case

<u>$M \geq 2$ and $\bar{z} \neq 0$</u>

Since $g(r) \to \bar{z}$ for $r \to$ we obviously have

$$\gamma_2(r) = r \sin \frac{g}{b\sqrt{r}} = \frac{g}{b} \sqrt{r}(1 + O(r^{-1})) = \left(\frac{\bar{z}}{b} + o(1)\right)\sqrt{r}$$

and $\gamma_1(r)/r \to 1$ for $r \to \infty$ and therefore $G(x_1) = (\bar{z}/b + o(1))\sqrt{x_1}$ for large x_1 verifying (2.7).

To prove the monotonicity of $G(x_1)$ it suffices (because of (5.7)) to show the monotonicity of $\gamma_2(r)$: Since

$$\gamma_2'(r) = \cos f(rf' + \operatorname{tg} f) = \frac{\cos f}{\sqrt{r}}(r^{3/2}f' + \sqrt{r}\, f + O(r^{-1})) \qquad (5.8)$$

we have to investigate the asymptotics of f': Taking into account (2.4) and (2.5) of Theorem 2.2 we conclude

$$r^{1+M/2}\,\frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=f(r)} = M(M-1)(2b)^{-M}\,H_{M-2}(\bar{z}) + o(1) \tag{5.9}$$

and note that $H_{M-2}(\bar{z}) \neq 0$. Applying (5.9) and (5.5) to (5.4) we arrive at

$$f'(r) = -\,r^{-3/2}\Big(\frac{M-1}{2b}\,\frac{H_{M-2}(\bar{z})}{H_{M-1}(\bar{z})} + o(1)\Big)\ . \tag{5.10}$$

Noting that $H_M(\bar{z}) = 0 = 2\bar{z}\,H_{M-1}(\bar{z}) - 2(M-1)\,H_{M-2}(\bar{z})$, (5.10) leads to

$$f'(r) = \Big(-\frac{\bar{z}}{2b} + o(1)\Big)r^{-3/2}\ . \tag{5.11}$$

Combining (5.11) with (5.8) and taking into account that $\sqrt{r}\,f \to \bar{z}/b$ for $r \to \infty$ we obtain

$$\operatorname{sgn}\,\gamma_2'(r) = \operatorname{sgn}\,(\bar{z}) \quad \text{for large } r\ . \tag{5.12}$$

(5.12) together with (5.7) shows that $\gamma_2 \circ \gamma_1^{-1}$ is strictly monotonously increasing for $\bar{z} > 0$ respectively decreasing for $\bar{z} < 0$.

Finally we have to consider the case

$M = 2m+1,\ m \in \mathbb{N}$ and $\bar{z} = 0$:

Due to (2.5) of Theorem 2.2 we have

$$\begin{aligned} r^{1+M/2}\,\frac{\partial u}{\partial r}(r,\omega)\Big|_{\omega=g(r)/(b\sqrt{r})} = {} & \frac{d_1}{\sqrt{r}}\big(1 + O(r^{-\varepsilon})\big) + O(r^{-1/2-a}) + \\ & + \frac{M(M-1)}{(2b)^M}\,H_{M-2}(g(r)) + o(g(r)) \end{aligned} \tag{5.13}$$

with $d_1 = (-1)^m \dfrac{(M+1)!}{m!}\,d\,(2b)^{-M-1}$,

and via Theorem 2.1

$$U_M^{(1)}(r,g) = (2b)^{-M}\,2M\,H_{M-1}(g) + o(1)\ . \tag{5.14}$$

Applying (5.13) and (5.14) to (5.4) and taking into account that

$$f' = \frac{1}{b\sqrt{r}}\,(g' - \frac{g}{2r})$$

we obtain

$$g' = \frac{g}{2r} - r^{M/2}\,\frac{\partial u}{\partial r}(r,\omega)\big|_{\omega=f(r)}(U_M^{(1)}(r,g(r)))^{-1} =$$

$$= \frac{1}{2r}\Big(g - \frac{d_1(1+O(r^{-\epsilon})) + O(r^{-a})}{\sqrt{r}((2b)^{-M}H_{M-1}(0) + o(1))} - (M-1)\,\frac{H_{M-2}(g) + o(|g|)}{H_{M-1}(g) + o(1)}\Big)\ . \qquad (5.15)$$

Since for $k \in \mathbb{N}$, $H_{2k}(0) = (-1)^k(2k)!/k!$ and $H'_{M-2}(0) = 2(M-2)H_{M-3}(0)$,

$$\frac{H_{M-2}(g)}{H_{M-1}(g)} = g\,\frac{H'_{M-2}(0) + O(|g|^2)}{H_{M-1}(0) + O(|g|^2)} = -\,g(1 + o(1))$$

we obtain from (5.15) for some $\delta_1(r),\delta_2(r) \to 0$ for $r \to \infty$ that for large r

$$g' = \frac{1}{r}\,[\frac{M}{2}\,g(1 + \delta_1(r)) - (\frac{d(M+1)}{4b} + \delta_2(r))r^{-1/2}]\ . \qquad (5.16)$$

In the following we show with the help of (5.16) that

$$\lim_{r\to\infty} \sqrt{r}\; g = \frac{d}{2b}\ . \qquad (5.17)$$

Let us first consider the case $d > 0$:
Suppose that for some $\bar{r}$ large $g(\bar{r}) < 0$, then because of (5.16), for $r \geq \bar{r}$ $g < 0$ and g strictly monotonously increasing follows, contradicting $g \to 0$ for $r \to \infty$. Therefore $g > 0$ for large r. Let

$$\bar{c}_1 = \frac{M}{2}(1 + \bar{\delta}_1) \qquad \text{and} \qquad \underline{c}_2 = \frac{d(M+1)}{4b} - \underline{\delta}_2$$

for some $\bar{\delta}_1, \underline{\delta}_2 > 0$ arbitrarily small, then due to (5.16)

$$g' \leq \frac{1}{r}\,g\,\bar{c}_1 - \underline{c}_2\,r^{-3/2} \qquad \text{for large enough } r\ .$$

Further let

$$h(r) = \frac{\underline{c}_2}{\bar{c}_1 + \frac{1}{2}} r^{-1/2} ,$$

then

$$h' = \frac{1}{r} \bar{c}_1 h - \underline{c}_2 r^{-3/2}$$

and hence

$$(h-g)' \geq \bar{c}_1 \frac{1}{r} (h-g) \quad \text{for large } r .$$

Suppose there exists $\bar{\bar{r}}$ (arbitrarily large) with $(h-g)(\bar{\bar{r}}) > 0$. Then the above inequality implies that $0 < h-g$ and $h-g$ strictly monotonously increasing for $r \geq \bar{\bar{r}}$, which contradicts $h-g \to 0$ for $r \to \infty$. Hence for large r, $g \geq h$ and therefore with some $\bar{\delta} > 0$ arbitrarily small

$$g(r) \geq \left(\frac{d}{2b} - \bar{\delta}\right) r^{-1/2} \quad \text{for large } r . \tag{5.18}$$

Combining (5.16) with (5.18) we obtain for some $\underline{\delta}_1$, $\bar{\delta}_2$, $\delta > 0$ arbitrarily small

$$g' \geq \left[\frac{M}{2}(1-\underline{\delta}_1)\left(\frac{d}{2b} - \bar{\delta}\right) - \left(\frac{d(M+1)}{4b} + \bar{\delta}_2\right)\right] r^{-3/2} \geq -\left(\frac{d}{4b} + \delta\right) r^{-3/2}$$

for large r. Integrating the above inequality leads to

$$g \leq \left(\frac{d}{2b} + \underline{\delta}\right) r^{-1/2} \tag{5.19}$$

with some $\underline{\delta} > 0$ arbitrarily small for $r \to \infty$. (5.18) and (5.19) imply (5.17) for $d > 0$.

The case $d < 0$ follows in the same way.

For $d = 0$ $\lim_{r\to\infty} \sqrt{r}\, g = 0$ can be seen by the following: Suppose there exists $r_n \to \infty$ for $n \to \infty$ such that $\sqrt{r_n}\, g(r_n) \to k$ for $n \to \infty$ with $0 < k \leq \infty$. Then because of (5.16)

$$g' = (\sqrt{r}\, g\, \frac{M}{2}(1 + \delta_1(r)) - \delta_2(r))r^{-3/2}$$

and therefore $g > 0$ and g is strictly monotonously increasing for large r, contradicting $g \to 0$ for $r \to \infty$. For $0 > k \geq -\infty$ the conclusion is analogously. This proves (5.17).

By (5.17) we finally obtain

$$\gamma_2(r) = r \sin \frac{g}{b\sqrt{r}} = \frac{\sqrt{r}g}{b} + O((g\sqrt{r})^3\, r^{-2}) = \frac{d}{2b^2} + o(1)$$

and since $\gamma_1(r)/r \to 1$ for $r \to \infty$ this verifies (2.8) and finishes the proof of Theorem 2.3.

References

[1] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and J. Swetina, Continuity and nodal properties near infinity for solutions of N-dimensional Schrödinger equations, Duke Math. J. 53 (1986) 271 - 306.

[2] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer Verlag, Berlin, Heidelberg, New York, 1977.

[3] M. Abramovitz and I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1968.

[4] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and J. Swetina, Pointwise bounds on the asymptotics of spherically averaged L^2-solutions of one-body Schrödinger equations, Ann. Inst. H. Poincaré 42 (1985) 341 - 361.

[5] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and J. Swetina, Asymptotics and continuity properties near infinity of solutions of Schrödinger equations in exterior domains, Ann. Inst. H. Poincaré, in press.

[6] L. Bers, Local behaviour of solutions of general elliptic equations, Comm. Pure Appl. Math. 8 (1955) 473 - 496.

[7] S.Y. Cheng, Eigenfunctions and nodal sets, Comment. Math. Helvetici 51 (1976) 43 - 55.

[8] L.A. Cafarelli and A. Friedmann, Partial regularity of the zero-set of solutions of linear and super-linear elliptic equations, J. Diff. Equations 60 (1985) 420 - 433.

[9] J.H. Albert, Generic properties of eigenfunctions of elliptic partial differential operators, Trans. Am. Math. Soc. 238 (1978) 341 - 354.

[10] K. Uhlenbeck, Generic properties of eigenfunctions, Am. J. Math., Vol. 98, 4 (1976) 1059 - 1078.

[11] M. Hoffmann-Ostenhof and T. Hoffmann-Ostenhof, On the asymptotics of nodes of L^2-solutions of Schrödinger equations, in preparation.

[12] D.S. Mitrinovic, Analytic Inequalities, Die Grundlehren d. math. Wiss., Bd. 165, Springer Verlag, Berlin, Heidelberg, New York, 1970.

[13] M. Reed and B. Simon, Methods of Modern Mathematical Physics I, Functional analysis, Academic Press, 1972.