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ABSTRACT

A solution of a pair of monopoles in 5d Kaluza-Klein theory without matter fields is given. The scale of the internal space is the same as that of the outer space. Its Euclidean version is useful for calculating the wave function of the universe with monopoles.

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It was pointed out that there are topological stable monopole solutions in the 5 dimensional Kaluza-Klein theory (1,2,3) and K-K theories of higher dimensions (4,5). These 5 dimensional solutions can be expressed by the instanton solutions in 4 dimensional Euclidean gravity. Unlike the Dirac monopole, K-K monopole does not need the Dirac string, since the spatial 4d space is not a S^1 bundle over R^3.

It is the purpose of the present paper to generalize the monopole solutions to the cosmological ones. By "cosmological" we mean the spatial subspace is compact and closed. We will discuss the simplest case of monopole-anti-monopole solutions. As argued in ref.(7) the Euler characteristic and signature of the 4d spatial submanifold are 2 and 0 respectively. To see this, let us first review the single monopole solution in Euclidean space.

If we take the cosmological constant as being zero and $g_{00} = -1$, $g_{0i} = 0$, then g_{ij} themselves satisfy the vacuum Euclidean Einstein equation. Especially we have the so-called Taub-NUT solution

$$dS_4^2 = V\,(dx^5 + 4m(1-\cos\theta)d\varphi)^2 + \frac{1}{V}(dr^2 + r^2 d\Omega^2)$$

$$\frac{1}{V} = 1 + \frac{4m}{r} \qquad (1)$$

This solution has two singularities. One is at $\theta = \pi$, another is at $r = 0$. To avoid the first singularity, let us take the transformation $x^5 \to x^5 + 8m\varphi$ around $\theta = \pi$, then $dx^5 + 4m(1-\cos\theta)d\varphi \to dx^5 + 4mx(1+\cos\theta)d\varphi$, the latter metric is well behaving.

Since the period of ψ is 2π, the period of x^5 then is $16m\pi$. As for the second singularity, when $r = 0$, the singular part of metric approaches

$$\frac{r}{4m}dx^{5\,2} + \frac{4m}{r}\left(dr^2 + r^2 d\Omega^2\right) \tag{2}$$

Introduce $r = \lambda^2$, eq.(2) becomes

$$16m\left[d\lambda^2 + \left(\frac{\lambda}{8m}\right)^2 dx^{5\,2}\right] + 4m\lambda^2 d\Omega^2 \tag{3}$$

It is easy to see that if the period of x^5 is $16m\pi$, there is no singularity at $r = 0$. The circle characterized by x^5 shrinks to a point when $r \to 0$. This fact explains why we need not introduce a Dirac string: for S^1 shrinks to a point at $r = 0$, manifold M_4 is not a S^1 bundle over R^3. The magnetic charge of the monopole is

$$g = \frac{4m}{\sqrt{16\pi G}} = \frac{4m}{\ell_p}$$

Because the radius of x^5 is $8m$, it must be equal to $\ell p/e$, so we have $ge = 1/2$. This is the Dirac quantization condition.

One can further construct multi-monopole solutions. They take the form of

$$dS_5^2 = -dt^2 + V\left(dx^5 + A_i dx^i\right)^2 + \frac{1}{V}\left(dr^2 + r^2 d\Omega^2\right) \tag{4-a}$$

$$\frac{1}{V} = \sum_{i=1}^{N} 4m/|\vec{x} - \vec{x}_i| + 1$$

$$\partial_j A_k - \partial_k A_j = -\epsilon_{ijk}\,\partial_i\left(\frac{1}{V}\right) \tag{4-b}$$

Changing the sign of r.h.s. of (4-b), we get a multi-anti-monopole solution. The Euler characteristic and signature of N-monopole solution are $\chi = N$, $\tau = N-1$. For N-anti-monopole solution, $\chi = N$, $\tau = -(N-1)$. For a pair of monopole and anti-monopole, $\chi = 2$ and $\tau = 0$. All the above solutions are over non-compact manifolds M_4.

But the universe may be compact and closed, we have to ask about the monopole solutions when M_4 is compact? Solution (1) can be explained as when $r \to \infty$, we have a non-trivial S^1 bundle over the infinite S^2; when $r \to 0$, S^1 shrinks to a point. The simplest generalization to compact M_4 is to consider a nontrivial S^1 bundle over the equator S^2 of S^3. In this case, when S^2 approaches the north pole or the south pole of S^3, S^1 must shrink to a point. As argued in ref.(7) we thus have a pair of monopoles and anti-monopoles. As for more complicated cases, the total number of monopoles is equal to the total number of anti-monopoles. In fact, the solution of a pair of monopole and anti-monopole can be viewed as patching solution (1) and its anti-solution together. So there are $\chi(M_4) = 2$ and $\tau(M_4) = 0$.

It is interesting to search for compact 5d Euclidean solutions. In this case M_4 is the boundary of a compact M_5. One can then use the formalism of quantum cosmology of Hartle and Hawking[9] to calculate the wave function of the universe with some K-K monopoles. Furthermore, after making a Wick rotation, we get a Minkowskian solution from a Euclidean one.

Mathematical cobordism theory tells us that M_4 is the boundary of M_5 if and only if $\tau(M_4) = 0$. So in the universe, the amount of monopoles and that of anti-monopoles are equal. Let us first look for the Euclidean solutions and assume that

$$dS_5^2 = d\tau^2 + F(\tau,\chi)\,(dx^5 + 4m(1-\cos\theta)d\varphi)^2 + G(\tau,\chi)\,(d\chi^2 + \sin^2\chi\, d\Omega^2) \tag{5}$$

Now we analysis the general features of ansatz (5). This ansatz has a symmetry SO(2). There may be only monopoles located at $\chi = 0, \pi$. We are interested in the special case in which when $\chi \to \pi - \chi$, functions F and G are invariant. When $\chi \to 0$, according to the analysis about the singularity, $F(\tau,\chi) \to F(\tau)\chi$ and $G(\tau,\chi) \to G(\tau)\chi^{-1}$, the singular part of metric(5) is

$$F(\tau)\chi\, dx^{5^2} + G(\tau)\chi^{-1}d\chi^2 = G(\tau)\left[\frac{F(\tau)}{G(\tau)}\chi\, dx^{5\,2} + \chi^{-1}d\chi^2\right] \tag{6}$$

$F(\tau)/G(\tau)$ must be constant and equal to $1/16m^2$. The singularity at $\chi = 0$ is now avoided and there is also no singularity at $\chi = \pi$ guaranteed by symmetry $\chi \to \pi - \chi$.

We are going to consider the case of $F(\tau,\chi) = F^2(\tau)V(\chi)^2$ and $G(\tau,\chi) = F^2(\tau)U(\chi)^2$. The metric (5) now reads

$$dS_5^2 = d\tau^2 + F^2(\tau)\, dS_4^2 \tag{7-a}$$

$$dS_4^2 = V^2(\chi)(dx^5 + 4m(1-\cos\theta)d\varphi)^2 + U^2(\chi)(d\chi^2 + \sin^2\chi\, d\Omega^2) \tag{7-b}$$

Now we use the tetrad formalism to calculate curvature

$$R^{(5)}_{ijkl} = [R^{(4)}_{ijkl} - F'^2(\delta_{ki}\delta_{lj} - \delta_{li}\delta_{kj})]/F^2$$

$$R^{(5)}_{0i0j} = -\frac{F''}{F}\delta_{ij} \tag{8}$$

where $R^{(4)}_{ijkl}$ is the curvature tensor of 4 dimensional metric ds_4^2. The Ricci tensor is

$$R^{(5)}_{ij} = \frac{1}{F^2}[R^{(4)}_{ij} - 3\delta_{ij}F'^2] - \delta_{ij}\frac{F''}{F}$$

$$R^{(5)}_{00} = -4F''/F \tag{9}$$

Solving Einstein equation $R^{(5)}_{\mu\nu} = \delta_{\mu\nu}\bar{\Lambda}$ with cosmological constant $\bar{\Lambda}$, we obtain

$$F = a\sin\left(\sqrt{\frac{\bar{\Lambda}}{4}}\,\tau\right) \tag{10}$$

and

$$R^{(4)}_{ij} = \delta_{ij}\frac{3\bar{\Lambda}}{4}a^2 \tag{11}$$

so the 4 dimensional metric satisfies Einstein equation with cosmological constant $\Lambda = \frac{3\lambda}{4}a^2$. Redefine $\tau \to \sqrt{\frac{\lambda}{4}}\,\tau$, the metric is

$$dS_5^2 = \frac{4}{\lambda}\left(d\tau^2 + \frac{\Lambda}{3}\sin^2\tau\, dS_4^2\right) \tag{12}$$

To solve eq.(11), we first list the non-vanishing curvature components

$$\begin{aligned} R^{(4)}_{11} &= 2R^{(4)}_{1212} + R^{(4)}_{1515} \\ R^{(4)}_{22} &= R^{(4)}_{33} = R^{(4)}_{1212} + R^{(4)}_{2323} + R^{(4)}_{2525} \\ R^{(4)}_{55} &= R^{(4)}_{1515} + 2R^{(4)}_{2525} \\ R^{(4)}_{15} &= -R^{(4)}_{3513} \end{aligned} \tag{13}$$

$$\begin{aligned} R^{(4)}_{1212} &= -\frac{1}{U^2\sin\chi}\left[\frac{1}{U}(U\sin\chi)'\right]' \\ R^{(4)}_{1515} &= -\frac{1}{UV}\left(\frac{1}{U}V'\right)' \\ R^{(4)}_{2323} &= \left(1-\frac{12m^2V^2}{(U\sin\chi)^2}\right)\frac{1}{(U\sin\chi)^2} - \frac{1}{U^2}\left[(\ln(U\sin\chi))'\right]^2 \\ R^{(4)}_{2525} &= 4m^2V^2/(U\sin\chi)^4 - \frac{1}{U^2}(\ln(U\sin\chi))'(\ln V)' \\ R^{(4)}_{3513} &= \frac{1}{U^2\sin\chi}\left(2mV/U\sin\chi\right)' \end{aligned} \tag{14}$$

From $R^{(4)}_{15} = 0$, we derive

$$\begin{aligned} U\sin\chi &= BV \\ B &= \text{const.} \end{aligned} \tag{15}$$

Substitute eq.(15) into eq.(14), we deduce $R^{(4)}_{1212} = R^{(4)}_{1515}$. From equations $R^{(4)}_{11} = \Lambda$ and eqs.(13), $R^{(4)}_{1212} = R^{(4)}_{2323} = R^{(4)}_{2525}$, we immediately obtain

$$\begin{aligned} R^{(4)}_{1212} &= -\frac{1}{B^2}\frac{\sin\chi}{V^2}\left[\frac{\sin\chi}{V}V'\right]' \\ R^{(4)}_{2323} &= \left(1-\frac{12m^2}{B^2}\right)\frac{1}{B^2V^2} - \frac{\sin^2\chi}{B^2V^4}V'^2 \\ R^{(4)}_{2525} &= \frac{4m^2}{B^4V^2} - \frac{\sin^2\chi}{B^2V^4}V'^2 \end{aligned} \tag{16}$$

by use of eq.(15). Now it is easy to solve the Einstein equation. At last, we have

$$B^2 = 16m^2 \tag{17-a}$$

$$V^2 = \frac{3}{64m^2\Lambda}U^2 \tag{17-b}$$

$$\frac{2U'}{U}\Big/\sqrt{1-U^2} = \pm\frac{1}{\sin\chi} \tag{17-c}$$

Integrate eq.(17-c), then

$$U^2 = \begin{cases} 4a\,\mathrm{tg}\frac{\chi}{2}\Big/\left(1+a\,\mathrm{tg}\frac{\chi}{2}\right)^2 \\ 4b\,\mathrm{ctg}\frac{\chi}{2}\Big/\left(1+b\,\mathrm{ctg}\frac{\chi}{2}\right)^2 \end{cases} \tag{18}$$

Consider first the first solution in eq.(18), when $\chi \to 0$, $V^2 \to 3a\chi/32m^2\Lambda$ and $U^2 \to 3a/2\Lambda\chi$. To avoid the singularity at $r = 0$, a must equal $8m\Lambda/3$. When $\chi \to \pi$, define $\chi = \pi - \gamma$, then $V^2 \to 3\gamma/32m^2\Lambda a$ and $U^2 \to 3/2\Lambda a\gamma$, we thus have $a = 1 = 8m\Lambda/3$.

When the same arguments applies to the second solution, we also have $b = 1 = 3m\Lambda/3$. The second solution is nothing but the first solution under transformation $\chi \to \pi - \chi$. Therefore we have only one solution

$$n^2 = 4\,\mathrm{tg}\frac{\chi}{2} \Big/ \left(1+\mathrm{tg}\frac{\chi}{2}\right)^2$$

$$\frac{8}{3} m \Lambda = 1$$

Because S^1 shrinks to zero whenever $\chi \to 0$ or $\chi \to \pi$, this solution is just the solution of a pair of monopoles and anti-monopoles. We expect that the 4-dimensional spatial space has $\chi = 2$ and $\tau = 0$. If we use the following formulas

$$\chi = \frac{1}{128\pi^2} \int_{M_4} \sqrt{g^{(4)}}\, \epsilon_{abef} R^{(4)}_{efgh} \epsilon_{cdgh} R^{(4)}_{abcd}\, d^4x \qquad (19)$$

$$\tau = \frac{1}{96\pi^2} \int_{M_4} \sqrt{g^{(4)}}\, R^{(4)}_{abcd} \epsilon_{cdef} R^{(4)}_{abef}\, d^4x \qquad (20)$$

we can actually get the expected result.

At $\chi = \frac{\pi}{2}$, the scale of x^5 is $\sqrt{\frac{4}{\Lambda}}\sin\tau$, and the scale of the 3d space is $\sqrt{\frac{1}{\Lambda}}\sin\tau$. So the two scales are the same. Under Wick rotation, solution (12) transformed into the Minkowskian sector

$$ds_5^2 = \frac{4}{\Lambda}\left(-dt^2 + \frac{\Lambda}{3}\,\mathrm{ch}^2 t\, ds_4^2\right) \qquad (21)$$

The scales thus are $\sqrt{\frac{1}{\Lambda}}\,\mathrm{ch}\,t$. The fifth dimension and the 3 dimensions expand equally.

We conclude that to get a solution in which the fifth dimension approaches constant, one must not use ansatz(7). Another possible way to obtain a satisfatory solution is to modify the Einstein equation.

At last, let us show that when $\Lambda = 0$, there is no regular solution. From eq.(9), $F' = $ const. and $R^{(4)}_{ij} = 3\delta_{ij}F'^2 = \Lambda$. If $F' \neq 0$, solution $R^{(4)}_{ij} = \Lambda > 0$ is the same as above. While $F = a\tau + b$, so $\tau = -b/a$ is a singular point. If $F' = 0$, the $R^{(4)}_{ij} = 0$. From eq.(16), solutions are $V^2 = 6(\mathrm{tg}\frac{\chi}{2})^{\pm 1}$. These solutions cannot be regular both at $\chi = 0$ and $\chi = \pi$.

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