IC/86/307

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OSCILLATORY AND NONOSCILLATORY PROPERTIES OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT DEVIATING ARGUMENTS

B.G. Zhang

and

N. Parhi

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1986 MIRAMARE-TRIESTE

IC/86/307

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OSCILLATORY AND NONOSCILLATORY PROPERTIES OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT DEVIATING ARGUMENTS *

B.G. Zhang ** and N. Parhi ***
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

In this paper, sufficient conditions have been obtained for oscillation and non-oscillation of solutions of first order differential equations with piecewise constant deviating arguments. These equations occur in mathematical models of certain biomedical problems.

MIRAMARE – TRIESTE
November 1986

* To be submitted for publication.

** Permanent address: Department of Mathematics, Shandong College of Oceanography, Qingdao, People's Republic of China.

*** Permanent address: Department of Mathematics, Berhampur University, Berhampur-760007, Orissa, India.

1. A great deal of work has been done and a large number of research articles have been published on the oscillation theory of first order delay differential equations of the form

$$y'(t) + p(t) f(y(g(t))) = 0 \tag{1}$$

with $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$. For details, the reader is referred to the survey article [6] and the monograph [5]. In a recent paper [1], Aftabizadeh and Wiener have obtained sufficient conditions for the oscillation of all solutions of the first order linear differential equations with piecewise constant deviating arguments of the type

$$y'(t) + q(t) y(t) + p(t) y([t]) = 0 \tag{2}$$

where p and q are real-valued continuous functions on $[0,\infty)$ and $[t]$ is the greatest integer function. We may note that $[t] \leq t$ and $\to\infty$ as $t \to \infty$. In most of the studies related to (1), $g(t)$ is not of the type such that $g(t) \leq t$. The equations of the type (2) are similar in structure to those found in [2]. The study of Eq.(2) is interesting because they occur in a natural way in mathematical models of some biological problems [2].

In this paper we consider

$$y'(t) + p(t) f(y([t])) = 0 \tag{3}$$

and forced equations

$$y'(t) + p(t) f(y([t])) = h(t), \tag{4}$$

where p and h are real-valued continuous functions on $[0,\infty)$ and $f : R \to R$ is continuous such that $yf(y) > 0$ for $y \neq 0$. Sufficient conditions have been obtained for nonoscillation of (3) and (4) in the second section. The third section deals with oscillatory properties of solutions of (3), (4), of differential equations with several delays of the form

$$y'(t) + \sum_{i=0}^{m} p_i(t) y([t-i]) = 0, \tag{5}$$

where $p_i(t)$, $i = 0,1,\dots,m$ is a real-valued continuous function on $[0,\infty)$, and of logistic equations with piecewise constant deviating arguments of the type

$$y'(t) + C(t)(1+y(t))\,y([t]) = 0, \tag{6}$$

where C is a real-valued continuous function on $[0,\infty)$ such that $C(t) > 0$. A comparison of a differential equation with piecewise constant deviating argument with a differential equation with constant delay has been given in the last section. It appears that the behaviour of solutions of (2) is not sufficiently close to that of the delay differential equations of the type (1). In Sec. 2, we prove that all solutions of (3) are nonoscillatory if $p(t) \leqslant 0$. However, this is not the case for an equation of the form (1). For example, the equation

$$y'(t) - y\left(t - \frac{3\pi}{2}\right) = 0$$

admits an oscillatory solution $y(t) = \sin t$.

By a solution of (3) or (4) on $[0,\infty)$ we mean a real-valued function $y(t)$ that satisfies the conditions: (i) $y(t)$ is continuous on $[0,\infty)$; (ii) the derivative $y'(t)$ exists at each point $t \in [0,\infty)$ with the possible exception of the points $[t] \in [0,\infty)$ where the right-hand derivative exists; (iii) Eq. (3) or (4) is satisfied on each interval $[n, n+1) \subset [0,\infty)$ with integral end points.

In this work we assume that Eqs. (3) or (4) admit a solution $y(t)$ on $[N_y,\infty)$, $N_y \geq 0$ is an integer, such that, for every $T \geqslant N_y$, $\sup\{|y(t)| : t \geqslant T\} > 0$. By a solution we mean a solution of this type.

A solution $y(t)$ of (3) or (4) is said to be oscillatory if there exists a sequence $\langle t_n \rangle$ such that $t_n \to \infty$ as $n \to \infty$ and $y(t_n) = 0$. $y(t)$ is said to be nonoscillatory if it is not oscillatory.

2. In this section we obtain sufficient conditions so that all solutions of (3) or (4) are nonoscillatory

Theorem 2.1 If $p(t) \leqslant 0$, then all solutions of (3) are nonoscillatory.

Proof Let $y(t)$ be a solution of (3) on $[N_y,\infty)$, where $N_y \geqslant 0$ is an integer. Let $y(n) = 0$ for some integer $n \geqslant N_y$. So, for $t \in [n, n+1)$, $y'(t) = 0$ and hence $y(t)$ is constant. Consequently, if $y(n) = 0$ for every integer $n \in [N_y,\infty)$, then from the continuity of $y(t)$ it follows that $y(t) \equiv 0$ on $[N_y,\infty)$. Since $y(t)$ is nontrivial in any neighbourhood

of infinity, there exists an integer $n_1 \geqslant N_y$ such that $y(n_1) \neq 0$. Let $y(n_1) > 0$. For $t \in [n_1, n_1+1)$, $y'(t) = -p(t)f(y(n_1)) \geqslant 0$. So $y(t) \geqslant y(n_1)$ for $t \geqslant n_1$. Hence $y(n_1+1) > 0$. This in turn implies that $y(n_1+2) > 0$ and so on. Hence $y(t) > 0$ for $t \geqslant n_1$. Similarly, $y(n_1) < 0$ implies that $y(t) < 0$ for $t \geqslant n_1$. Hence $y(t)$ is nonoscillatory.

This completes the proof of the theorem.

Remark The above theorem holds for

$$y'(t) + q(t)\,y(t) + p(t)\,f(y([t])) = 0,$$

where p and q are real-valued continuous functions on $[0,\infty)$ such that $p(t) \leqslant 0$.

In [1], it has been proved that

$$\limsup_{n\to\infty} \int_n^{n+1} p(t) \exp\left(\int_n^t q(s)\,ds\right) dt > 1 \tag{7}$$

implies that all solutions of (2) are oscillatory. When p and q are nonzero constants, then the condition (7) reduces to

$$p(e^q - 1)/q > 1 \tag{8}$$

and Eq. (2) takes the form

$$y'(t) + q\,y(t) + p\,y([t]) = 0. \tag{9}$$

They have proved that the condition (8) is necessary and sufficient for all solutions of (9) to be oscillatory. They have achieved this by showing that (9) admits only the trivial solution when $p(e^q-1) = q$ and

$$p(e^q - 1)/q < 1 \tag{10}$$

implies that all solutions of (9) are nonoscillatory. In the following we prove a result which generalizes the above result for (9) to (2).

Theorem 2.2 If $p(t) \geqslant 0$ and

$$\limsup_{n\to\infty} \int_n^{n+1} p(t) \exp\left(\int_n^t q(s)\,ds\right) dt < 1,$$

then all solutions of (2) are nonoscillatory.

Proof Let $y(t)$ be a solution of (2) on $[N_y,\infty)$, where $N_y \geq 0$ is an integer. From the given condition it follows that there exists an integer $N^* \geq N_y$ such that

$$\int_n^{n+1} p(t)\exp\left(\int_n^t q(s)ds\right)dt < 1$$

for $n \geq N^*$. Since $y(t)$ is nontrivial in any neighbourhood of infinity, there exists an integer $n_1 \geq N^*$ such that $y(n_1) \neq 0$. Let $y(n_1) > 0$. Eq.(2) may be written as

$$x'(t) + p(t)\exp\left(\int_{[t]}^t q(s)ds\right)x([t]) = 0,$$

where $x(t) = y(t)\exp\left(\int_{N^*}^t q(s)ds\right)$. Clearly $x(n_1) > 0$. For $t \in [n_1, n_1+1)$, $x'(t) \leq 0$ and

$$x(t) = x(n_1) - \int_{n_1}^t p(s)\exp\left(\int_{[s]}^s q(\theta)d\theta\right)x([s])ds$$

$$= x(n_1)\left\{1 - \int_{n_1}^t p(s)\exp\left(\int_{n_1}^s q(\theta)d\theta\right)ds\right\}$$

and hence

$$x(n_1+1) = x(n_1)\left\{1 - \int_{n_1}^{n_1+1} p(t)\exp\left(\int_{n_1}^t q(s)ds\right)dt\right\} > 0.$$

Consequently, $x(t) > 0$ for $t \in [n_1, n_1+1]$. Proceeding as above we may show that $x(n_1+2) > 0$. So $x(n_1) > 0$ implies that $x(t) > 0$ for $t \geq n_1$, that is, $y(n_1 > 0)$ implies that $y(t) > 0$ for $t \geq n_1$. Similarly it can be shown that $y(n_1) < 0$ implies that $y(t) < 0$ for $t \geq n_1$. So $y(t)$ is nonoscillatory.

Hence the proof of the theorem is completed.

Theorem 2.3 If $p(t) \geq 0$, $0 < f(x)/x \leq M$ for $x \neq 0$ and

$$\limsup_{n\to\infty} \int_n^{n+1} p(t)dt < 1/M,$$

then all solutions of (3) are nonoscillatory.

Proof Let $y(t)$ be a solution of (3) on $[N_y,\infty)$, where $N_y \geq 0$ is an integer. Clearly, there exists an integer $n_1 \geq N_y$ and $0 < \epsilon < 1/M$ such that

$$\int_n^{n+1} p(t)dt < \frac{1}{M} - \epsilon$$

for $n \geq n_1$. Since $y(t)$ is nontrivial in any neighbourhood of infinity, there exists an integer $n_2 \geq n_1$ such that $y(n_2) \neq 0$. Let $y(n_2) > 0$. Integrating (3) from n_2 to $t \in (n_2, n_2+1)$, we obtain

$$y(t) = y(n_2) - f(y(n_2))\int_{n_2}^t p(s)ds$$

$$\geq y(n_2) - f(y(n_2))\int_{n_2}^{n_2+1} p(s)ds$$

$$\geq y(n_2)\left[1 - \frac{f(y(n_2))}{y(n_2)}\int_{n_2}^{n_2+1} p(t)dt\right]$$

$$> 0.$$

Hence $y(t) > 0$ for $t \in [n_2, n_2+1]$. Repeating this process we can show that $y(t) > 0$ for $t \in [n_2+1, n_2+2]$. Hence $y(n_2) > 0$ implies that $y(t) > 0$ for $t \geq n_2$. Similarly, it can be proved that $y(n_2) < 0$ implies that $y(t) < 0$ for $t \geq n_2$.

Hence the theorem.

Remark Theorem 2.3 does not include the super linear or the sub-linear case. But the following theorem includes the super linear case.

Theorem 2.4 Let $p(t) \geq 0$. Let $|x| \leq K$ implies that $f(x)/x \leq M$ for $x \neq 0$. If

$$\lim_{n\to\infty} \int_n^{n+1} p(t)dt = 0,$$

then all bounded solutions of (3) are nonoscillatory.

The proof of this theorem is the same as that of Theorem 2.3 and hence is omitted.

Remark The proof of Theorem 2.3 or 2.4 runs smoothly if $p(t)$ is locally integrable instead of being continuous. In the following we give two examples to illustrate these theorems.

Example Consider

$$y'(t) + \frac{[t]}{t^2\left(1+\sin^2\frac{1}{[t]}\right)}\, y([t])\left(1+\sin^2 y([t])\right) = 0 \quad \text{for } t \geq 1$$

From Theorem 2.3 it follows that all solutions of the equation are nonoscillatory. In particular, $y(t) = \frac{1}{t}$ is a nonoscillatory solution of the equation.

Example $y'(t) + q[t]^3\, t^{-10}\, y^3([t]) = 0,\ t \geq 1.$

From Theorem 2.4 it follows that all bounded solutions of the equation are nonoscillatory. In particular, $y(t) = t^{-9}$ is a bounded nonoscillatory solution of the equation.

Theorem 2.5 Let $h(t) > 0$ and $\lim_{t\to\infty} \frac{h(t)}{|p(t)|} = \infty$, whenever it is defined. Then all bounded solutions of (4) are nonoscillatory.

Proof Let $y(t)$ be a bounded solution of (4) on $[N_y,\infty)$ such that $|y(t)| \leq M$ for $t \in [N_y,\infty)$, where $N_y \geq 0$ is an integer. Since f is continuous, there exists a constant $K > 0$ such that $|f(u)| \leq K$ for $u \in [-M,M]$. From the given hypothesis it follows that there exists a $T \geq N_y$ such that $h(t) \geq K|p(t)|$ for $t \geq T$. If $y(t)$ is oscillatory, then there exists a sequence $\langle t_m \rangle$ such that $y(t_m) = 0$ and $t_m \to \infty$ as $m \to \infty$. Choose m_1 sufficiently large so that $t_{m_1} > T$. Now integrating (4) from t_{m_1} to t_{m_1+1}, we get

$$0 = \int_{t_{m_1}}^{t_{m_1+1}} [\,h(t) - p(t) f(y([t]))\,]\,dt$$

$$\geq \int_{t_{m_1}}^{t_{m_1+1}} [\,h(t) - K|p(t)|\,]\,dt$$

$$> 0,$$

a contradiction.

Hence the theorem.

Example Consider

$$y'(t) - e^{3[t]-5t}\, y^3([t]) = e^{-t} + e^{-5t}, \quad t \geq 0.$$

Clearly,

$$\frac{h(t)}{|p(t)|} \geq e^t + e^{-3t}.$$

From Theorem 2.5 it follows that all bounded solutions of the equation are nonoscillatory. $y(t) = -e^{-t}$ is a bounded nonoscillatory solution of the equation.

3. This section is concerned with oscillatory behaviour of solutions of Eq.(3), (4), (5) and (6).

Theorem 3.1 If $p(t) \geq 0$

$$\lim_{x\to 0}\,(x/f(x)) = M < \infty \quad \text{and} \quad \limsup_{n\to\infty} \int_n^{n+1} p(t)\,dt > M,$$

then all solutions of (3) are oscillatory.

Proof Let $y(t)$ be a solution of (3) on $[N_y,\infty)$, where $N_y \geq 0$ is an integer. If possible, let $y(t)$ be nonoscillatory. We may assume, without any loss of generality, that $y(t) > 0$ for $t \geq N^* \geq N_y$. So $y'(t) \leq 0$ for $t \geq N^*$ and hence $\lim_{t\to\infty} y(t) = \alpha \geq 0$. Let $\alpha > 0$. From the given condition it follows that there exists an $\epsilon > 0$ and a sequence $\langle n_j \rangle$ such that $n_j \to \infty$ and

$$\int_{n_j}^{n_j+1} p(t)\,dt > M + \epsilon.$$

Since $f(u)$ is continuous, $f(y(n_j)) \to f(\alpha)$ as $n_j \to \infty$. So, for $0 < \eta < f(\alpha)$, there exists an integer N such that $f(y(n_j)) > f(\alpha) - \eta$ for $n_j \geq N$. Now integrating (3) from N to $N+m$, where $m > 0$ is an integer, we obtain

$$y(N+m) - y(N) = -\int_N^{N+m} p(t) f(y([t]))\,dt$$

$$= -f(y(N))\int_N^{N+1} p(t)dt - f(y(N+1))\int_{N+1}^{N+2} p(t)dt - \cdots - f(y(N+m-1))\int_{N+m-1}^{N+m} p(t)dt,$$

that is, $y(N+m) \leq y(N) - m(f(\alpha) - \eta)(M+\epsilon)$. This in turn implies that $y(N+m) < 0$ for sufficiently large m, a contradiction. Suppose that $\alpha = 0$. Choosing $n_j \geq N^*$ and integrating (3) from n_j to n_j+1, we get

$$-y(n_j) < y(n_j+1) - y(n_j) = -f(y(n_j)) \int_{n_j}^{n_j+1} p(t)\,dt\,,$$

so

$$\limsup_{n_j\to\infty} \int_{n_j}^{n_j+1} p(t)\,dt \le \limsup_{n_j\to\infty} \frac{y(n_j)}{f(y(n_j))} = M\,,$$

a contradiction again.

Hence the theorem.

Remark We may note that Theorem 3.1 includes sublinear case but does not include superlinear case. Moreover, in Theorem 3.1, $p(t)$ need not be continuous everywhere in $[0,\infty)$. It is enough if $p(t)$ is continuous on $[0,\infty)$ except possibly at integral points.

The nonhomogeneous equation corresponding to (2) is given by

$$y'(t) + q(t)y(t) + p(t)y([t]) = h(t), \qquad (11)$$

where h is a real-valued continuous function on $[0,\infty)$.

Theorem 3.2 Let $p(t) \ge 0$ and $q(t) \ge 0$. Suppose there exists a function H, two constants a and b and two sequences $\langle s_m\rangle$ and $\langle t_m\rangle$ such that $H'(t) = h(t)$ everywhere on $[0,\infty)$ except possibly at integral points, $H(s_m) = a$, $H(t_m) = b$, $s_m \to \infty$, $t_m \to \infty$ and $a \le H(t) \le b$. If

$$\limsup_{n\to\infty} \int_n^{n+1} p(t) \exp\left(\int_n^t q(s)ds\right) dt > 1\,,$$

then all solutions of (11) are oscillatory.

Proof Let $y(t)$ be a nonoscillatory solution of (11) on $[N_y,\infty)$, $N_y \ge 0$ is an integer. Let $y(t) > 0$ for $t \ge N \ge N_y$. Setting $x(t) = y(t)-H(t)$, we get

$$x'(t) = y'(t) - h(t) = -q(t)y(t) - p(t)y([t]) \le 0$$

for $t \ge N$. There exists $N^* \ge N$ such that $x(t) + a > 0$ for $t \ge N^*$. If not, we can find a large t_1 such that $x(t_1) + a \le 0$. Since $x(t)$ is nonincreasing, $x(t) + a < 0$ for large t. But, for a large s_m, $x(s_m) + a = y(s_m) > 0$, a contradiction. Hence our claim holds. Setting $z(t) = x(t) + a$, we obtain

$$\begin{aligned} z'(t) &= -q(t)y(t) - p(t)y([t]) \\ &= -q(t)(x(t)+H(t)) - p(t)(x([t]) + H([t])) \\ &\le -q(t)z(t) - p(t)z([t])\,. \end{aligned}$$

So $z'(t) + q(t)z(t) + p(t)z([t]) \le 0$ admits an ultimately positive solution, a contradiction (see [1]).

A similar contradiction may be obtained when $y(t)$ is ultimately negative.

Hence the theorem.

Theorem 3.3 Suppose that $p(t) \ge 0$

$$\liminf_{n\to\infty} \int_n^{n+1} p(t) \exp\left(\int_n^t q(s)ds\right) dt > 1$$

$$\liminf_{n\to\infty} \int_n^{n+1} h(t) \exp\left(\int^t q(s)ds\right) dt < 0$$

and

$$\limsup_{n\to\infty} \int_n^{n+1} h(t) \exp\left(\int^t q(s)ds\right) dt > 0\,.$$

Then all solutions of (11) are oscillatory.

Proof Let $y(t)$ be a nonoscillatory solution of (11). We assume that $y(t) > 0$ for $t \ge N$. The proof for the case when $y(t) < 0$ for $t \ge N$ is similar. From the given condition it follows that there exists a sequence $\langle n_j\rangle$ such that $n_j \to \infty$ and

$$\int_{n_j}^{n_j+1} h(t) \exp\left(\int^t q(s)ds\right) dt < 0\,.$$

Setting $x(t) = y(t)\exp\left(\int_N^t q(s)ds\right)$, we may write (11) as

$$x'(t) + p(t)\exp\left(\int_{[t]}^t q(s)ds\right) x([t]) = h(t)\exp\left(\int_N^t q(s)ds\right). \qquad (12)$$

Choose $n_j \geq N$. From (12) we get

$$0 \leq x(n_j+1) = x(n_j)\left\{1 - \int_{n_j}^{n_j+1} p(t)\exp\left(\int_{n_j}^{t} q(s)ds\right)dt\right\}$$

$$+ \int_{n_j}^{n_j+1} h(t)\exp\left(\int_{N}^{t} q(s)ds\right)dt$$

$$< 0,$$

a contradiction.

This completes the proof of the theorem.

<u>Theorem 3.4</u> If $p(t) \geq 0$,

$$\liminf_{t\to\infty} \int_0^t h(s)ds = -\infty \quad \text{and} \quad \limsup_{t\to\infty} \int_0^t h(s)ds = \infty,$$

then all solutions of (4) are oscillatory.

<u>Proof</u> Let $y(t)$ be a solution of (4) on $[N_y,\infty)$ such that $y(t) > 0$ for $t \geq N \geq N_y$, where $N_y \geq 0$ is an integer. So $y([t]) > 0$ for $t \geq N+1$. Consequently, from (4) we obtain

$$y(t) \leq y(N+1) + \int_{N+1}^{t} h(s)ds.$$

so $\liminf\limits_{t\to\infty} y(t) < 0$, a contradiction. A similar contradiction is obtained when $y(t) < 0$ for $t \geq N$.

Hence the theorem.

<u>Theorem 3.5</u> Let $f(u)$ be nondecreasing. Let $p(t) \geq 0$ be such that

$$\int^{\infty} p(t)dt = \infty.$$

If there exists an oscillatory function H such that $H'(t) = h(t)$ everywhere on $[0,\infty)$ except possibly at integral points and $\lim\limits_{t\to\infty} H(t)$ exists, then a solution of (4) tends to zero as $t \to \infty$ or is oscillatory.

<u>Proof</u> Let $y(t) > 0$ for $t \geq N \geq N_y$, where $y(t)$ is a solution of (4) on $[N_y,\infty)$ and $N_y \geq 0$ is an integer. Setting $x(t) = y(t) - H(t)$, we obtain $x'(t) = -p(t)f(y([t]))$ everywhere except possibly at integral points in $[0,\infty)$. Clearly, $x(t) > 0$ for large t. For $t \geq N+1$, $x'(t) \leq 0$ and hence $\lim\limits_{t\to\infty} x(t)$ exists. Consequently, $\lim\limits_{t\to\infty} y(t) = \lambda$ exists. Suppose that $\lambda > 0$. For $0 < \epsilon < \lambda$, there exists an integer $n > N+1$ such that $y(t) > \lambda - \epsilon$ for $t \geq n$. Now, for $t \geq n+1$,

$$x(t) \leq x(n+1) - f(\lambda-\epsilon)\int_{n+1}^{t} p(s)ds.$$

This in turn implies that $x(t) < 0$ for large t, a contradiction. A similar contradiction is obtained if $y(t) < 0$ for large t.

Hence the theorem is proved.

<u>Theorem 3.6</u> If $p_i(t) \geq 0$ for $i = 0,1,\ldots,m$ and

$$\limsup_{n\to\infty} \sum_{i=0}^{m} \left(\int_{n+m}^{n+m+1} p_i(t)dt\right) > 1, \tag{13}$$

then all solutions of (5) are oscillatory.

<u>Proof</u> Let $y(t) > 0$ for $t \geq N$. Choose $n \geq N+m$. For $t \in [n,n+1)$ we get

$$y'(t) = -\sum_{i=0}^{m} p_i(t)\, y(n-i) \leq 0$$

Again, for $t \in [n+1, n+2)$

$$y'(t) = -\sum_{i=0}^{m} p_i(t)\, y(n+1-i) \leq 0$$

and so on. So $y(t)$ is nonincreasing for $t \geq n$. From (5) we obtain, for $t \in [n+m, n+m+1)$,

$$0 < y(n+m+1) = y(n+m) - \int_{n+m}^{n+m+1}\left(\sum_{i=0}^{m} p_i(t)\, y(n+m-i)\right)dt$$

$$\leq y(n+m)\left\{1 - \int_{n+m}^{n+m+1}\left(\sum_{i=0}^{m} p_i(t)\right)dt\right.$$

so

$$\limsup_{n\to\infty} \int_{n+m}^{n+m+1}\left(\sum_{i=0}^{m} p_i(t)\right)dt \leq 1,$$

a contradiction.

Hence the theorem.

Remark Theorem 3.6 can be extended to

$$y'(t) + q(t)y(t) + \sum_{i=0}^{m} p_i(t)\, y([t-i]) = 0,$$

where $p_i(t) \geq 0$, $i = 0,1,2,\ldots,m$. In this case the condition (13) is replaced by the condition

$$\limsup_{n\to\infty} \sum_{i=0}^{m} \left(\int_{n+m}^{n+m+1} p_i(t) \exp\left(\int_{n+m-i}^{t} q(s)ds \right) dt \right) > 1.$$

In the following we prove a result concerning oscillatory behaviour of solutions of logistic Eq. (6).

Theorem 3.7 If

$$\limsup_{n\to\infty} \int_n^{n+1} C(t)dt > 1,$$

then all solutions of (6) are oscillatory.

Proof Let $y(t)$ be a solution of (6) on $[0,\infty)$. From the ecology theory it follows that $1 + y(0) > 0$. Integrating (6) we obtain $1 + y(t) > 0$ for $t \geq 0$.

If possible, let $y(t) > 0$ for $t \geq N > 0$. Choosing $n \geq N+1$, we see that $y'(t) \leq 0$ for $t \in [n, n+1)$. Integrating (6) from n to $n+1$, we get

$$y(n+1) - y(n) = -y(n) \int_n^{n+1} C(t)(1+y(t))\,dt$$

$$\leq -y(n)(1+y(n+1)) \int_n^{n+1} C(t)dt.$$

Hence

$$0 < y(n+1) \leq y(n) \left\{ 1 - (1+y(n+1)) \int_n^{n+1} C(t)dt \right\}.$$

Consequently,

$$(1+y(n+1)) \int_n^{n+1} C(t)dt < 1,$$

that is,

$$\int_n^{n+1} C(t)dt < \frac{1}{1+y(n+1)} < 1.$$

so

$$\limsup_{n\to\infty} \int_n^{n+1} C(t)dt \leq 1,$$

a contradiction. If $y(t) < 0$ for $t \geq N > 0$, then $y'(t) > 0$ for $t \geq N+1$. Hence $\lim_{t\to\infty} y(t) = \alpha \leq 0$ exists. Suppose that $\alpha = 0$. Choosing $n \geq N+1$ and proceeding as above we arrive at

$$0 > y(n+1) = y(n) \left\{ 1 - (1+y(n)) \int_n^{n+1} C(t)dt \right\}.$$

This in turn gives us

$$\limsup_{n\to\infty} \int_n^{n+1} C(t)dt \leq \lim_{n\to\infty} \frac{1}{1+y(n)} = 1,$$

a contradiction. Next suppose that $\alpha < 0$. Since $1 + y(t)$ is positive and increasing, $\lim_{t\to\infty}(1 + y(t)) > 0$, that is, $1 + \alpha > 0$. Choosing $0 < \epsilon < \min\{1+\alpha, -\alpha\}$, we obtain $-y([t])(1 + y(t)) > \beta > 0$, where $\beta = -(\alpha+\epsilon)(1+\alpha-\epsilon)$, for $t \geq t_0 \geq N+2$. Hence integrating (6) from t_0 to t, we get

$$y(t) > y(t_0) + \beta \int_{t_0}^{t} C(s)ds.$$

From the given condition it follows that

$$\int_{t_0}^{\infty} C(t)dt = \infty.$$

Hence $y(t) > 0$ for large t, a contradiction.

This completes the proof of the theorem.

4. In this section we compare a differential equation with piecewise constant deviating argument

$$y'(t) + p\, y([t]) = 0 \tag{14}$$

with the differential equation with average delay

$$y'(t) + p\,y(t - \tfrac{1}{2}) = 0, \tag{15}$$

where p is a non-zero constant. We call (15) a differential equation with average delay because the delay $t - [t]$ in (14) satisfies

$$\int_{n}^{n+1} \{ t - [t] \}\, dt = \frac{1}{2}$$

for all n. If $p < 0$, then all solutions of (14) are nonoscillatory (Theorem 2.1). But (15) may admit an oscillatory solution. Indeed,

$$y'(t) - 3\pi\, y(t - \tfrac{1}{2}) = 0$$

admits an oscillatory solution $y(t) = \cos 3\pi t$. If $0 < p < 1$, then all solutions of (14) are nonoscillatory (Theorem 2.2). If $1 > p > 2/e$, then all solutions of (15) are oscillatory (see [4]). For $0 < p \leq 2/e$, Eq.(15) admits at least one nonoscillatory solution (see [4]). For $p = 1$, Eq.(14) does not admit a nontrivial solution. Indeed, integrating (14) from n to $n+1$, we get $y(n+1) = 0$. For $t \in [n+1, n+2)$, $y'(t) = 0$ and hence $y(t) = 0$ for $t \in [n+1, n+2]$. Proceeding as above we get $y(t) = 0$ for $t \geq n+1$. On the other hand, all solutions of (15) with $p = 1$ are oscillatory because $e > 2$ (see [4]). If $p > 1$, then all solutions of (14) are oscillatory (see [1]) as well as all solutions of (15) are oscillatory (see [4]).

Hence solutions of (15) are more oscillatory in nature than those of (14). So it appears that the delay $t - [t]$ cannot be replaced by its average $\frac{1}{2}$.

ACKNOWLEDGMENTS

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] A.R. Aftabizadeh and J. Wiener, "Oscillatory properties of first order linear functional differential equations", Applicable Analysis <u>20</u> (1985), 165-187.

[2] S. Busenberg and K.L. Cooke, "Models of vertically transmitted diseases with sequential continuous dynamics", in <u>Nonlinear Phenomena in Mathematical Sciences</u>, Ed. V. Lakshmikantham (Academic Press, New York, 1982), 179-187.

[3] R.D. Driver, <u>Ordinary and Delay Differential Equations</u> (Springer Verlag, New York, Berlin, 1977).

[4] G. Ladas, "Sharp conditions for oscillations caused by dealys", Applicable Analysis <u>9</u> (1979) 93-98.

[5] V. Lakshmikantham, G.S. Ladde and B.G. Zhang, <u>Oscillation Theory of Differential Equations with Deviating Arguments</u> (Marcel Dekker Inc., New York, to appear).

[6] B.G. Zhang, "A survey of oscillation of solutions to first order differential equations with deviating arguments" in <u>Trends in the Theory and Practice of Nonlinear Analysis</u>, Ed. V. Lakshmikantham (1985) 475-483.