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ABSTRACT

By enlarging the functional space to include nonlocal fields which are sensitive to the space-time asymptotics of the configurations we can formally construct the θ-sectors in the OS-Hilbert space. On two quantum mechanical examples and in the case of non-Abelian gauge theories we study the question of inequivalence of the different θ-sectors.

АННОТАЦИЯ

Включая при расширении функционального пространства нелокальные поля, чувствительные к пространственно-временной асимптотике конфигураций, имеется возможность формально конструировать θ-секторы в пространстве OS-Гильберта. На двух квантовомеханических примерах, а также в случае некоммутирующего калибровочного поля изучается вопрос неэквивалентности разных θ-секторов.

KIVONAT

A konfigurációs téren értelmezett funkcionálokat ugy kiterjesztve, hogy olyan nemlokális tereket is tartalmazzon, amelyek érzékenyek a konfigurációk téridő aszimptotikájára, formálisan megkonstruáljuk az OS-Hilbert térben a θ-szektorokat. Két kvantummechanikai példában és nem-Abeli mértékelméletben tanulmányozzuk a különböző θ-szektorok inekivalenciájának a kérdését.

1. Introduction

It has been well known for a long time that non-Abelian gauge theories in four dimensions and U(1) gauge theories in two dimensions have non-trivial topological structure [1]. Several authors argued how this topological structure may affect the quantum theory [2]. The expectation is that there exist different representations of the quantum field algebra corresponding to the so called θ-sectors. So far it has been rigorously established only in 2-dimensional Abelian gauge theories [3].

The severe problem obstructing a precise treatment is that the most efficient method to control the ultraviolet behaviour of the theory, the lattice approach does not admit a representation of the topological structure on the lattice configuration space. Since we cannot solve this problem either we concentrate in this paper on the question how the presence of a topological structure in the configuration space modifies the OS-construction [4], the most promising method in constructive quantum field theory. In this way we can point out the conditions which lead to the existence of θ-sectors as different representations of the local quantum field algebra. For pedagogical reasons it will be useful to compare the cases of 1) quantum mechanics in a periodical potential, 2) quantum pendulum and 3) non-Abelian gauge theories in 4-dimensions which will be discussed in Sections 2,3 and 4 respectively.

Although the results obtained for those three models are widely known we believe that our methodically new approach may be useful in the future in a rigorous construction of the θ-sectors.

2. θ-sectors in quantum mechanics:periodic potential

Consider a particle in one space dimension moving in a bounded periodic potential V,that is:

$$m \geq V(x) \geq 0 \;; \quad x \in \mathbf{R}, \tag{2.1a}$$

$$V(x+d) = V(x) \;; \quad x \in \mathbf{R}. \tag{2.1b}$$

We denote the minima of the potential by $x_i, i \in Z$; of course they also show the periodicity: $x_i = x_0 + id,\ i \in Z$. We assume that $V(x_i) = 0$.

The Minkowskian Lagrangian and the physical weight in the path integral measure:

$$L_M = \frac{1}{2} \cdot \dot{x}^2 - V(x), \tag{2.2a}$$

$$\exp(iS_M) = \exp(i\int L_M). \tag{2.2b}$$

Because we want to make the OS-construction to get the quantum mechanics of the model we need the Euclidean version of these quantities:

$$L_E = \frac{1}{2} \cdot \dot{x}^2 + V(x), \tag{2.3a}$$

$$\exp(-S_E) = \exp(-\int L_E). \tag{2.3b}$$

We define the configuration space of the classical paths as:

$$\mathfrak{C} \equiv \bigcup_{T<\infty} \mathfrak{C}^T \quad , \tag{2.4a}$$

$$\mathfrak{C}^T \equiv \{ x : \mathbf{R} \to \mathbf{R} \mid x \in C^0(\mathbf{R}); \quad i,j \in \mathbb{Z} : x(t)=x_i, \ x(-t)=x_j \text{ for } t \geq T \}. \tag{2.4b}$$

At first sight it may seem insufficient to consider this configuration space $\mathfrak{C}$ the paths of which all have finite actions S_E. The main argument against it is that it is a zero measure set in the larger space $\tilde{\mathfrak{C}} = C^0(\mathbf{R})$ equipped with the pointwise convergece topology and the corresponding Borel σ-algebra of measurable sets. However we have two reasons for not using measures on $\tilde{\mathfrak{C}}$ only on $\mathfrak{C}^T$, $T<\infty$. The first is a pragmatic one: everything is constructed through the thermodynamical limit (in our quantum mechanical example through the $T \to \infty$ limit) therefore it is irrelevant whether the limit of integrals on $\mathcal{C}^T$ is again an integral on $\tilde{\mathcal{C}}$ or defines only a state. The other reason is the more decisive. We want to give a definite meaning to the "winding number" $W[x] = x(\infty)/d$, of a path x and to the corresponding "Pontrjagin number", $x(\infty)/d - x(-\infty)/d$.

Let $W: \tilde{\mathcal{C}} \longrightarrow \mathbf{R}$ be a function yielding a kind of a winding number $W[x]$ for a path $x \in \tilde{\mathcal{C}}$. The natural requirement for this quantity is that it should depend only on the asymptotics of x at $t \to \infty$. That is if one allows x to vary with the condition that $x|_{(-\infty, T]}$ is fixed one has to recover all the possible values for $W[x]$, whether but finite value T was. Now we prove that if W is not constant then it cannot be Borel-measurable. Namely, in that case we can decompose $\tilde{\mathcal{C}}$ into two disjoint non-empty sets

$$= W^{-1}((-\infty, w\ ]) \cup W^{-1}((w\ , \infty))$$

for some $w \in \mathbf{R}$. The simple sets from which the Borel algebra is built up have the general form:

$$\mathcal{U}_x(t_1, \ldots, t_n | \delta) = \{ y \in \tilde{\mathcal{C}} \mid \delta > |y(t_i) - x(t_i)| \ ; \ i = 1, \ldots, n \}.$$

From the above requirement on the winding number W and from the fact that in the simple sets the configuration is restricted only at finite number of points it follows that any simple set $\mathcal{U}$ intersects both $W^{-1}((-\infty, w\ ])$ and $W^{-1}((w\ , \infty))$.

Therefore both sets have internal measure zero and external measure one, which means that W is not measurable.

Now we define the class of local functionals on $\mathfrak{C}$:

$$\mathfrak{C}^*_{loc} \equiv \{ f : \mathfrak{C} \to \mathbf{C} \mid |\text{sens}\, f| < \infty \} \qquad (2.5)$$

Here $|\text{sens}\, f|$ denotes the Lebesgue measure of the set sens f which is itself the "sensitivity domain" of the functional and has the following properties:

i) sens $f \subset D_x = \mathbf{R}$,

ii) for any $Q \subset \mathbf{R}$, if $x|_Q = y|_Q$ implies $f(x) = f(y)$, then $Q \supset$ sens f .

The functionals in $\mathfrak{C}^*_{loc}$ being local do not know about the asymptotics of a path from $\mathfrak{C}$. Thus we will use a wider class of functionals, the dual space of $\mathfrak{C}$, which contains nonlocal functionals as well.

$$\mathfrak{C}^* \equiv \{ f : \mathfrak{C} \to \mathbf{C} \}. \qquad (2.6)$$

Now we define $\mathfrak{C}^*_+$, which is the subspace of those functionals which are sensitive only for positive times:

$$\mathfrak{C}^*_+ \equiv \{ f \in \mathfrak{C}^* \mid \text{sens}\, f \subset [0,\infty) \}. \qquad (2.7)$$

We can divide $\mathfrak{C}$ and $\mathfrak{C}^*_+$ into disjoint sets from their asymptotic behaviour point of view:

$$\mathcal{C} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{C}_{ij} \; ; \; \mathcal{C}_{ij} = \{ x \in \mathcal{C} \mid \lim x(-t) = x_i ,$$

$$\lim x(t) = x_j \}, \tag{2.8a}$$

$$\mathcal{C}^*_+ = \bigoplus_j \mathcal{C}^*_{+j} \; ; \mathcal{C}^*_{+j} = \{ f \in \mathcal{C}^*_+ \mid \operatorname{supp} f = \bigcup_i \mathcal{C}_{ij} \}. \tag{2.8b}$$

The time reflection on $\mathcal{C}$ and $\mathcal{C}^*$ is

$$(\Theta x)(t) = x(-t) \; ; \; x \in \mathcal{C} , \tag{2.9a}$$

$$(\Theta f)[x] = \overline{f[\Theta x]} \; ; \; f \in \mathcal{C}^* . \tag{2.9b}$$

Now we define a pre-Hilbert space from $\mathcal{C}^*_+$ if there is a "measure" on the configuration space (more precisely a state on $\mathcal{C}^*_+$) which has the property of reflection positivity, that is:

$$\langle (\Theta f) f \rangle_{\mathcal{C}} \geq 0 \; , \forall f \in \mathcal{C}^*_+ . \tag{2.10}$$

The inner product on $\mathcal{C}^*_+$ then is defined by

$$\langle f , \psi \rangle \equiv \langle (\Theta f) \psi \rangle_{\mathcal{C}} , \tag{2.11}$$

which is positive semidefinite due to (2.10). We do not construct the "measure" on $\mathcal{C}$, but we will assume the following properties of it:

i) reflection positivity,

ii) $\langle \varphi_i , \psi_j \rangle = F(\varphi_0 , \psi_0 ; |i-j|)$; $\varphi_i \in \mathcal{C}^*_{+i}$,
$\psi_j \in \mathcal{C}^*_{+j}$.

where $\varphi_0 = U^{-i} \varphi_i$, $\psi_0 = U^{-j} \psi_j$ and U is the translation by the period d of the potential. So property ii) is a restricted translational invariance.

Using the inner product defined by (2.11) we can give a seminorm on $\mathcal{C}^*_+$:

$$\| \varphi \|^2 = \langle \varphi , \varphi \rangle . \tag{2.12}$$

Let us denote the subspace of the functionals with zero norm by $\mathcal{N}$. Then we get the OS-Hilbert space after factorizing $\mathcal{C}^*_+$ by $\mathcal{N}$ and make this factor space complete:

$$\mathcal{H} = (\mathcal{C}^*_+ / \mathcal{N})^{compl} . \tag{2.13}$$

If the $\mathcal{C}^*_{+i}$ sectors were orthogonal then we could factorize in every subspace independently, and our Hilbert space would be the direct sum of these factor spaces. One can see that easily. Let $\eta \in \mathcal{N}$ then we know from (2.8b) that

$$\eta = \bigoplus_i \eta_i \quad ; \quad \eta_i \in \mathcal{C}^*_{+i} , \tag{2.14}$$

and this decomposition is unique. If the subspaces are orthogonal then

$$0 = \langle \eta , \eta \rangle = \sum_i \langle \eta_i , \eta_i \rangle , \tag{2.15}$$

and it follows that $\langle \eta_i, \eta_i \rangle = 0$, $\eta_i \in \mathcal{N}$; $i \in Z$. That is

$$\mathcal{N} = \bigoplus_i \mathcal{N}_i \, . \tag{2.16}$$

Therefore

$$\mathcal{H} = \Big(\bigoplus_{i \in Z} \; \mathcal{C}^*_{+i} / \mathcal{N}_i \Big)^{\mathrm{compl}} \, . \tag{2.17}$$

However the existence of instanton or kink solutions of the classical (Euclidean) equation of motion suggests that the subspaces $\mathcal{C}^*_{+i}$ should not be orthogonal.

Let $f_i \in \mathcal{C}^*_{+i}$, $\psi_j \in \mathcal{C}^*_{+j}$ and examine their inner product:

$$\langle f_i , \psi_j \rangle = \langle (\Theta f_i) \psi_j \rangle = F(f_0 , \psi_0 ; |i-j|). \tag{2.18}$$

If $|i-j| = 1$ there is a path being a classical solution which contribute to the integral. It starts from x_i at $t=-\infty$ and reaches x_j as $t \to \infty$. So in the quasiclassical approximation the inner product (2.18) is not zero. We assume that this is valid also in the exact calculations, that is the quantum fluctuations do not obliterate this property. But due to translational and time reflection invariances the inner product depends on the relative asymptotics of the functionals only, and we can use this property to define orthogonal sectors on the functional space $\mathcal{C}^*_+$.

Suppose that we know the functional measure $d\mu^T(x)$ on the configuration space $\mathcal{C}_{nm}(T)$ consisting of paths such that

$$x(t) = \begin{cases} x_m , & t \geq T \\ x_n , & t \leq -T . \end{cases} \tag{2.19}$$

Then we can define the inner product on $\mathcal{C}_+^*$ by a limit prorcedure:

$$\langle \varphi , \psi \rangle \equiv \langle (\Theta \varphi) \psi \rangle_{\mathcal{C}} \equiv \lim_{T\to\infty} \lim_{K\to\infty} \lim_{N\to\infty} \frac{1}{Z_{NK}^T} \cdot \\ \cdot \sum_{k=-K}^{K} \sum_{n=-N}^{N} \int_{\mathcal{C}_{n,n+k}(T)} d\mu^T(x) \; \overline{\varphi[\theta x]} \; \psi[x] , \tag{2.20}$$

where Z_{NK}^T is a normalization factor which will be fixed later. We expect that the orthogonal sectors in $\mathcal{C}_+^*$ will be the "θ-sectors". We define functionals - φ_θ - from the functionals - φ - in the $\mathcal{C}_{+0}^*$ subspace by the formal sum:

$$\varphi_\theta \equiv \sum_{k\in\mathbb{Z}} \exp(i\theta k) \; U^k \varphi \quad . \tag{2.21}$$

Their values are well defined for every path in $\mathcal{C}$ because all but one of the terms are zero:

$$\varphi_\theta[x] \equiv \sum_k \exp(i\theta k) \; (U^k \varphi)[x] = \\ = \exp(i\theta m) \; (U^m \varphi)[x] \; ; \; x \in \mathcal{C}_{nm} . \tag{2.22}$$

f_θ has the useful property:

$$(U f_\theta)[x] = \sum_k \exp(i\theta k) \; (U^{k+1} f)[x] =$$

$$= \exp(-i\theta) \sum_k \exp(i\theta k) \; (U^k f)[x] = \qquad (2.23)$$

$$= \exp(-i\theta) \; f_\theta[x] \; ; \quad \forall x \in \mathcal{C} .$$

Because

$$(U^n f_\theta)[x] = f_\theta[U^{-n} x] = \exp(-i\theta n) \cdot f_\theta[x], \qquad (2.24)$$

if $\theta \neq \theta'$ then

$$\langle f_\theta , \Psi_{\theta'} \rangle = \lim_{T\to\infty} \lim_{K\to\infty} \lim_{N\to\infty} \frac{1}{Z^T_{NK}} \sum_{k=-K}^{K} \int_{\mathcal{C}_{0,k}(T)} d\mu^T[x] \cdot$$

$$\overline{f_\theta[\theta x]} \; \Psi_{\theta'}[x] \sum_{n=-N}^{N} \exp(in(\theta'-\theta)) =$$

$$(2.25)$$

$$= \{ \lim_{T\to\infty} \lim_{K\to\infty} \frac{1}{Z^T_K} \sum_{k=-K}^{K} \int_{\mathcal{C}_{0,k}(T)} d\mu^T[x] \; \overline{f_\theta[\theta x]} \; \Psi_{\theta'}[x] \}$$

$$\{ \lim_{N\to\infty} \frac{1}{Z_N} \sum_{n=-N}^{N} \exp(in(\theta'-\theta)) \} = 0 ,$$

due to the limit in N. In (2.25) we factorized Z_{NK} as: Z^T_{NK} = Z_K Z_N. If $\theta = \theta'$ we have to decide whether we want the functionals to have finite norm or not.

In the first case we get a nonseparable space of the functionals because we have an uncountable orthonormal set in it:

$$\{ \frac{1}{\|f_\theta\|} \cdot f_\theta \mid \theta \in [0,2\pi) \}. \tag{2.26}$$

In this case we have to choose $Z_N = 2N+1$ – thus the limit value of the second bracket in (2.25) is one– and have to ensure by Z_K the finitness of the limit in the first bracket of (2.25), which we denote by $\langle f_\theta , \psi_\theta \rangle_\theta$.

In the second case the functional space is the direct integral space of the θ-sectors. Namely in this case we choose $Z_N = 1$ and keep $\langle f_\theta , \psi_\theta \rangle_\theta$ finite as before. Then from the identities

$$f_n = \int \frac{d\theta}{2\pi} \cdot \exp(-i\theta n) \cdot f_\theta \ , \tag{2.27a}$$

$$\langle f_n , f_n \rangle = \iint \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} \exp(-in(\theta-\theta')) \langle f_\theta , f_{\theta'} \rangle =$$

$$= \iint \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} \cdot \exp(-in(\theta-\theta'))\ 2\pi\ \delta(\theta-\theta') \cdot \tag{2.27b}$$

$$\cdot \langle f_\theta , f_\theta \rangle_\theta = \int \frac{d\theta}{2\pi} \langle f_\theta , f_\theta \rangle_\theta$$

the direct integral structure follows.

Since the f_θ functionals are "eigenfunctionals" of the space translation **U**, in the case of the periodic potential we do not want them to be in the physical state space because no realizable translation invariant state exists in quantum mechanics. So we choose the second possibility, namely that our functional space is a direct integral space. That is:

$$\mathcal{H} = \int_{\oplus} \frac{d\Theta}{2\pi} \left(\mathcal{C}^{*}_{+\Theta} / \mathcal{N}_{\Theta} \right)^{compl} = \int_{\oplus} \frac{d\Theta}{2\pi} \mathcal{H}_{\Theta} \qquad , \qquad (2.28)$$

where $\mathcal{N}_{\Theta}$ are those functionals from $\mathcal{C}^{*}_{+\Theta}$ which have zero norm with respect to the inner product $\langle \, , \, \rangle_{\Theta}$.

Of course we look for the irreducible representation of the canonical commutational relations, so it is not enough to deal with the representation space only. We have to look at the operators in the representation as well, whether they can be also decomposed as a direct integral.

We use the Weyl form of the canonical commutational relations that is the algebra $\mathcal{A}_{w}$ we want to represent is generated by the set

$$\{ V(\lambda), U(a) \mid \lambda, a \in \mathbf{R} \}. \qquad (2.29)$$

with the multiplication rules

$$V(\lambda_1)\cdot V(\lambda_2) = V(\lambda_1 + \lambda_2) \;, \qquad (2.30a)$$

$$U(a_1)\cdot U(a_2) = U(a_1 + a_2) \;, \qquad (2.30b)$$

$$V(\lambda)\cdot U(a) = U(a)\cdot V(\lambda)\cdot \exp(-i\lambda a) \;. \qquad (2.30c)$$

The elements $V(\lambda)$, $U(a)$ can be thought as the exponentiated version of the coordinate and the momentum. Because the momentum is the generator of the space translation we will have a "natural" representation of the algebra on the functional space $\mathcal{C}^*_+$:

$$(\hat{V}(\lambda)\mathfrak{f})[x] = \exp(-i\lambda x(0))\cdot \mathfrak{f}[x] \; , \qquad (2.31a)$$

$$(\hat{U}(f)\mathfrak{f})[x] = \exp(-S_+[x-f]+S_+[x])\cdot \mathfrak{f}[x-f] \; , \qquad (2.31b)$$

where $\lambda \in R$, $x \in \mathcal{C}$, $\mathfrak{f} \in \mathcal{C}^*_+$ and

$$-S_+[x-f]+S_+[x] = \lim_{T\to\infty} \int_0^T dt \; (-L_*[x-f]+L_*[x]). \qquad (2.32)$$

The $f: R \to R$ has to be an even function such that $x-f \in \mathcal{C}$ should be valid whenever $x \in \mathcal{C}$. About the $\hat{U}(f)$ operators we will show the following facts:

i) they form a group,

ii) they are constant on the equivalence classes: $\mathfrak{f}+\mathcal{N}$,

iii) they depend only on $f(0)$ and are isometries.

The first property is a trivial consequence of the definition (2.31b). To prove ii) it is enough to show that

$$\langle \mathfrak{f}, \hat{U}(f)\nu \rangle = 0 \; ; \quad \forall \mathfrak{f} \in \mathcal{C}^*_+ \; , \; \nu \in \mathcal{N}. \qquad (2.33)$$

Using the definition of the scalar product we get

$$< \varphi ,\hat{U}(f)\nu >^T \equiv \int d\mu^T(x) \quad (\Theta\varphi)[x] \quad (\hat{U}(f)\nu)[x] =$$

$$= \int d[x(t)]_T \; \exp(-S_-[x])\cdot(\Theta\varphi)[x]\cdot \exp(-S_+[x])\cdot$$

$$\cdot\exp(S_+[x])\; \nu[x-f]\; \exp(-S_+[x-f]) \;=\; \int d[x(t)]_T\cdot$$

$$\cdot\exp(-S_-[x+f])\cdot\{\; \exp(S_-[x])\; \overline{\varphi[\Theta x+\Theta f]}\; \exp(-S_-[x+f])\;\}\cdot \tag{2.34a}$$

$$\cdot\exp(-S_+[x])\;\nu[x] \;=\; <\hat{U}(-\Theta f)\varphi\,,\nu> \;=\; 0\;,$$

where we decomposed the measure: $d\mu(x)^T = d[x(t)]_T\exp(-S[x])$ and we used for $g = -\Theta f$ the following identity:

$$(\Theta\hat{U}(g)\varphi)[x] \;=\; \exp(S_-[x])\cdot\; \overline{\varphi[\Theta x-g]}\; \exp(-S_-[x-\Theta g])\,. \tag{2.34b}$$

To prove iii) we will show that if $f(0) = f'(0)$ then

$$<\hat{U}(f)\psi\;,\hat{U}(f')\varphi>^T \;=\; <\psi,\;\varphi>^T\;; \tag{2.35}$$

where $\psi,\;\varphi\in \mathcal{C}^*_+$ and the functions f,f' have the properties mentioned before. Using again the definitions and (2.34b):

$$<\hat{U}(f)\psi\;,\hat{U}(f')\varphi>^T \;=\; \int d[x(t)]_T\exp(-S_-[x])\cdot(\Theta(\hat{U}(f)\psi)[x]\cdot$$

$$\cdot\exp(-S_+[x])\cdot(\hat{U}(f')\varphi)[x] \;=\; \int d[x(t)]_T\exp(-S_-[x-\Theta f])\cdot$$

$$\cdot\overline{\psi[\theta x-f]}\ \exp(-S_{+}[x-f'])\cdot \varphi[x-f'] = \int d[x(t)]_{T}\cdot \tag{2.36}$$

$$\cdot\exp(-S_{-}[x-f])\ \overline{\psi[\theta(x-f)]}\cdot\exp(-S_{+}[x-f'])\cdot \varphi[x-f'] =$$

$$= \langle \psi , \varphi \rangle^{T} .$$

i)-iii) mean that we get the unitary representation of the subalgebra (2.30b) on $\mathcal{H}$ with the identification a=f(0). Now it is a trivial task to show that (2.30c) is valid using (2.31a-b).

Let us examine whether this representation of the algebra $\mathcal{A}_{*}$ can arise as a direct integral representation on the θ-sectors. The answer is trivially negative. The θ-sectors come from the spectral decomposition of the unitary operator $\hat{U}(d)$. But this operator does not commute with the $\hat{V}(\lambda)$ operators, that is the $\mathcal{C}^{*}_{+\theta}$ "subspaces" are not invariant with respect to the actions of the $\hat{V}(\lambda)$ operators. Because $U = \hat{U}(d)$, from (2.30c)

$$\hat{V}(\lambda)\cdot U = U\cdot\hat{V}(\lambda)\cdot\exp(-i\lambda d) \tag{2.37}$$

follows. It means that $\hat{V}(\lambda)$ transforms the θ-sectors to the $(\theta-\lambda d)$-sectors (mod 2π), because using (2.23) the

$$U\,\hat{V}(\lambda)\varphi_{\theta} = \exp(i\lambda d)\cdot\hat{V}(\lambda)\cdot U\varphi_{\theta} = \exp(-i(\theta-\lambda d))\cdot$$

$$\cdot\hat{V}(\lambda)\varphi_{\theta} \tag{2.38}$$

equation is valid. Because λ is arbitrary the representation is not decomposable to the θ-sectors.

3. θ-sectors in quantum mechanics: quantum pendulum

This problem is very similar to the case of the periodic potential. We consider again a particle in one space dimension moving in a bounded potential V, but now the space is not the real line but a circle.

Yet we can use the results of the previous paragraph. Because R is the universal covering space of S^1 we can spread out the configuration space of the pendulum, and so we arrive to the problem of the periodic potential. We can use the path space, the functional space, ... etc. defined in the previous paragraph to find the irreducible representations of the quantum pendulum. But there is a substantial difference: we have to represent another algebra. Because the translation by d (d is equal to the perimeter of the circle) is the identical transformation of the circle, it is required to commute with all of the other elements of the algebra. Of course this requirement gives the algebra of the quantum pendulum as a subalgebra of the algebra $\mathcal{A}_m$ discussed in the previous paragraph. From (2.30c) with a=d we get:

$$V(\lambda)\cdot U(d) = U(d)\cdot V(\lambda)\, \exp(-i\lambda d) \qquad (3.1)$$

From the requirement that U(d) be in the center of the algebra follows the equality

$$\exp(-i\lambda d) = 1 \,. \qquad (3.2)$$

So the values of λ are restricted to the set:

$$\{\, 2\pi k/d \mid k\in \mathbb{Z} \,\} \,. \qquad (3.3)$$

Thus the algebra of the quantum pendulum - which we denote by $\mathcal{A}_\bullet$ - is generated by the set:

$$\{\, U(a),\ V(\lambda) \mid a\in \mathbb{R},\ \lambda\in \frac{2\pi}{d}\mathbb{Z} \,\} \,. \qquad (3.4)$$

The multiplicative rules of course are the same as in (2.30a-c).

Let us examine the representations of this algebra. Because it is a subalgebra of $\mathcal{A}_R$, the representation of $\mathcal{A}_R$ defines a representation of $\mathcal{A}_\bullet$ as well. Of course if this representation is irreducible with respect to $\mathcal{A}_R$ it is not necessarily valid for $\mathcal{A}_\bullet$. Let us consider the representation on the direct integral of the θ-sectors defined by (2.31a-b). This representation must be reducible because U = = U(d) being an element of the center of $\mathcal{A}_\bullet$ is not a constant operator: it is the multiplication by $\exp(-i\theta)$ on each θ-sector. But $\mathcal{H}_\theta$ is an invariant subspace because in the case of the quantum pendulum (2.38) reads as:

$$U(\hat{V}(\lambda)f_\theta) = \exp(-i(\theta-\lambda d))\ (\hat{V}(\lambda)f_\theta) =$$
$$= \exp(-i\theta)\cdot(\hat{V}(\lambda)f_\theta) \qquad (3.5)$$

due to (3.2). So the representation defined by (2.31a-b) with respect to $\mathcal{A}_\bullet$ is reducible, it is a direct integral of inequivalent representations. They are trivially inequivalent because the values of an element of the center - U(d) - are different on the θ-sectors, namely exp(-iθ).

We can see the inequivalency on the spectrum of the momentum, too. Because the strongly continous unitary operators $\{\hat{U}(a) \mid a\in\mathbf{R}\}$ form a group, there is a selfadjoint generator $\hat{P}$ of this group:

$$\hat{U}(a) = \exp(ia\hat{P}) \ . \qquad (3.7)$$

Because on a θ-sector we have

$$\exp(id\hat{P}) = \exp(-i\theta) \quad , \qquad (3.8)$$

the possible eigenvalues of $\hat{P}$ can be:

$$p = \frac{2\pi}{d}\,(k-\frac{\theta}{2\pi})\ ;\ k\in\mathbf{Z}\ . \qquad (3.9)$$

Thus they are trivially different in different θ-sectors.

In this way we represented the algebra $\mathcal{A}_{\bullet}$ on functionals with different periodicity properties, but the measure which was used to define the inner product of these functionals was the same. Alternatively we can change the roles and can shift the difference from the functionals upon the measure. If $f \in \mathcal{H}_{\theta}$, we define $\tilde{f}$ as follows:

$$\tilde{f}[x] \equiv \exp(-i\theta x(\varphi)/d) \cdot f[x] \ . \tag{3.10}$$

It is easy to see that $\tilde{f} \in \mathcal{H}_{\theta=0}$, independently of in whichever $\mathcal{H}_{\theta}$ f was:

$$(U\tilde{f})[x] = \tilde{f}[x-d] = \exp(-i\theta(x(\infty)-d)/d) \cdot f[x-d] =$$

$$= \exp(-i\theta x(\infty)/d) \ f[x] = \tilde{f}[x] \ . \tag{3.11}$$

Now let $f, \psi \in \mathcal{H}_{\theta}$. The inner product of these functionals can be rewritten as:

$$\langle f, \psi \rangle \equiv \lim_{T\to\infty} \int d[x(t)]_T \exp(-S[x]) \ (\theta f)[x] \cdot \psi[x] =$$

$$= \lim_{T\to\infty} \int d[x(t)]_T \exp(-S[x]) \cdot \exp\{ i\theta/d \cdot \int_{-T}^{T} dt \ \dot{x}(t) \} \cdot$$

$$\cdot (\theta\tilde{f})[x] \cdot \tilde{\psi}[x] \equiv (\tilde{f}, \tilde{\psi})_{\theta} \ . \tag{3.12}$$

That is, the measures will be different when we change the θ parameter because the physical weight of a configuration becomes θ-dependent:

$$S_\theta[x] = S[x] - i\theta/d \cdot \int_{-\infty}^{\infty} dt\ \dot{x}(t)\ .$$

We denoted the measure dependence of the inner product (,) by the suffix θ.

4. θ-sectors in quantum field theory: non-Abelian gauge theories

It is well known about non-Abelian gauge theories that they have topologically distinct sectors in the space of local gauge equivalence classes of the gauge field configurations. Thus if we want to quantize by functional integral we have to sum for these topologically distinct sectors. But we have to decide which configurations are allowed because the characterization of the topologically distinct sectors depend on the properties of the configurations.

Let us denote the topological charge by ν. Its definition is:

$$\nu[A] = \frac{g^2}{64\pi^2} \varepsilon_{\alpha\beta\gamma\delta} \cdot \int d^4x\ F^a_{\alpha\beta}(x) \cdot F^a_{\gamma\delta}(x) = \frac{g^2}{32\pi^2} \int d^4x\ F \cdot F^* \ . \qquad (4.1)$$

This expression is valid in four dimensional Euclidean space-time where the metric is $\delta_{\alpha\beta}$; α,β = 1,2,3,4 and the totally antisymmetric tensor $\varepsilon_{\alpha\beta\gamma\delta}$ is given by $\varepsilon_{1234}=1$. We want to restrict the configuration space to those gauge configurations which have the same asymptotics as of the classical solutions with finite Euclidean actions. There are arguments (first reference in [1]) that $\nu[A]$ should be an integer if the action S[A] is finite. But it is certainly true if the configuration $A^a_\mu(x)$ can be smoothly mapped onto the unit hypersphere in five dimensions. This is a consequence of the Atiyah-Singer index theorem [5]. We will restrict ourselves to this case.

We can cover S^4 by two patches which are homeomorphic to D^4 . On the intersection of the patches the gauge field configurations differ only by a pure gauge. If one of the patches is shrinking to one point corresponding to the points at infinity of R , then using the relation

$$\partial_\mu K_\mu = \frac{g^2}{32\pi^2} \cdot F \cdot F^* , \tag{4.2}$$

where

$$K_\mu = \frac{g^2}{32\pi^2} \varepsilon_{\mu\alpha\beta\gamma} \cdot A^a_\alpha \cdot (F^a_{\beta\gamma} - \frac{1}{3} g \cdot f^{abc} \cdot A^b_\beta A^c_\gamma) \tag{4.3}$$

one obtains the topological charge as a surface integral on the boundary of the other patch:

$$\nu[A] = \int_{S^3} d\sigma_\mu K_\mu(x) . \tag{4.4}$$

In order to adopt the S^3 of (4.4) to the use of the temporal gauge $A_0 = 0$, we realize it as $S^3 = D^3 \cup D^3 \cup I \times S^2$ that is the boundary of a 4-cylinder with symmetry axis pointing towards to the time direction. We also require that the gauge transformation G occuring in the expression $g A_\mu = i \cdot G^{-1} \partial_\mu G$ valid on the surface of the cylinder is constant, let us say G = 1 on the cylinder-jacket. In this case the topological charge becomes the difference of the winding numbers:

$$\nu[A] = \lim_{T\to\infty} \int d^3x \{K_0(T,x) - K_0(-T,x)\} = n[A]-m[A]. \quad (4.5)$$

It is easy to see that $\nu[A]$ as well as $m[A]$ and $n[A]$ are integers.

Thus the configuration space is again a union of distinct sectors as in the previous paragraphs:

$$\mathcal{C} = \bigcup_{n,m\in\mathbb{Z}} \mathcal{C}_{nm}, \quad \mathcal{C}_{nm} = \bigcup_{T<\infty} \mathcal{C}_{nm}(T) \quad (4.6a)$$

$$\mathcal{C}_{nm}(T) = \{ A_\mu | \; gA_\mu(x) = iG^{-1}\partial_\mu G, \text{ if either } |x| \geq T \text{ or } |x| \geq T \} \quad (4.6b)$$

where G satisfies the above mentioned property.

Because we have a "big" time independent gauge transformation U, as a "step operator" on winding numbers:

$$n[UA_i] = n[A] + 1 \tag{4.7}$$

we can repeat the whole construction of the θ-sectors considered in the previous paragraph. Of course we use the invariance property of the measure for the "big" gauge transformation which is provided by the invariance of the action.

The difference is again in the algebra we want to represent. In this case this is the quasilocal algebra $\mathcal{A}_{YM}$ of locally gauge invariant fields. It means that the automorphism of a local gauge transformation $\alpha(U_{loc})$ acts trivially on the algebra:

$$\alpha(U_{loc})\, A = A\,, \qquad \forall A \in \mathcal{A}_{YM}\,. \tag{4.8}$$

But we don't want the "big" gauge transformation to make any physical difference, thus as in the case of the quantum pendulum we want U to be in the center of the algebra $\mathcal{A}_{YM}$:

$$UA = AU\,, \qquad \forall A \in \mathcal{A}_{YM}\,. \tag{4.9}$$

Thus the θ-sectors will be invariant and orthogonal subspaces in the functional space $\mathcal{C}^*_+$. But now the question of inequivalency is not so simple. Of course the values of U on the θ-sectors are different, but due to its non-locality we cannot think of U as an observable. So from the physical point of view these representations will be inequivalent only in that case when one can find an observable which gives different expectation value in the different θ-sectors, just like the momentum gives in the case of the quantum pendulum.

We can shift again the θ-dependence from the functionals upon the measure, that is we change the physical weight, the action:

$$S_\theta[A] = S[A] - i\theta\nu = S[A] - i\theta\frac{g^2}{32\pi^2}\cdot\int d^4x\; F\cdot F^* \; . \qquad (4.10)$$

Because the second term is not invariant under space reflection the breaking of this reflection symmetry makes a physical difference between the θ=0 and the $\theta \neq 0$ sectors. In addition to this the expectation value of $F\cdot F^*$ is conjectured to be θ-dependent because the topological susceptibility

$$-i\frac{\partial}{\partial\theta}\left\langle\frac{g^2}{32\pi^2}F\cdot F^*(x)\right\rangle_\theta = \left(\frac{g^2}{32\pi^2}\right)^2\int d^4y\;\{\langle F\cdot F^*(x)\cdot F\cdot F^*(y)\rangle_\theta -$$

$$-\langle F\cdot F^*(x)\rangle_\theta\,\langle F\cdot F^*(y)\rangle_\theta\,\} \qquad (4.11)$$

is non-zero in general. In this case the θ-sectors turn out to be inequivalent representations of $\mathcal{A}_{YM}$.

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