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PRECISE ERROR PROPAGATION FROM DETECTOR MEASUREMENTS TO CHARGED TRACKS PARAMETERS IN SIMULATION STUDIES

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Résumé :

Un algorithme numérique direct (sans recours à la méthode de Monte-Carlo) est proposé pour le calcul des matrices d'erreurs sur un estimateur optimal des paramètres des trajectoires dans un détecteur de traces, en supposant des erreurs gaussiennes sur les mesures brutes, et en incluant, au besoin, l'effet de la diffusion multiple. Des approximations sont données pour le cas de trajectoires hélicoïdal.

Abstract :

A direct numerical algorithm (without Monte-Carlo evaluation) is proposed to compute the error matrices on optimal estimator of trajectory parameters in a track detector, assuming gaussian errors on the raw measurements, and including, if needed, the effect of multiple scattering. Approximations are given for the case of helix trajectories.

1. Introduction :

In order to evaluate by simulation the precision available on quantities of physical interest, one needs an estimation of the error matrices on the geometrical parameters of each track, starting from the errors on each point measurement. This is generally obtained by approximate analytical expressions (e.g. assuming equally spaced points measured with the same precision), or by a Monte Carlo simulation. We propose a direct method to calculate these matrices (within the gaussian approximation), using the additivity of weight matrices : Moreover this calculation can be extended to the case where random deviations occur along the trajectory, such as multiple scattering (in this case the weights are no more additive) ; it follows closely the method proposed elsewhere to reconstruct tracks and fit their parameters [1,2,3].

2. Deterministic trajectory - Additivity of weight matrix :

2.1. Principle :

We suppose in this section that the trajectory is exactly determined by the initial parameters (5 param. P_1 P_2...P_5 for curved tracks), i.e. there is no multiple scattering, no fluctuations in energy loss.

It is measured at n points : two coordinates (x_i, y_i) *at each point i, with a covariance matrix*

$$\sigma_i = \begin{pmatrix} \sigma_{i\,xx} & \sigma_{i\,xy} \\ \sigma_{i\,xy} & \sigma_{i\,yy} \end{pmatrix}$$

or, in a more convenient form, with a weight matrix $w_i = \sigma_i^{-1}$. When only one coordinate (e.g. x) is measured, we write :

$$w_i = \begin{pmatrix} 1/\sigma_x^2 & 0 \\ 0 & 0 \end{pmatrix}$$

The least squares estimator of $P_1....P_5$ from these measurements has a well-known weight matrix :

$$\mathbf{W} = \sum_i \mathbf{D}_i^t \, w_i \, \mathbf{D}_i \quad (1)$$

where $\mathbf{D}_i$ is the (2x5) matrix of derivatives of x_i and y_i w.r.t. $P_1...P_5$.

Each term of (1) represents the weight matrix of the point i relative to the parameters. It should be noted that these elementary matrices are singular, but W is inversible as soon as n is large enough for example 3 measurements of both coordinates).

The matrix $\mathbf{D}_i$ can be expressed as a product $\mathbf{D}_i = U_i \, T_i$, where T_i is the (5x5) matrix of derivatives of the local parameters (at point i) w.r.t. the initial ones, and $\mathbf{U}_i$ is the (2x5) matrix of derivatives of the measured coordinates w.r.t. the local parameters (in favourable cases, there coordinates are merely two out of the local parameters ; see appendix 1 for more general cases).

2.2. Practical implementation :

An interesting feature of Eq.(1) is that different detectors encountered by a trajectory can be handled independently, and the determination of the error matrix on $P_1...P_5$ for a given particle is decomposed in 3 steps :

(a) calculation of the trajectory (geometrical step)

(b) for each detector on the trajectory : determination of meas.errors, calculation of w_i , whence $\mathbf{W}_{det} = \sum_i D_i^t w_i D_i$ (detector dependent step).

(c) Summation over the detectors and inversion of $\mathbf{W}$.

Steps (a) and (c) are general and need no discussion.

Step (b) requires a good knowledge of the detector performances, but at the level of a point measurement only : the precision on each point is evaluated from prototype studies, accounting for all circumstances : position, angles, edge effects, ambiguities, overlapping of two tracks, etc... Inefficiencies, ionization fluctuations and other random local phenomena can be inserted at this stage for a Monte-Carlo evaluation (simulating many times the same track, i.e. with the same initial parameters).

The error matrix obtained in this way gives the uncertainties to be expected in an actual track reconstruction with an optimal estimator.

The actual computation of D_i is not needed and can be replaced by a step-by-step decomposition of Eq.(1) ; each step is a propagation from a measured point to the next one in the same detector (thus over a short length) so that first order expressions can be used (see appendix 2 for explicit expressions in some useful parametrizations). For example, we want to calculate the weight matrix on the parameters at first point in a detector :

$$\mathbf{W} = \sum_{i=1}^{n} \mathbf{T}_{i,1}^{t}\, \mathbf{U}_i^{t}\, w_i\, \mathbf{U}_i\, \mathbf{T}_{i,1} = \sum_{i=1}^{n} \mathbf{T}_{i,1}^{t}\, v_i\, \mathbf{T}_{i,1}$$

where $\mathbf{T}_{i,j}$ is the matrix of derivatives of parameters at point i w.r.t. the parameters at point j.

Since $\mathbf{T}_{i,1} = \mathbf{T}_{i,i-1} \cdots \mathbf{T}_{2,1}$, we obtain :

$$\begin{aligned}\mathbf{W} &= v_1 + \mathbf{T}_{2,1}^{t}\, v_2\, \mathbf{T}_{2,1} + \mathbf{T}_{2,1}^{t}\, \mathbf{T}_{3,2}^{t}\, v_3\, \mathbf{T}_{3,2}\, \mathbf{T}_{2,1} + \cdots \\ &= v_1 + \mathbf{T}_{2,1}^{t} \left\{ v_2 + \mathbf{T}_{3,2}^{t} \left[v_3 + \mathbf{T}_{4,3}^{t} (v_4 + \cdots) \mathbf{T}_{4,3} \right] \mathbf{T}_{3,2} \right\} \mathbf{T}_{2,1}\end{aligned}$$

Hence a practical algorithm of computation :

- start with $\mathbf{W} = v_n$
- from $i = n-1$ to 1
  - (1) compute $\mathbf{T}_{i,i+1}$ (first order approx.)
  - (2) replace $\mathbf{W}$ by $\mathbf{T}_{i+1,i}^{t}\, \mathbf{W}\, \mathbf{T}_{i+1,i}$
  - (3) add v_i to $\mathbf{W}$

3. Accounting for random deviations (mult. scattering)

3.1. Formalism :

In this case the errors on the points are correlated ; so their contributions to W are no more additive. However the algorithm developed in sect.2 can be extended in a very natural way.

Let W_{i+1} be the information matrix on the parameters at i+1, obtained from the measurements at i+1, i+2, ...n, accounting for all sources of errors (including the multiple scattering between i+1 and n) : W_{i+1} is the weight matrix of an <u>optimal</u> estimator of these parameters. We want to express W_i as a function of W_{i+1}, w_i (weight of point i) and the multiple scattering between i and i+1. Let us consider $W_i' = T^t_{i+1,i} W_i T_{i+1,i}$, the information matrix on the parameters <u>at point i</u> from the points <u>i+1 to n</u> , neglecting the multiple scattering. The effect of mult. scatt. is to add extra terms to $\Sigma_i' = (W_i')^{-1}$; if the distance between i and i+1 is small, and if we choose as parameters the angles α and β in two planes tangent to the trajectory and perpendicular to each other, these terms are merely the well known variance of the angle deviation :

$$\Delta\theta^2_{i,i+1} = \left(\frac{15\ \mathrm{MeV}}{p v}\right)^2 \xi_{i,i+1}$$

($\xi_{i,\,i+1}$ is the number of radiation lengths between i and i+1). So the covariance matrix, including the multiple scattering is :

$$\Sigma_i'' = \Sigma_i' + M_{i,i+1}\,, \quad \text{with } (M_{i,i+1})_{\alpha\alpha} = (M_{i,i+1})_{\beta\beta} = \Delta\theta^2_{i,i+1}$$

and all other terms of $M_{i,i+1} = 0$.

Now we can express W_i as the <u>sum</u> of $(\Sigma_i'')^{-1}$, information from points i+1 to n, and w_i, information from point i, because the measurement errors on point i are clearly independent of all errors involved in Σ_i'' (measurement errors on points i+1 to n, and mult. scatt. errors between i and n).

To summarize :

$$W_i = \left[\left(T^t_{i+1,i} \; W_{i+1} \, T_{i+1,i} \right)^{-1} + M_{i,i+1} \right]^{-1} + v_i \qquad (2)$$

3.2. Algorithm :

With measurement points close to each other, we just insert Eq.(2) in the loop described and the end of sect.2 ; so we get :

- start with $W = v_n$
- from $i = n-1$ to 1 :

$$\begin{cases} (1) \text{ compute } T_{i,i+1} \\ (2) \text{ replace } W \text{ by } T^t_{i+1,i} \; W \; T_{i+1,i} \\ (2') \text{ replace } W \text{ by } (W^{-1} + M_{i,i+1})^{-1} \\ (3) \text{ add } v_i \text{ to } W \end{cases}$$

N.B. Steps (2) and (2') can be exchanged.

In more complex cases, when two points are separated by a long distance and a large amount of material (for example beteen different detector) it is necessary to split the propagation step (2) in several parts, and to insert a mult. scatt. step (2') at intermediate points.

In principle this algorithm must be applied throughout all detectors. However, if the effects of mult. scatt. inside the detectors are much smaller than between them, it can be split and applied in each detector independently ; the detectors are then linked in this way :

- start with $W = W_1^{(N)}$ (weight matrix at first point in the N^{th} detector)

- For $j = N-1$ to 1 (loop over the detectors)

  (i) propagate W to the wall(s) between det. j+1 and j
  (i') replace W by $(W^{-1}+M)^{-1}$ at this wall
  } repeat if necessary
  (ii) propagate W to the first point meas. in det. j
  (iii) add $W_1^{(j)}$ to W (weight at first point in det. j)

If the mult. scatt. inside a detector is not quite negligible, a step (ii') is to be inserted after (ii) :

(ii') replace W by $\left(W^{-1}+M_j\right)^{-1}$

Approximate expressions for M can be found in [1,2].

4. Using vertex information :

When several tracks are issuing from the same vertex, a geometrical constraint increases the information on each track. We want to show briefly how this is expressed in terms of a global information matrix.

For each track independently we have a set of 5 param. ($Pi_1 \ldots Pi_5$) and their weight matrix W_i. All tracks together are defined by 3 general parameters XYZ (coordinates of the vertex) and 3 param. ($q_{i_1}, q_{i_2}, q_{i_3}$) for each track (e.g. the components of the momentum,or two angles + the curvature). The 5 param. P_{i_j} are functions of X,Y,Z and $q_{i_1}, q_{i_2}, q_{i_3}$. Let $\mathcal{D}_i$ be the (5x6) derivative matrix of these functions ; the information matrix on X,Y,Z, q_{i_1}, q_{i_2}, q_{i_3} from track i is $\mathcal{D}_i^t W_i \mathcal{D}_i$ and the information matrix on all parameters $(X, Y, Z, q_{11}, q_{12}, q_{13}, \ldots q_{ij} \ldots)$ reads :

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}_{00} & \mathcal{W}_{10} & \cdots & \mathcal{W}_{i0} & 0 & \cdots \\ \mathcal{W}_{10}^t & \mathcal{W}_{11} & 0 & \cdots & 0 & \cdots \\ 0 & & \ddots & & \vdots & \\ \vdots & & & & & \\ \mathcal{W}_{i0}^t & 0 & \cdots & \cdots & \mathcal{W}_{ii} & 0 \cdots \\ \vdots & & & & 0 & \\ & & & & \vdots & \end{pmatrix}$$

where the $\mathcal{W}_{k\ell}$ are (3x3) blocks, and $\mathcal{W}_{k\ell} = 0$ if $k \neq 0, \ell \neq 0$ and $k \neq \ell$

$\mathcal{W}^{-1}$ is the covariance matrix of the parameters :

$$\mathcal{W}^{-1} = \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{10} & \cdots & \mathcal{C}_{i0} & \cdots \\ \mathcal{C}_{10}^{t} & \mathcal{C}_{11} & & \vdots & \\ \vdots & & \ddots & \vdots & \\ \mathcal{C}_{i0}^{t} & \cdots & \cdots & \mathcal{C}_{ii} & \\ \vdots & & & & \end{pmatrix}$$

with

$$\mathcal{C}_{00} = \left(\mathcal{W}_{00} - \sum_i \mathcal{W}_{i0} \mathcal{W}_{ii}^{-1} \mathcal{W}_{i0}^{t} \right)^{-1}$$

$$\mathcal{C}_{i0} = - \mathcal{C}_{00} \mathcal{W}_{i0} \mathcal{W}_{ii}^{-1}$$

$$\mathcal{C}_{ij} = \delta_{ij} \mathcal{W}_{ii}^{-1} + \mathcal{W}_{ii}^{-1} \mathcal{W}_{i0}^{t} \mathcal{C}_{00} \mathcal{W}_{j0} \mathcal{W}_{jj}^{-1}$$

Of course all parameters are correlated to each other : to obtain the uncertainty on a physical combination of them (such as an invariant mass, a total energy, etc...) one must, in principle, use the whole covariance matrix of the subset of parameters involved.

5. Conclusions :

We obtained a practical algorithm to calculate the global information matrix, i.e. the weight matrix on the physical parameters of a track, or several tracks originating from the same vertex ; the ingredients are :

- weights of elementary measurements (local physical properties of the detectors)

- transport matrices along the trajectory (purely geometrical calculation)

- If needed, loss of information due to random deviations (multiple scattering)

In certain conditions this method can be applied to each detector independently, and the matrices then linked with the same basic tools.

The advantages of this approach are :

- precision : it accounts for local circumstances which influence the uncertainties on raw measurements, and for local inefficiencies (missing points, overlapping tracks, etc...).

- universality and flexibility : it allows various parametrizations, and decouples the local handling of errors from the geometrical propagation.

It could be used, either to make realistic and detailed predictions on the geometrical and kinematical resolution for charged particles, or to verify on real tracks (without Monte-Carlo simulation) that all sources of errors have *been correctly handled, better than with* χ^2 *tests only* (e.g. *comparing with* predictions the gap between two portions of the same track in different detectors, or the dispersion at a vertex, or the reconstructed mass of a V°, etc...)

References

[1] P. Billoir Thèse LPC-T 83-01

[2] P. Billoir N.I.M. 225 (1984) p.352

[3] P. Billoir DELPHI 84-18

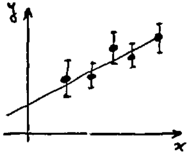
Appendix I - Matrix U

1. 2d example : let us begin with a very simple example : a straight line fit, with one coordinate measured at each point : so T is (2x2), U is (1x2), w is scalar.

Equation of trajectory : $y = a_0 + a_1 x$ (param. a_0, a_1)

Local parameters at x_i: $\begin{cases} \alpha_0 = a_0 + a_1 x & \text{(local coordinate)} \\ \alpha_1 = a_1 & \text{(local slope)} \end{cases}$

1st case:



measurement of y at fixed x_i with error σ_i

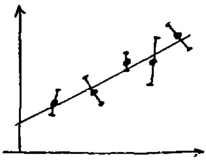
$$T_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \qquad U_i = (1\ 0)$$

$$\rightarrow D_i = U_i T_i = (1\ x_i)$$

$$\rightarrow W_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix} (1/\sigma_i^2)(1\ x_i) = \begin{pmatrix} 1/\sigma_i^2 & x_i/\sigma_i^2 \\ x_i/\sigma_i^2 & x_i^2/\sigma_i^2 \end{pmatrix}$$

$$\rightarrow W = \sum_i W_i = \begin{pmatrix} \sum_i 1/\sigma_i^2 & \sum_i x_i/\sigma_i^2 \\ \sum_i x_i/\sigma_i^2 & \sum_i x_i^2/\sigma_i^2 \end{pmatrix} \quad \text{(well known!)}$$

2nd case:



measurement of $t_i = \mu_i x - \lambda_i y$ (with error σ_i) on the line defined by $\lambda_i x + \mu_i y = 0$ $(\lambda_i^2 + \mu_i^2 = 1)$ i.e. "oblique" measurement

In the local frame, the equation of the trajectory is :

$$y = \alpha_1 x + \alpha_0$$

This equation and the condition $\underbrace{\quad}_{}$ $\lambda x + \mu y = 0$ give x and y as functions of α_0 and α_1, whence $t = \alpha_0 / (\lambda + \mu\alpha_1)$

and the derivatives :

$$\frac{\partial t}{\partial \alpha_0} = \frac{1}{\lambda_i + \mu_i \alpha_1} \quad ; \quad \frac{\partial t}{\partial \alpha_1} \simeq 0 \qquad \text{For} \quad \alpha_0 \simeq 0$$

(this is true if the fitted trajectory is close to the measured point).

So we get : $$U_i = \left(\frac{1}{\lambda_i + \mu_i \alpha_1} \quad 0 \right)$$

whence

$$D_i = U_i T_i = \left(\frac{1}{\lambda_i + \mu_i \alpha_1} \quad \frac{x_i}{\lambda_i + \mu_i \alpha_1} \right)$$

and

$$W_i = D_i^t \, w_i \, D_i = \begin{pmatrix} 1/\sigma_i'^2 & x_i/\sigma_i'^2 \\ x_i/\sigma_i'^2 & x_i^2/\sigma_i'^2 \end{pmatrix}$$

with $\sigma_i' = (\lambda_i + \mu_i \alpha_1)\, \sigma_i$,

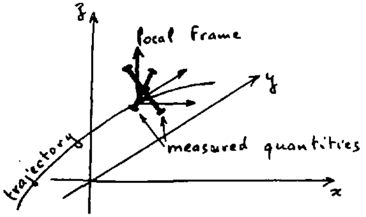
(σ_i' is the projected uncertainty on the y axis along the direction of the trajectory).

2. 3d generalization :

We consider now 3d trajectories (curved or not)

We choose as local parameters around point i :

$$\underbrace{\underbrace{y, z}_{\text{Coord.}} \, , \, \underbrace{\frac{dy}{dx} \, , \, \frac{dz}{dx}}_{\text{slopes}}}_{\text{at Fixed } x_i} \, , \, \underbrace{[\, c \,]}_{\text{param. of curvature (if any)}}$$



We measure, in the local frame :

$$\begin{cases} t' = \lambda'_i x + \mu'_i y + \nu'_i z \\ t'' = \lambda''_i x + \mu''_i y + \nu''_i z \end{cases}$$

in the plane $\lambda_i x + \mu_i y + \nu_i z = 0$

$\begin{pmatrix} \lambda_i & \mu_i & \nu_i \\ \lambda'_i & \mu'_i & \nu'_i \\ \lambda''_i & \mu''_i & \nu''_i \end{pmatrix}$ is an orthogonal matrix .

Local parametrization of the trajectory $\begin{cases} y = \alpha_1 x + \alpha_0 & (\alpha_0 \simeq 0) \\ z = \beta_1 x + \beta_0 & (\beta_0 \simeq 0) \end{cases}$

After elimination of x, y, z we find :

$$\frac{\partial t'}{\partial \alpha_0} = \frac{\nu'' - \lambda'' \beta_1}{\lambda + \alpha_1 \mu + \beta_1 \nu} \qquad \frac{\partial t'}{\partial \beta_0} = \frac{\lambda'' \alpha_1 - \mu''}{\lambda + \alpha_1 \mu + \beta_1 \nu}$$

$$\frac{\partial t''}{\partial \alpha_0} = \frac{\lambda' \beta_1 - \nu'}{\lambda + \alpha_1 \mu + \beta_1 \nu} \qquad \frac{\partial t''}{\partial \beta_0} = \frac{\mu' - \lambda' \alpha_1}{\lambda + \alpha_1 \mu + \beta_1 \nu}$$

other derivatives $\simeq 0$

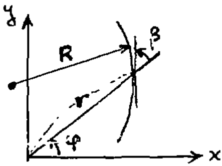
whence $$U = \frac{1}{\lambda + \alpha_1 \mu + \beta_1 \nu} \begin{pmatrix} \nu'' - \lambda'' \beta_1 & \lambda'' \alpha_1 - \mu'' & 0 & 0 & 0 \\ \lambda' \beta_1 - \nu' & \mu' - \lambda' \alpha_1 & 0 & 0 & 0 \end{pmatrix}$$

instead of $U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ when y and z are measured at fixed x.

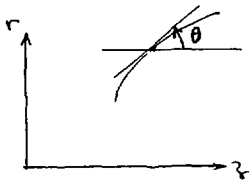
Appendix 2 - Matrix T

We consider the trajectory as an helix (radius R, axis along z)

1. Cylindric coordinates ; measurements at fixed r :



r-φ plane



r-z plane

Position defined by φ, z

Direction defined by β, θ

Curvature defined by $c = 1/R$

Local parameters : $\underline{\varphi, z, \beta, t, c}$ at fixed r ($t = \cot\theta$)

Variation of parameters from r to $r + \Delta r$ along the trajectory (at first order in Δr ; this is valid for $\Delta r \ll r$ and $\Delta r \ll R$) :

$$\Delta\varphi = \frac{\Delta r}{r} \tan\beta \quad ; \quad \Delta z = \frac{t\,\Delta r}{\cos\beta} \quad ; \quad \Delta\beta = \frac{\Delta r}{\cos\beta}\left(c - \frac{\sin\beta}{r}\right)$$

and, of course :

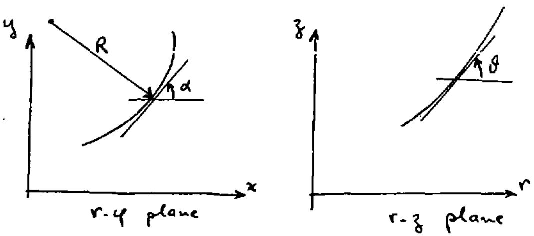
$$\Delta t = 0 \quad ; \quad \Delta c = 0$$

Hence the matrix of derivatives $\partial P(r+\Delta r) / \partial P(r)$:

$\partial\downarrow/\partial\rightarrow$	z	φ	β	t	c
z	1	0	$\frac{t\,\Delta r \sin\beta}{\cos^2\beta}$	$\frac{\Delta r}{\cos\beta}$	0
φ	0	1	$\frac{\Delta r}{r\cos^2\beta}$	0	0
β	0	0	$1+\frac{\Delta r}{\cos^2\beta}\left(c\sin\beta - 1/r\right)$	0	$\frac{\Delta r}{\cos\beta}$
t	0	0	0	1	0
c	0	0	0	0	1

N.B. If β is not too large and $\Delta r \ll r$, $\frac{\partial\beta(r+\Delta r)}{\partial\beta(r)} \simeq 1$

2. Cartesian coordinates ; measurements at fixed z



Position defined by x, y

Direction defined by α, θ

Curvature defined by $c = 1/R$

local parameters : x, y, α, u, c at fixed z ($u = \tan\theta$)

Variation of parameters from z to $z + \Delta z$:

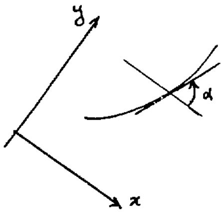
$$\Delta\alpha = c\,u\,\Delta z \quad ; \quad \Delta x = u\cos\alpha\,\Delta z \quad ; \quad \Delta y = u\sin\alpha\,\Delta z$$

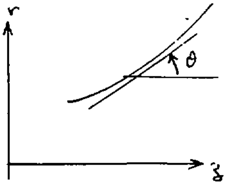
$$\Delta u = 0 \quad ; \quad \Delta c = 0$$

Hence the matrix $\partial \underline{P}(z+\Delta z) / \partial \underline{P}(z)$

$\partial\downarrow / \partial\rightarrow$	x	y	α	u	c
x	1	0	$-\Delta y$	$\cos\alpha\,\Delta z$	0
y	0	1	Δx	$\sin\alpha\,\Delta z$	0
α	0	0	1	$c\,\Delta z$	$u\,\Delta z$
u	0	0	0	1	0
t	0	0	0	0	1

3. Cartesian coordinates ; measurements at fixed x in a rotated frame





Position defined by y, z
Direction defined by α, θ
Curvature defined by $c = 1/R$

local parameters : y,z,s,t,c at fixed x
($s = \sin\alpha$, $t = \cot\theta$)

Variation of parameters from x to x+Δ x :

$$\Delta s = c\,\Delta x \;\; ; \quad \Delta y = \frac{s}{\sqrt{1-s^2}}\,\Delta x \quad ; \quad \Delta z = \frac{t}{\sqrt{1-s^2}}\,\Delta x$$

Hence the matrix $\partial P(x+\Delta x)\,/\,\partial P(x)$:

$\partial\downarrow/\partial\rightarrow$	y	z	s	t	c
y	1	0	$q^3\Delta x$	0	0
z	0	1	$-q^3 s t\,\Delta x$	$q\,\Delta x$	0
s	0	0	1	0	Δx
t	0	0	0	1	0
c	0	0	0	0	1

where $q = \dfrac{1}{\sqrt{1-s^2}} = \dfrac{1}{\cos\alpha}$