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TRANSVERSE BEAM BREAK UP IN A PERIODIC LINAC *

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INTRODUCTION

In this paper, the problem of cumulative beam break up in a periodic linac for a general impedance is discussed, with the effects of acceleration included. The transverse equations of motion for a set of identical point like bunches moving along the length of the linac are cast into a simple form using a smooth approximation. This results in a working formula that is used to analyze beam breakup. Explicit expressions for the transverse motion in the case of a single resonance impedance are found using saddle point integration. This is done first with no external focusing, and again in the strong focusing limit.

EQUATION OF MOTION

Consider a sequence of equally spaced point like bunches moving down a linac with positions defined by

$$z_M = ct - Mc\tau, \qquad M = 0, 1, \cdots \infty. \qquad (1)$$

Here τ is the bunch separation in seconds, c is the speed of light, and M is a bunch labelling index.

The equation of motion for the transverse displacement x_M of bunch M is given by

$$\frac{d}{dz}\,\gamma(z)\,\frac{dx_M(z)}{dz} + K(z)\,x_M(z) = F_M(z), \qquad z \geq 0. \quad (2)$$

Here $\gamma(z)$ is the beam energy in units of the rest energy. The function $K(z)$ describes the external transverse focusing and is equal to γ times the focusing function used in circular accelerator theory.

$F_M(z)$ is the transverse wake field force resulting from the passage of earlier bunches:

$$F_M(z) = e^2 N_B \sum_{m=0}^{\infty} \sum_{N=0}^{\infty} \delta(z - NL)\, S_{M-m}\, x_m(z) \qquad (3)$$

where e is the electronic charge, N_B is the number of particles per bunch, and

$$S_{M-m} = G\,[(M-m)\tau]\,. \qquad (4)$$

$G(t)$ is the transverse wake function, vanishing for $t<0$ due to causality.

Formula (3) assumes that the accelerating cavities are placed periodically at positions $z = NL$, $N = 0, 1, \cdots$, and that they have infinitessimal length, acting like thin lenses.

THE SMOOTH APPROXIMATION

In the case of a coasting beam ($\gamma =$ const.), a formal solution of eqn. (3) can be found for the case where $K(z)$ is periodic with period L. This is not easily done when acceleration is included, and certain simplifying assumptions are useful. First, assume that the sum of delta functions in eqn. (3) can replaced by $1/L$. This serves to smooth out the effect of the transverse kicks of the cavities. Second, assume uniform acceleration, i.e. $\gamma = \gamma_0 + \gamma' z$, where γ_0 and γ' are constants.

With the above assumptions, a change of independent variable from z to $u = \sqrt{\gamma}$ allows eqn. (2) to be rewritten as

$$\frac{1}{u}\frac{d}{du}\,u\frac{dx_M(u)}{du} + \left[\frac{2}{\gamma'}\right]^2 K(u)\,x_M(u) =$$
$$= \left[\frac{2}{\gamma'}\right]^2 \frac{e^2 N_B}{L} \sum_{m=0}^{\infty} S_{M-m}\, x_m(u) \qquad (5)$$

Because we are dealing with an initial value problem, the sum on the right hand side of eqn. (5) extends from $m = 0$ to ∞. For the case of an infinite number of equally spaced bunches extending from $z = -\infty$ to ∞ (i.e. steady state), it would be replaced by a convolution sum from $-\infty$ to $+\infty$. Considering the convolution sum as the product of an ∞ - dimensional matrix S_{M-m} with a vector x_m, the matrix can be diagonalized using eigenfunctions

$$x_m(u) = \frac{1}{2\pi}\,\Xi(u, \theta)\, e^{-im\theta},$$
$$m = -\infty, \cdots \infty, \quad 0 \leq \theta \leq 2\pi\,. \qquad (6)$$

The solution of the initial value problem can be expressed as a superposition of these eigenfunctions:

$$x_m(u) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\;\Xi(u,\theta)\, e^{-im\theta}, \qquad (7)$$

or, equivalently,

$$\Xi(u,\theta) = \sum_{m=0}^{\infty} e^{im\theta}\, x_m(u)\,. \qquad (8)$$

Insertion of eqn. (8) into the equation of motion (5) results in a differential equation for the transformed function $\Xi(u,\theta)$.

$$\frac{1}{u}\frac{d}{du}\,u\frac{d}{du}\,\Xi(u,\theta) +$$
$$+ \left[\frac{2}{\gamma'}\right]^2 [K(u) - \Delta(\theta)]\,\Xi(u,\theta) = 0 \qquad (9)$$

where

$$\Delta(\theta) = \frac{e^2 N_B}{L} \sum_{m=0}^{\infty} S_m\, e^{im\theta} \qquad (10)$$

Equations (9) and (7) are the working formulae from which the transverse trajectories $x_M(u)$ can be found given a focusing function $K(u)$ and wake function S_{M-m}.

INITIAL CONDITIONS

The initial conditions for transverse bunch position and angle are translated into initial conditions for $\Xi(u,\theta)$ using eqn. (8). Consider the case where the leading bunch ($M = 0$) is initially offset by an amount x_0, with successive bunches entering the linac on axis ($x_M(u_0) = 0$, $M \neq 0$). Assuming that all the bunches start out moving parallel to the acceleration direction ($dx_M/dz = 0$), it is seen that

$$\Xi(u_0, \theta) = x_0\,, \qquad \text{(leading bunch initially offset).} \qquad (11)$$

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Here $u_0 = \sqrt{\gamma_0}$, with γ_0 being the injection energy in units of the particle rest energy.

Because the equation of motion is linear, the solution of a more general initial value problem can be written as a superposition of solutions using an initial condition of the form of eqn. (11), provided that an appropriate permutation of indices is made.

A second case of interest is the situation where all bunches are initially offset by an amount x_0. In this case, the initial value of Ξ is given by

$$\Xi(u_0, \theta) = x_0 \sum_{m=0}^{\infty} e^{im\theta}$$

$$= \frac{x_0}{1 - e^{i\theta}} \quad \text{(all bunches initially offset).} \quad (12)$$

This corresponds to a transverse translation of the injector relative to the linac by an amount x_0.

SOLUTION FOR SMOOTH FOCUSING

Focusing in a linac is usually done with quadrupole magnets placed at fixed positions along the z-axis, so that the focusing function $K(u)$ is a piece wise continuous function. In the spirit of the smooth approximation, the focusing here is assumed to vary smoothly with the variable u, having no step discontinuities. The most general form of focusing function $K(u)$ for which eqn. (9) can be readily solved is given by

$$K(u) = K_0 + \frac{K_1}{u} + \frac{K_2}{u^2}, \quad (13)$$

where K_0, K_1, and K_2 are constants.

This function decreases with increasing energy, assuming the constants K_i are positive (recall $u = \sqrt{\gamma}$). This is reasonable behaviour for $K(u)$, because for constant focusing, the transverse beam size naturally decreases with increasing energy due to adiabatic damping, so that a constant beam size could in principle be maintained if $K(u)$ were to decrease with increasing energy.

The solution to eqn. (9) using the focusing function of eqn. (13) can be written as

$$\Xi(u, \theta) = u^{-1/2}\, w(\kappa, \mu, \xi) \quad (14)$$

where

$$\kappa = \frac{1}{\gamma'} \frac{K_1}{[\Delta(\theta) - K_0]^{1/2}}, \quad (15)$$

$$\mu = i\left[\frac{2}{\gamma'}\right]\sqrt{K_2}, \quad (16)$$

$$\xi = \frac{4}{\gamma'}\,[\Delta(\theta) - K_0]^{1/2}\, u$$

$$= \frac{4}{\gamma'}\,[(\Delta(\theta) - K_0)\gamma]^{1/2}, \quad (17)$$

and $w(\kappa, \mu, \xi)$ is a solution to Whittaker's standard form of the confluent hypergeometric equation:

$$\frac{d^2 w}{d\xi^2} + \left[-1/4 + \frac{\kappa}{\xi} + \frac{1/4 - \mu^2}{\xi^2}\right] w = 0 \quad (18)$$

If the constant K_1 in eqn. (13) is assumed to vanish, the function $\Xi(u, \theta)$ can be written in terms of Bessel functions:

$$\Xi(u, \theta) = A(u_0, \theta)\, H_\nu^{(1)}(\alpha u) + B(u_0, \theta)\, H_\nu^{(2)}(\alpha u), \quad (19)$$

where the $H_\nu^{(i)}$ are Hankel functions of order nu,

$$\alpha = \frac{2}{\gamma'}[K_0 - \Delta(\theta)]^{1/2}, \quad (20)$$

and

$$\nu^2 = -\left[\frac{2}{\gamma'}\right]^2 K_2 \quad (21)$$

The quantities A and B are determined from the initial conditions according to

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{i\pi\alpha u_0}{4} \begin{pmatrix} H_\nu^{(2)\prime}(\alpha u_0) & -H_\nu^{(2)}(\alpha u_0) \\ -H_\nu^{(1)\prime}(\alpha u_0) & H_\nu^{(1)}(\alpha u_0) \end{pmatrix} \begin{pmatrix} \Xi(u_0, \theta) \\ \frac{1}{\alpha}\frac{d\Xi}{du}(u_0, \theta) \end{pmatrix} \quad (22)$$

SINGLE RESONANCE MODEL

When the deflecting wakefields inside the accelerating cavities are modelled by a single resonance, the wake function $G(t)$, using (4) is given by

$$S_m = \frac{1}{2m_p c}\frac{R}{Q}\, e^{-\frac{m\omega\tau}{2Q}} \sin(m\,\omega\tau) \quad (23)$$

where m_p is the particle mass, c the speed of light, R the shunt impedance, and Q the mode quality factor.

Using eqn. (10), it is seen that

$$\Delta(\theta) = \frac{\epsilon \sin(\omega\tau)}{\cos(\theta + \frac{i\omega\tau}{2Q}) - \cos(\omega\tau)} \quad (24)$$

where

$$\epsilon = \frac{e^2 N_B}{4 m_p c L}\frac{R}{Q} \quad (25)$$

BREAK UP WITHOUT FOCUSING

For the case where $K(u) = 0$, the function $\Xi(u, \theta)$ reduces to a linear combination of modified Bessel functions of order zero. An asymptotic formula using the single resonance model can be found by expanding the Bessel functions for large argument, and using saddle point integration. The method of Gluckstern, Cooper, and Channell [1] is followed here.

Using the initial conditions of eqn. (11) (leading bunch offset), and expanding the Bessel functions, one arrives at the following:

$$\frac{x_M(u)}{x_0} = \frac{\sqrt{(u_0/u)}}{4\pi}\int_0^{2\pi} d\theta\, e^{-iM\theta}\left[e^{i\alpha(u - u_0)} + (\alpha \to -\alpha)\right] \quad (26)$$

where α is given by eqn. (20), with K_0 set equal to zero.

Using the function $\Delta(\theta)$ of the preceding section, the exponent giving the largest contribution to the integral, and its first derivative, can be written as

$$f(\theta) = -iM\theta + \frac{2(u - u_0)}{\gamma'}\sqrt{\Delta(\theta)}, \quad (27)$$

$$\frac{df}{d\theta} = -iM + \frac{(u - u_0)}{\gamma'\,\epsilon}\frac{\sin(\theta + i\omega\tau/2Q)}{\sin(\omega\tau)}[\Delta(\theta)]^{3/2} \quad (28)$$

Saddle points are located by setting $df/d\theta$ equal to zero. In this way one obtains

$$\sqrt{\Delta_s} = \left[\frac{iMa\gamma'}{u-u_0}\,\frac{\sin(\omega\tau)}{\sin(\theta_s + i\omega\tau/2Q)}\right]^{1/3} \tag{29}$$

The root having the largest positive real part is the most important, since it appears in an exponent.

Assuming $|a/\Delta_s| << 1$, one can estimate the value of θ_s from eqn. (24):

$$\theta_s + \frac{i\omega\tau}{2Q} \approx \omega\tau - \frac{a}{\Delta_s} \tag{30}$$

Inserting this result into eqns. (29) and (27), one finds the value of the exponent at the dominant saddle point, finally arriving at the result

$$\frac{x_M(u)}{x_0} \approx \frac{\sqrt{u_0/u}}{2M\sqrt{3\pi}}\sqrt{E}\; e^{-iM\omega\tau - \frac{M\omega\tau}{2Q} + 3E} + c.c. \tag{31}$$

where

$$E = \left[\frac{Ma(u-u_0)^2}{(\gamma')^2}\right]^{1/3} e^{i\pi/6}. \tag{32}$$

The second term in (31) results from the solution where the right hand side of eqn. (30) changes sign. The exponent (32) is the result stated by Helm and Loew [2] using a technique of Panofsky, in the limit $u >> u_0$.

A necessary condition for the validity of the solution (31) is that the higher order terms in the Taylor expansion of $f(\theta)$ are small compared to those which are kept. First write

$$f(\theta) \approx f(\theta_s) + f''(\theta_s)\frac{(\theta-\theta_s)^2}{2}\left[1 + \frac{f'''(\theta_s)}{3f''(\theta_s)}(\theta-\theta_s) + \cdots\right] \tag{33}$$

The second term in square brackets in (33) must be small compared to one over the region where the integrand is non-negligible (i.e. near the saddle point). In this region, $(\theta-\theta_s)$ has a magnitude less than or equal to $\sqrt{2/f''(\theta_s)}$, so that the condition becomes

$$\left[\frac{(\gamma')^2}{Ma(u-u_0)^2}\right]^{1/6} = |E|^{-1/2} << 1. \tag{34}$$

One must also require that the arguments of the Bessel functions near the saddle point be large compared to unity. This gives

$$\frac{2u_0}{u-u_0}\,|E| >> 1, \tag{35}$$

$$\frac{2u}{u-u_0}\,|E| >> 1. \tag{36}$$

The assumption $|a/\Delta_s| << 1$, used to derive eqn. (30), reduces to

$$\frac{1}{M}\,|E| << 1. \tag{37}$$

STRONG FOCUSING

If $K(u) = const. = K_0$, an asymptotic solution can be obtained using eqn. (26), with α given by eqn. (20). Strong focusing will be taken to mean that K_0 is larger in magnitude than $\Delta(\theta)$ over the region of integration. Proceeding as before, the saddle points are determined from

$$\Delta_s^2 = -\frac{M\gamma'(K_0-\Delta_s)^{1/2}}{u-u_0}\,\frac{\sin(\omega\tau)}{\sin(\theta_s + i\omega\tau/2Q)}. \tag{38}$$

In the strong focusing limit, the quantity Δ_s can be dropped relative to K_0 on the right hand side.

Using eqn. (30) as before, one arrives at the result

$$\frac{x_M(u)}{x_0} \approx \frac{\sqrt{u_0/u}}{4M\sqrt{\pi}}\sqrt{E_1}\; e^{-iM\omega\tau + i\Psi - \frac{M\omega\tau}{2Q} + 2E_1} + c.c. \tag{39}$$

where

$$\Psi = \frac{2(u-u_0)\sqrt{K_0}}{\gamma'}, \tag{40}$$

and

$$E_1 = \left[\frac{aM(u-u_0)}{\gamma'\sqrt{K_0}}\right]^{1/2}. \tag{41}$$

Here the second term in eqn. (39) comes from the $(\alpha \rightarrow -\alpha)$ term in eqn. (26).

The condition analagous to (34) is that $|E_1|^{-1/2} << 1$. Relations (35) and (36) become

$$\frac{2}{\gamma'}\sqrt{K_0}\,u >> 1, \tag{42}$$

$$\frac{2}{\gamma'}\sqrt{K_0}\,u_0 >> 1, \tag{43}$$

and (37) has the same form, but with $E \rightarrow E_1$.

INJECTION OFFSET ERROR

If all of the bunches were assumed to enter the linac offset by an amount x_0 in the preceding examples, the integrand in eqn. (26) would have to be multiplied by $(1-e^{i\theta})^{-1}$, using (12). To first approximation, the results (31) and (39) can be generalized to this situation simply by multiplication by $(1-e^{i\theta_s})^{-1}$, since the integrand in (26) is presumably a highly peaked function of θ near θ_s. This approximation will break down if θ_s is near $2k\pi, k =$ integer.

In the case of zero focusing, the first term in eqn. (31) must be multiplied by

$$\left[1 - e^{i\omega\tau + \frac{\omega\tau}{2Q} - \frac{E}{M}}\right]^{-1}. \tag{44}$$

The second term is the complex conjugate of the first, as before.

For strong focusing, the same is done to eqn. (39), but with $E \rightarrow E_1$.

REFERENCES

[1] R.L. Gluckstern, R.K. Cooper, P.J. Channell, "Cumulative Beam Break Up in RF Linacs", Particle Accelerators, 16, 125 (1985).

[2] P.M. Lapostolle, A.L. Septier, Linear Accelerators, Amsterdam: North-Holland Publishing Company (1970), pp. 173 - 221.