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CHERN-SIMONS FORMS AND FOUR-DIMENSIONAL N=1 SUPERSPACE GEOMETRY

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ABSTRACT

The complete superspace geometry for Yang-Mills, chiral U(1) and Lorentz Chern-Simons forms is constructed. The analysis is completely off-shell and covers the cases of minimal, new minimal and 16-16 supergravity. Supersymmetry is guaranteed by construction. Invariant superfield actions are proposed.

1. INTRODUCTION

Green and Schwarz[1] have proposed a modification of the system of ten dimensional N=1 supersymmetric Yang-Mills theory coupled to supergravity[2] (viewed as the low-energy effective field theory[3] of superstring theory) which is free of Yang-Mills and of gravitational anomalies by chosing as gauge group either SO(32) or $E_8 \times E_8$ and by adding suitable local interactions. The local counterterms are higher derivative terms parametrised in terms of Chern-Simons forms[4,5] for both the Yang-Mills gauge potentials and the Lorentz connection. The counterterms proposed by Green and Schwarz are purely bosonic and thus violate supersymmetry. Green and Schwarz speculated "that restoring it would require introducing the entire infinite expansion of the superstring effective action".

The complications encountered in the attempts to supersymmetrize the Green-Schwarz anomaly cancellation mechanism are mainly due to the Chern-Simons forms associated with the Lorentz group. Partial answers, to low orders in α', have been obtained both in component fields[6,7,8] and in the framework of superspace geometry[9,10,11,12] in ten and six dimensions.

It is of course suggestive to study Chern-Simons forms in four dimensional supersymmetric theories[13,14,15,16,17] which are much better understood than the ten dimensional ones. Besides being technically less complicated we found this to be an interesting problem in its own right.

In this article we describe the complete four dimensional superspace geometry for Yang-Mills as well as supergravitational and chiral U(1) Chern-Simons forms in N=1 supergravity. The knowledge of the complete superspace geometry allows to obtain supersymmetry as well as BRS transformations[18] for the component fields in a compact way and is essential for the construction of invariant superfield actions[17,18].

In chapter two we give a compact review of U(1) superspace geometry, in chapter three we transcribe Chern's formula and the triangular equation, obtained from methods of algebraic geometry[20], to superspace geometry. In chapter four we present the complete superspace geometry for Lorentz, chiral U(1) and Yang-Mills Chern-Simons forms in curved U(1) superspace and in the conclusions we discuss some details of different N=1 supergravities and comment on the construction of invariant superspace actions[19].

2. SUPERSPACE GEOMETRY AND LOCAL R-INVARIANCE

In contrast to other $N=1$ supergravity theories new minimal supergravity[21] possesses local R-invariance[22]. For its geometric description it is therefore natural to start with a superspace geometry which contains the Lorentz group as well as a chiral $U(1)$ as structure group. It was pointed out by Müller[14] that such a superspace geometry can account for old minimal[23] and 16-16 supergravity[24,25] as well by imposing suitable torsion constraints and by removing afterwards the local $U(1)$ symmetry in particular ways. The important point is that there exists a set of natural torsion constraints which corresponds to a reducible supergravity multiplet of 16 bosonic and 16 fermionic components[14]. The reducible multiplet contains a component field of (mass) dimension two and cannot therefore be used for the construction of a Poincaré supergravity theory. Its superspace geometry can however be further specified to describe, alternatively, old minimal, new minimal or 16-16 supergravity. Besides this advantage of covering three supergravity theories at the same time, $U(1)$ superspace has a rather simple geometrical structure. This is in particular useful for our present purpose, the discussion of Chern-Simons forms especially of the supergravitational type where complicated higher polynomial expressions are expected to show up. For this reason and in order to be self contained we shall briefly review the properties of $U(1)$ superspace in this section and come back to the various subcases later on.

The basic superfields of $U(1)$ geometry are the supervielbein $E_M{}^A(z)$ and the superconnections $\phi_{MB}{}^A(z)$ and $A_M(z)$ for the Lorentz group and the chiral $U(1)$, respectively. From the corresponding one forms

$$\begin{aligned} E^A &= dz^M E_M{}^A \\ \phi_B{}^A &= dz^M \phi_{MB}{}^A \\ A &= dz^M A_M \end{aligned} \tag{2.1}$$

and the exterior derivative d in superspace one defines torsion, curvature and $U(1)$ field strength

$$\begin{aligned} T^A &= dE^A + E^B\phi_B{}^A + w(E^A)E^A A \\ R_B{}^A &= d\phi_B{}^A + \phi_B{}^C\phi_C{}^A \\ F &= dA \end{aligned} \tag{2.2}$$

as two forms, covariant with respect to local Lorentz and chiral U(1) transformations of the basic superfields.

The chiral U(1) weight $w(E^A)$ for the vielbein form is different for different values of the superindex $A \sim (a,\alpha,\dot{\alpha})$:

$$w(E^a) = 0 \quad , \quad w(E^\alpha) = 1 \quad , \quad w(E_{\dot{\alpha}}) = -1 \tag{2.3}$$

The nonvanishing parts of the Lorentz connection,

$$\phi_B{}^A \sim (\phi_b{}^a , \phi_\beta{}^\alpha , \phi^{\dot{\beta}}{}_{\dot{\alpha}}) \tag{2.4}$$

are related among each other as usual:

$$\begin{aligned} \phi_\beta{}^\alpha &= -\frac{1}{2}(\sigma^{ba})_\beta{}^\alpha \phi_{ba} \\ \phi^{\dot{\beta}}{}_{\dot{\alpha}} &= -\frac{1}{2}(\bar{\sigma}^{ba})^{\dot{\beta}}{}_{\dot{\alpha}} \phi_{ba} \end{aligned} \tag{2.5}$$

Remember that with our choice of structure group the Lorentz curvature and the U(1) field strengths

$$R_B{}^A = \frac{1}{2} E^C E^D R_{DCB}{}^A \tag{2.6}$$

$$F = \frac{1}{2} E^C E^D F_{DC} \tag{2.7}$$

are completely defined in terms of the coefficients of the torsion two form

$$T^A = \frac{1}{2} E^B E^C T_{CB}{}^A \tag{2.8}$$

and their covariant derivatives[26] as a consequence of the superspace Bianchi identities

$$DT^A - E^B R_B{}^A - w(E^A) E^A F = 0 \tag{2.9}$$

In addition to the particular choice of structure group certain natural torsion constraints are imposed[14)]. They allow to express all the coefficients of torsion, curvature and U(1) field strength in terms of the covariant superfields R, R^+ and G_a of canonical dimension one and $W_{\underbrace{\gamma\beta\alpha}}$, $W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}}$ of canonical dimension 3/2 (x^m has dimension -1) and their covariant derivatives. A detailed discussion of the torsion constraints and their consequences can be found in ref. 14). Here we just present the catalogue of nonvanishing torsion coefficients. At dimension zero one has

$$T_\gamma{}^{\dot\beta a} = -2i(\sigma^a\epsilon)_\gamma{}^{\dot\beta} \tag{2.10}$$

All the torsion coefficients of canonical dimension 1/2 vanish whereas the superfields R, R^+ and G_a are located in the dimension one torsions

$$T_{\gamma b\dot\alpha} = -i\,\sigma_{b\gamma\dot\alpha}R^+ \;,\quad T^{\dot\gamma}{}_b{}^\alpha = -i\,\bar\sigma^{b\dot\gamma\alpha}R \tag{2.11}$$

$$T_{\gamma b}{}^\alpha = \frac{i}{2}(\sigma_c\bar\sigma_b)_\gamma{}^\alpha G^c \;,\quad T^{\dot\gamma}{}_{b\dot\alpha} = -\frac{i}{2}(\bar\sigma_c\sigma_b)^{\dot\gamma}{}_{\dot\alpha}G^c \tag{2.12}$$

The Weyl spinor superfields $W_{\underbrace{\gamma\beta\alpha}}$ and $W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}}$ appear in the dimension 3/2 torsion coefficients $T_{cb}{}^\alpha$ and $T_{cb\dot\alpha}$. This becomes most transparent in spinor notation with vector indices replaced by pairs of undotted and dotted spinor indices,

$$T_{\gamma\dot\gamma\,\beta\dot\beta}{}^A = \sigma^c_{\gamma\dot\gamma}\sigma^b_{\beta\dot\beta}T_{cb}{}^A \;,\quad G_{\alpha\dot\alpha} = \sigma^a_{\alpha\dot\alpha}G_a \tag{2.13}$$

The dimension 3/2 torsions are then given by the following expressions:

$$\begin{aligned}
T_{\gamma\dot\gamma\,\beta\dot\beta\,\alpha} &= 2\epsilon_{\dot\gamma\dot\beta}W_{\underbrace{\gamma\beta\alpha}} + \frac{2}{3}\epsilon_{\dot\gamma\dot\beta}\left(\epsilon_{\alpha\beta}S_\gamma + \epsilon_{\alpha\gamma}S_\beta\right) - 2\epsilon_{\gamma\beta}T_{\underbrace{\dot\gamma\dot\beta}\alpha} \\
T_{\underbrace{\dot\gamma\dot\beta}\alpha} &= -\frac{1}{4}\left(\mathcal{D}_{\dot\gamma}G_{\alpha\dot\beta} + \mathcal{D}_{\dot\beta}G_{\alpha\dot\gamma}\right) \\
S_\gamma &= -\mathcal{D}_\gamma R + \frac{1}{4}\mathcal{D}^{\dot\gamma}G_{\gamma\dot\gamma}
\end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
T_{\gamma\dot\gamma\,\beta\dot\beta\,\dot\alpha} &= 2\epsilon_{\dot\gamma\dot\beta}\, T_{\underset{\smile}{\gamma\beta}\,\dot\alpha} - 2\epsilon_{\gamma\beta}\, \bar W_{\underset{\smile}{\dot\gamma\dot\beta\dot\alpha}} - \tfrac{2}{3}\,\epsilon_{\gamma\beta}\left(\epsilon_{\dot\alpha\dot\beta} S_{\dot\gamma} + \epsilon_{\dot\alpha\dot\gamma} S_{\dot\beta}\right) \\
T_{\underset{\smile}{\gamma\beta}\,\dot\alpha} &= \tfrac{1}{4}\left(\mathcal{D}_\gamma G_{\beta\dot\alpha} + \mathcal{D}_\beta G_{\gamma\dot\alpha}\right) \qquad (2.15) \\
S_{\dot\gamma} &= \mathcal{D}_{\dot\gamma} R^{+} - \tfrac{1}{4}\, \mathcal{D}^{\gamma} G_{\gamma\dot\gamma}
\end{aligned}
$$

It should be noted that the covariant derivatives appearing in the last two equations are covariant with respect to Lorentz and chiral U(1) transformations. For a zero form superfield X_A of chiral weight $w(X_A)$ for instance, the covariant derivative is defined as

$$\mathcal{D}_B X_A = E_B{}^M \partial_M X_A - \phi_{BA}{}^C X_C + w(X_A) A_B X_A \qquad (2.16)$$

and, consequently, the graded commutator

$$\left(\mathcal{D}_B, \mathcal{D}_A\right) \equiv \mathcal{D}_B \mathcal{D}_A - (-)^{ab} \mathcal{D}_A \mathcal{D}_B \qquad (2.17)$$

of two covariant derivatives takes the form

$$\left(\mathcal{D}_C, \mathcal{D}_B\right) X_A = - T_{CB}{}^F \mathcal{D}_F X_A - R_{CBA}{}^F X_F + w(X_A) F_{CB} X_A \qquad (2.18)$$

The chiral weight of the covariant derivative is determined from that of the vielbein,

$$w\left(\mathcal{D}_A\right) = - w\left(E^A\right) \qquad (2.19)$$

If the vielbein transforms under a chiral U(1) transformation as

$$E^{A\,\prime} = E^A\, Y^{w(E^A)} \qquad (2.20)$$

then $\mathcal{D}_A$ and A_A transform under the same transformation as $(D_A \equiv E_A{}^M \partial_M)$

$$\mathcal{D}'_A = Y^{-w(E^A)}\mathcal{D}_A$$
$$A'_A = Y^{-w(E^A)}\left(A_A - Y^{-1}\mathcal{D}_A Y\right) \tag{2.21}$$

Similarly the chiral U(1) weights of the coefficients of the torsion two form (2.8) are determined from those of the vielbein through the relation

$$w\left(T_{CB}{}^A\right) = w\left(E^A\right) - w\left(E^C\right) - w\left(E^B\right) \tag{2.22}$$

The chiral weights of the basic covariant superfields describing torsions and curvatures follow from eqs (2.11-15):

$$w\left(R^+\right) = -2 \quad , \quad w(R) = +2 \; ,$$
$$w\left(G_a\right) = 0 \tag{2.23}$$
$$w\left(W_{\gamma\beta\alpha}\right) = +1 \quad , \quad w\left(\bar{W}_{\dot\gamma\dot\beta\dot\alpha}\right) = -1 \; .$$

As already mentioned above the coefficients of Lorentz curvatures and U(1) field strengths too are expressed in terms of these few superfields. At dimension one we obtain

$$R_{\delta\gamma\, ba} = 8\left(\sigma_{ba}\epsilon\right)_{\delta\gamma} R^+ \tag{2.24}$$

$$R^{\dot\delta\dot\gamma}{}_{ba} = 8\left(\bar\sigma_{ba}\epsilon\right)^{\dot\delta\dot\gamma} R \tag{2.25}$$

$$R_\delta{}^{\dot\gamma}{}_{ba} = -2i\, G^d \left(\sigma^c\epsilon\right)_\delta{}^{\dot\gamma}\, \varepsilon_{dcba} \tag{2.26}$$

for the Lorentz curvatures whereas the chiral U(1) field strengths are given by

$$F_{\beta\alpha} = 0 \quad , \quad F^{\dot\beta\dot\alpha} = 0 \tag{2.27}$$

$$F_{\beta}{}^{\dot\alpha} = 3(\sigma^a\epsilon)_{\beta}{}^{\dot\alpha} G_a \tag{2.28}$$

At dimension 3/2 one finds

$$R_{\delta c ba} = i\sigma_{c\delta\dot\delta} T_{ba}{}^{\dot\delta} + i\sigma_{b\delta\dot\delta} T_{ca}{}^{\dot\delta} + i\sigma_{a\delta\dot\delta} T_{bc}{}^{\dot\delta} \tag{2.29}$$

$$R^{\dot\delta}{}_{cba} = i\bar\sigma_c{}^{\dot\delta\delta} T_{ba\delta} + i\bar\sigma_b{}^{\dot\delta\delta} T_{ca\delta} + i\bar\sigma_a{}^{\dot\delta\delta} T_{bc\delta} \tag{2.30}$$

and

$$F_{\delta c} = \frac{3i}{2}\mathcal{D}_\delta G_c + \frac{i}{2}\sigma_{c\delta\dot\delta}\bar X^{\dot\delta} \tag{2.31}$$

$$F^{\dot\delta}{}_c = \frac{3i}{2}\mathcal{D}^{\dot\delta} G_c - \frac{i}{2}\bar\sigma_c{}^{\dot\delta\delta} X_\delta \tag{2.32}$$

with the definitions

$$X_\delta = \mathcal{D}_\delta R - \mathcal{D}^{\dot\delta} G_{\delta\dot\delta} \tag{2.33}$$

$$\bar X^{\dot\delta} = \mathcal{D}^{\dot\delta} R^+ + \mathcal{D}_\delta G^{\delta\dot\delta} \tag{2.34}$$

Finally, having expressed torsions, curvatures and U(1) field strengths in terms of few covariant superfields the Bianchi identities themselves are now represented by a small set of rather simple conditions for these superfields:

$$\mathcal{D}_\alpha R^+ = 0 \quad , \quad \mathcal{D}^{\dot\alpha} R = 0 \tag{2.35}$$

$$\mathcal{D}_\alpha \bar W_{\dot\gamma\dot\beta\dot\alpha} = 0 \quad , \quad \mathcal{D}_{\dot\alpha} W_{\gamma\beta\alpha} = 0 \tag{2.36}$$

$$\mathcal{D}_\alpha \bar X^{\dot\alpha} = 0 \quad , \quad \mathcal{D}^{\dot\alpha} X_\alpha = 0 \tag{2.37}$$

$$\mathcal{D}^\alpha X_\alpha - \mathcal{D}_{\dot\alpha}\bar X^{\dot\alpha} = 0 \tag{2.38}$$

$$\mathcal{D}^{\alpha} T_{cb\alpha} + \mathcal{D}_{\dot\alpha} T_{cb}{}^{\dot\alpha} = 0 \tag{2.39}$$

The expressions for the Lorentz curvatures and U(1) field strengths at dimension two and some additional notational informations are collected in the appendix. We conclude this section by showing how covariant redefinitions of the vectorial part A_a of the U(1) gauge potential change the form of eqs (2.12), (2.28) and (2.31-32). Call A_0 the U(1) gauge potential of the geometric situation described in this section and define

$$A_1 = A_0 + \Sigma \tag{2.40}$$

The superspace geometry of ref. 14) is then obtained from

$$\Sigma = -\frac{3i}{2} E^{a} G_a \tag{2.41}$$

In particular, $F_{\beta}{}^{\dot\alpha}(A_1) = 0$.

On the other hand the superspace geometry corresponding to old minimal supergravity[27] is obtained from $A_1 = 0$. As a consequence the torsions of eq.(2.12) become

$$T^{o.m.}{}_{\gamma b}{}^{\alpha} = \frac{3i}{2}\delta_{\gamma}^{\alpha} G_b + \frac{i}{2} G^{c}(\sigma_c\bar\sigma_b)_{\gamma}{}^{\alpha} \tag{2.42}$$

$$T^{o.m.\dot\gamma}{}_{b\dot\alpha} = -\frac{3i}{2}\delta^{\dot\gamma}_{\dot\alpha} G_b - \frac{i}{2} G^{c}(\bar\sigma_c\sigma_b)^{\dot\gamma}{}_{\dot\alpha} \tag{2.43}$$

and from $X_\delta = 0$, $\bar X^{\dot\delta} = 0$ and eqs (2.33),(2.34) it follows that

$$\mathcal{D}_{\delta} R = \mathcal{D}^{\dot\delta} G_{\delta\dot\delta} \quad , \quad \mathcal{D}^{\dot\delta} R^{+} = -\mathcal{D}_{\delta} G^{\delta\dot\delta} \tag{2.44}$$

in accordance with ref. 27).

New minimal and 16-16 supergravity contain antisymmetric tensor gauge fields. We postpone therefore the discussion of these cases after the description of the superspace geometry of the two form gauge potential.

3. CHERN'S FORMULA AND THE TRIANGULAR EQUATION IN SUPERSPACE

In this chapter we establish the superspace version of Chern's formula and the triangular equation[20] in a general curved superspace without assuming any constraints for the gravitational and Yang-Mills parts of superspace geometry. We use the triangular equation to calculate how the Chern-Simons forms change under gauge transformations and under covariant redefinitions of the connection.

Consider a superspace Yang-Mills potential

$$\mathcal{A} = E^A \mathcal{A}_A \tag{3.1}$$

subject to superspace gauge transformations,

$$\mathcal{A}' = X^{-1}(\mathcal{A} - d)X \tag{3.2}$$

Its covariant field strength

$$\mathcal{F}(\mathcal{A}) = d\mathcal{A} + \mathcal{A}\mathcal{A} \tag{3.3}$$

satisfies Bianchi identities

$$d\mathcal{F} - \mathcal{A}\tilde{\mathcal{F}} + \tilde{\mathcal{F}}\mathcal{A} = 0 \tag{3.4}$$

Chern's formula and the triangular equation in this geometrical setting are special cases of the superspace version of the extended Cartan homotopy formula of Mañes, Stora and Zumino[20].

For the super Chern formula take two super Yang Mills potentials $\mathcal{A}_0$ and $\mathcal{A}_1$. The convex combination

$$\mathcal{A}_t = (1-t)\mathcal{A}_0 + t\mathcal{A}_1 \tag{3.5}$$

with $t \in [0,1] = T_1$ is still a connection. The covariant field strength

$$\mathcal{F}(\mathcal{A}_t) \equiv \mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t\mathcal{A}_t \tag{3.6}$$

satisfies Bianchi identities

$$d\mathcal{F}_t - \mathcal{A}_t\mathcal{F}_t + \mathcal{F}_t\mathcal{A}_t = 0 \tag{3.7}$$

The symmetric invariant polynomials $J_n(\mathcal{F},\ldots,\mathcal{F})$ are closed with respect to the superspace exterior derivative due to the Bianchi identities and the super Chern formula reads

$$\begin{aligned} J_n(\mathcal{F}_0,\ldots,\mathcal{F}_0) - J_n(\mathcal{F}_1,\ldots,\mathcal{F}_1) &= dQ_{2n-1}(\mathcal{A}_0,\mathcal{A}_1) \\ Q_{2n-1}(\mathcal{A}_0,\mathcal{A}_1) &= n\int_{T_1} J_n(d_t\mathcal{A}_t,\mathcal{F}_t,\ldots,\mathcal{F}_t) \end{aligned} \tag{3.8}$$

In this equation the integration is over the unit interval and d_t denotes the exterior derivative with respect to t,

$$d_t^2 = 0 \quad , \quad dd_t + d_td = 0 \tag{3.9}$$

in particular:

$$d_t\mathcal{A}_t = dt\,(\mathcal{A}_0 - \mathcal{A}_1) \tag{3.10}$$

As can be directly seen from its definition the $2n-1$ form $Q_{2n-1}(\mathcal{A}_0,\mathcal{A}_1)$ is antisymmetric in its arguments,

$$Q_{2n-1}(\mathcal{A}_0,\mathcal{A}_1) = -Q_{2n-1}(\mathcal{A}_1,\mathcal{A}_0) \tag{3.11}$$

and invariant under simultaneous gauge transformations of $\mathcal{A}_0$ and $\mathcal{A}_1$.

From Chern's formula one sees immediately that

$$d\left\{Q_{2n-1}(\mathcal{A}_0,\mathcal{A}_1) + Q_{2n-1}(\mathcal{A}_1,\mathcal{A}_2) + Q_{2n-1}(\mathcal{A}_2,\mathcal{A}_0)\right\} = 0 \tag{3.12}$$

The triangular equation describes the exactness of this same combination,

$$Q_{2n-1}(\mathcal{A}_0,\mathcal{A}_1)+Q_{2n-1}(\mathcal{A}_1,\mathcal{A}_2)+Q_{2n-1}(\mathcal{A}_2,\mathcal{A}_0)=d\chi_{2n-2}(\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2)$$

$$\chi_{2n-2}(\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2)=\frac{n(n-1)}{2}\int_{T_2}\mathcal{J}_n\left(d_t\mathcal{A}_t,d_t\mathcal{A}_t,\cdot,\mathcal{F}_t\right) \tag{3.13}$$

where now $\mathcal{A}_t$ interpolates between the three connections $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$,

$$\mathcal{A}_t=\mathcal{A}_0+t^1(\mathcal{A}_1-\mathcal{A}_0)+t^2(\mathcal{A}_2-\mathcal{A}_0) \tag{3.14}$$

with $t^i\in[0,1]$ for $i=1,2$. d_t is the corresponding exterior derivative and the integration is over the standard two simplex in parameter space, T_2.

The derivation of Chern's formula and the triangular equation in ref. 20) is purely algebraic, employing the homotopy operation ℓ_t defined on $\mathcal{A}_t$, $\mathcal{F}_t$, $d_t\mathcal{A}_t$, $d_t\mathcal{F}_t$,

$$\ell_t\mathcal{A}_t=0\quad,\quad\ell_t\mathcal{F}_t=d_t\mathcal{A}_t \tag{3.15}$$

and satisfying

$$\ell_t d-d\,\ell_t=d_t \tag{3.16}$$

$$\ell_t d_t-d_t\,\ell_t=0 \tag{3.17}$$

It is therefore completely legitimate to apply these constructions to superspace geometry, as we are doing here. A useful relation we shall need later arises when taking $\mathcal{A}_0$ to vanish and $\mathcal{A}_1$ to be a pure gauge in eq.(3.8):

$$\mathcal{A}_0=0\ ,\ \mathcal{A}_1=-X^{-1}dX\equiv\Omega\ ,$$

$$d\Omega+\Omega\Omega=0 \tag{3.18}$$

It follows that

$$d\, Q_{2n-1}(0,\Omega) = 0 \tag{3.19}$$

We introduce a family of gauge transformations parametrised with $s \in [0,1]$ such that $s = 0$ corresponds to the identity, $X_o = 1$ and $X_1 = X$ and define

$$\Omega_s = - X_s^{-1} dX_s \tag{3.20}$$

Making use of the homotopy operation

$$\ell_s \Omega_s = - X_s^{-1} d_s X_s \tag{3.21}$$

we arrive at

$$Q_{2n-1}(0,\Omega) = d \int_{T_1} \ell_s Q_{2n-1}(0,\Omega_s) \tag{3.22}$$

In physical applications one usually encounters Chern-Simons forms which depend on one connection only,

$$Q_{2n-1}(\mathcal{A}) \equiv Q_{2n-1}(\mathcal{A},0) \tag{3.23}$$

and one may ask how this expression changes under a gauge transformation of $\mathcal{A}$ alone. This amounts to calculate

$$Q_{2n-1}(X^{-1}\mathcal{A}X - X^{-1}dX, 0) - Q_{2n-1}(\mathcal{A},0)$$

The answer is easily obtained from the triangular equation (3.13) and from eq. (3.22). In eq.(3.13) take

$$\begin{aligned} \mathcal{A}_1 &= {}^{X^{-1}}\!\mathcal{A}_o = X\mathcal{A}_o X^{-1} - X dX^{-1} \\ \mathcal{A}_2 &= \mathcal{A} \end{aligned} \tag{3.24}$$

and use the fact that

$$Q_{2n-1}\left({}^{\chi^{-1}}\mathcal{A}_0, \mathcal{A}_1\right) = Q_{2n-1}\left(\mathcal{A}_0, {}^{\chi}\mathcal{A}_1\right) \tag{3.25}$$

to arrive at

$$\begin{aligned} &Q_{2n-1}\left({}^{\chi}\mathcal{A}_0, \mathcal{A}_0\right) + Q_{2n-1}\left(\mathcal{A}_0, {}^{\chi}\mathcal{A}\right) \\ &+ Q_{2n-1}\left(\mathcal{A}, \mathcal{A}_0\right) = d\chi_{2n-2}\left(\mathcal{A}_0, {}^{\chi^{-1}}\mathcal{A}_0, \mathcal{A}\right) \end{aligned} \tag{3.26}$$

Then take $\mathcal{A}_0 = 0$ and use (3.22) to obtain

$$\begin{aligned} &Q_{2n-1}\left(0, {}^{\chi}\mathcal{A}\right) - Q_{2n-1}\left(0, \mathcal{A}\right) = d\,\Delta_{2n-2} \\ &\Delta_{2n-2} = \chi_{2n-2}\left(0, -\chi d\chi^{-1}, \mathcal{A}\right) + \sum_{T_1} \ell_s Q_{2n-1}\left(0, \Omega_s\right) \end{aligned} \tag{3.27}$$

This shows that the physicist's Chern-Simons form changes under a gauge transformation of the Yang-Mills potential by a total derivative,

$$Q_{2n-1}\left({}^{\chi}\mathcal{A}\right) - Q_{2n-1}\left(\mathcal{A}\right) = -\,d\,\Delta_{2n-2} \tag{3.28}$$

Observe that one might construct a gauge invariant object at the expense of introducing a $2n-2$ form gauge potential B_{2n-2} with gauge transformations

$$B'_{2n-2} = B_{2n-2} + \Delta_{2n-2} + d\,X_{2n-3} \tag{3.29}$$

The $2n-1$ form

$$H_{2n-1} = d\,B_{2n-2} + Q_{2n-1}\left(\mathcal{A}\right) \tag{3.30}$$

is then gauge invariant and satisfies Bianchi identities

$$d\,H_{2n-1} = \mathcal{I}_n\left(\mathcal{F}, \ldots, \mathcal{F}\right) \tag{3.31}$$

What is the effect of a covariant redefinition of the connection on eq.(3.30)? Suppose Γ to be some gauge covariant Lie algebra valued one form and define

$$\mathcal{A}_1 = \mathcal{A} + \Gamma \tag{3.32}$$

The Chern-Simons forms for the two connections are related through the triangular equation. Taking $\mathcal{A}_2 = 0$ in eq.(3.13) one obtains

$$Q_{2n-1}(\mathcal{A}_1) - Q_{2n-1}(\mathcal{A}) = Q_{2n-1}(\mathcal{A}_1, \mathcal{A}) + d\chi_{2n-2}(\mathcal{A}, \mathcal{A}_1, 0) \tag{3.33}$$

In eq.(3.30) $\chi_{2n-2}(\mathcal{A}, \mathcal{A}_1, 0)$ can be absorbed in a redefinition of B_{2n-2} and the gauge covariant quantity $Q_{2n-1}(\mathcal{A}, \mathcal{A}_1)$ gives rise to the new field strength

$$H_{2n-1}(\mathcal{A}_1) = H_{2n-1}(\mathcal{A}) + Q_{2n-1}(\mathcal{A}_1, \mathcal{A}) \tag{3.34}$$

For illustrative purpose we have, in this chapter, only considered the super Yang-Mills potential. It should be clear that the same arguments hold for the connection forms which correspond to the structure group of flexible (curved) superspace.

4. CHERN-SIMONS FORMS IN U(1) SUPERSPACE

In four dimensional superspace we are concerned with Chern-Simons forms

$$Q_3(\mathcal{A}) = J_2(\mathcal{A}, \mathcal{F} - \tfrac{1}{3}\mathcal{A}\mathcal{A}) \tag{4.1}$$

which satisfy

$$dQ_3(\mathcal{A}) = J_2(\mathcal{F}, \mathcal{F}) \tag{4.2}$$

These equations are obtained from (3.8) if one takes $n=2$, $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_1 = 0$ and performs explicitly the t-integration. $\mathcal{A}$ denotes the superspace connection of some supersymmetric Yang-Mills theory formulated in flexible (curved) U(1) superspace. In addition we introduce the Chern-Simons forms constructed from the superspace connections $\phi_B{}^A$ and A corresponding to local Lorentz and chiral U(1) transformations, respectively, as defined in section 2:

$$Q_3(\phi) = J_2(\phi, R - \tfrac{1}{3}\phi\phi) \tag{4.3}$$

$$Q_3(A) = J_2(A, F) \tag{4.4}$$

They satisfy, of course,

$$dQ_3(\phi) = J_2(R, R) \tag{4.5}$$

$$dQ_3(A) = J_2(F, F) \tag{4.6}$$

In section two it was pointed out that, as a consequence of the choice of structure group and torsion constraints, all the torsions and curvatures of U(1) superspace can be expressed in terms of the few superfields R, R^+, G_a, $W_{\underbrace{\gamma\beta\alpha}}$, $W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}}$ and their covariant derivatives. Likewise the field strengths of supersymmetric Yang-Mills theory,

$$\mathcal{F} = \tfrac{1}{2} E^A E^B \mathcal{F}_{BA} \tag{4.7}$$

can be expressed in terms of the superfields λ_α, $\bar\lambda^{\dot\alpha}$ as a consequence of the constraints

$$\mathcal{F}_{\beta\alpha} = 0 \quad , \quad \mathcal{F}^{\dot\beta\dot\alpha} = 0 \quad , \quad \mathcal{F}_\beta{}^{\dot\alpha} = 0 \tag{4.8}$$

The remaining field strengths take the form

$$\mathcal{F}_{\beta a} = \frac{i}{2}\sigma_{a\,\beta\dot\beta}\bar\lambda^{\dot\beta} \tag{4.9}$$

$$\mathcal{F}^{\dot\beta}{}_a = -\frac{i}{2}\bar\sigma_a{}^{\dot\beta\beta}\lambda_\beta \tag{4.10}$$

$$\mathcal{F}_{ba} = \frac{1}{4}(\epsilon\sigma_{ba})^{\beta\alpha}\mathcal{D}_\beta\lambda_\alpha + \frac{1}{4}(\bar\sigma_{ba}\epsilon)^{\dot\beta\dot\alpha}\mathcal{D}_{\dot\alpha}\bar\lambda_{\dot\beta} \tag{4.11}$$

and the Bianchi identities reduce to

$$\mathcal{D}^{\dot\alpha}\lambda_\alpha = 0 \quad , \quad \mathcal{D}_\alpha\bar\lambda^{\dot\alpha} = 0 \tag{4.12}$$

$$\mathcal{D}^\alpha\lambda_\alpha - \mathcal{D}_{\dot\alpha}\bar\lambda^{\dot\alpha} = 0 \tag{4.13}$$

The covariant derivatives are now supposed to be covariant with respect to local Lorentz, chiral U(1) and non-abelian gauge transformations in superspace. The chiral U(1) weights of λ_α, $\bar\lambda^{\dot\alpha}$ are

$$w(\lambda_\alpha) = +1 \quad , \quad w(\bar\lambda^{\dot\alpha}) = -1 \tag{4.14}$$

Following the ideas presented in the previous chapter, we introduce a two form gauge potential in superspace,

$$B = \frac{1}{2}dz^M dz^N B_{NM} \tag{4.15}$$

in order to construct an invariant three form

$$H = dB + Q_3(\mathcal{A}) + Q_3(\Phi) + Q_3(A) \tag{4.16}$$

The corresponding Bianchi identities then read

$$dH = \mathcal{J}_2(\mathcal{F},\mathcal{F}) + \mathcal{J}_2(R,R) + \mathcal{J}_2(F,F) \tag{4.17}$$

So far we have not considered constraints for the invariant three form

$$H = \frac{1}{3!} E^A E^B E^C H_{CBA} \tag{4.18}$$

In the case of vanishing Chern-Simons forms it is well known[28,29] how to impose the standard constraints which define consistently the superspace geometry of a multiplet containing a real scalar field, an antisymmetric tensor gauge field, a Majorana spinor and no auxiliary fields. It is however not guaranteed that the same constraints can be maintained in the presence of Chern-Simons forms. One has to convince oneself that the constraints in the various geometric sectors of superspace geometry are compatible with eqs (4.16) and (4.17).

For the case of Yang-Mills Chern-Simons forms alone it is known[13,14] that the usual constraints for supersymmetric gauge theory, eqs (4.8), and the standard constraints for the three form H are indeed compatible with (4.17), even in flexible (curved) superspace. In order to accommodate Lorentz Chern-Simons forms as well one has to allow for additional quadratic terms to appear in the covariant components of the three form H, as described in the following. We replace the invariant polynomials in (4.17) by symmetrized traces, e.g.

$$\mathcal{J}_2(\mathcal{F},\mathcal{F}) = t(\mathcal{F}\mathcal{F}) \tag{4.19}$$

$$\mathcal{J}_2(R,R) = \frac{\tau}{2} R_b{}^a R_a{}^b \tag{4.20}$$

$$\mathcal{J}_2(F,F) = \sigma FF \tag{4.21}$$

τ and σ are constants and t stands for some suitably normalized trace. The Bianchi identities (4.17) are equations which are four forms in superspace,

$$I = \frac{1}{4!} E^A E^B E^C E^D I_{DCBA} =$$
$$= \frac{1}{4!} E^A E^B E^C E^D \Big\{ 4 \mathcal{D}_D H_{CBA} + 6 T_{DC}{}^F H_{FBA} \qquad (4.22)$$
$$-2t(\mathcal{F}_{DC}\mathcal{F}_{BA}) - \tau R_{DCf}{}^e R_{BAe}{}^f - 2\sigma F_{DC} F_{BA} \Big\} = 0$$

It is straightforward to convince oneself that the constraints

$$H_{\gamma\beta\alpha} = H_{\gamma\beta}{}^{\dot\alpha} = H_\gamma{}^{\dot\beta\dot\alpha} = H^{\dot\gamma\dot\beta\dot\alpha} = 0 \qquad (4.23)$$

are compatible with the properties of F, $R_b{}^a$, $\mathcal{F}$ and eq.(4.22). Otherwise stated the Bianchi identities $I_{\delta\gamma\beta\alpha}$, $I_{\delta\gamma\beta}{}^{\dot\alpha}$ and their complex conjugates do not contain any quadratic terms in field strengths or curvatures. This is no longer true for the equation of index structure $I_{\delta\gamma\beta a}$ and its complex conjugate and we are therefore led to demand

$$H_{\gamma\beta\, a} = 16 i \tau\, G^d R^+ (\sigma_{ad}\epsilon)_{\gamma\beta} \qquad (4.24)$$

$$H^{\dot\gamma\dot\beta}{}_a = -16 i \tau\, G^d R\, (\bar\sigma_{ad}\epsilon)^{\dot\gamma\dot\beta} \qquad (4.25)$$

Furthermore we define

$$H_\gamma{}^{\dot\beta}{}_a = -2i (\sigma^b \epsilon)_\gamma{}^{\dot\beta} H_{ba} \qquad (4.26)$$

and require the antisymmetric part of H_{ba} to vanish,

$$H_{ba} = H_{ab} \qquad (4.27)$$

This can always be achieved by a covariant redefinition of the vectorial part of the two form gauge potential B. The traceless part of H_{ba} is then determined from the Bianchi identity $I^{\dot\delta\dot\gamma}{}_{\beta\alpha}$ and we parametrize the trace part such that

$$H_{\gamma}{}^{\dot\beta}{}_{a} = -2i(\sigma_a\epsilon)_{\gamma}{}^{\dot\beta}(L - 4\tau RR^{+} - \tau G^{b}G_{b}) + 2i(\sigma^{b}\epsilon)_{\gamma}{}^{\dot\beta}(\tau + \tfrac{9\sigma}{4})G_{b}G_{a} \tag{4.28}$$

L is a new independent, real superfield. The Bianchi identities $I_{\delta\gamma}{}^{\dot\beta}{}_{a}$ and $I^{\dot\delta\dot\gamma}{}_{\beta a}$ serve to determine $H_{\gamma ba}$ and $H^{\dot\gamma}{}_{ba}$,

$$H_{\gamma\,ba} = 2(\sigma_{ba})_{\gamma}{}^{\varphi}\mathcal{D}_{\varphi}L - 8\tau R^{+}T_{ba\gamma} + 4\tau\sigma^{c}_{\gamma\dot\gamma}G_{c}T_{ba}{}^{\dot\gamma} - i\nu\,\delta^{dc}_{ba}G_{d}\left(F_{\gamma c} - \tfrac{3i}{4}\mathcal{D}_{\gamma}G_{c}\right) \tag{4.29}$$

$$H^{\dot\gamma}{}_{ba} = 2(\bar\sigma_{ba})^{\dot\gamma}{}_{\dot\varphi}\mathcal{D}^{\dot\varphi}L - 8\tau R\,T_{ba}{}^{\dot\gamma} - 4\tau\bar\sigma^{c\dot\gamma\gamma}G_{c}T_{ba\gamma} - i\nu\,\delta^{dc}_{ba}G_{d}\left(F^{\dot\gamma}{}_{c} - \tfrac{3i}{4}\mathcal{D}^{\dot\gamma}G_{c}\right) \tag{4.30}$$

$$\nu = 3\sigma + 4\tau \quad , \quad \delta^{dc}_{ba} = \delta^{d}_{b}\delta^{c}_{a} - \delta^{c}_{b}\delta^{d}_{a} \tag{4.31}$$

The Bianchi identities $I_{\delta\gamma ba}$ and $I^{\dot\delta\dot\gamma}{}_{ba}$ then take the following compact form:

$$(\mathcal{D}^{\alpha}\mathcal{D}_{\alpha} - 8R^{+})L = \frac{t}{2}(\bar\lambda_{\dot\alpha}\bar\lambda^{\dot\alpha}) + \frac{\nu}{6}\bar\chi_{\dot\alpha}\bar\chi^{\dot\alpha} - 8\tau\,\bar W_{\dot\gamma\dot\beta\dot\alpha}\bar W^{\dot\gamma\dot\beta\dot\alpha} \tag{4.32}$$

$$(\mathcal{D}_{\dot\alpha}\mathcal{D}^{\dot\alpha} - 8R)L = \frac{t}{2}(\lambda^{\alpha}\lambda_{\alpha}) + \frac{\nu}{6}\chi^{\alpha}\chi_{\alpha} - 8\tau\,W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} \tag{4.33}$$

Observe that (as already indicated above) for $t = \tau = \nu = 0$ one obtains just the superspace geometry describing the supermultiplet of the antisymmetric tensor gauge field.

We still have to discuss the remaining Bianchi identities. From $I_{\delta}{}^{\dot\gamma}{}_{ba}$ we learn what the $\theta\bar\theta$ component of the superfield L looks like, it contains the field strength of the antisymmetric tensor and some non-linear terms:

$$
\begin{aligned}
&[\mathcal{D}_{\alpha}, \mathcal{D}_{\dot{\alpha}}] L - 4\sigma^{a}{}_{\alpha\dot{\alpha}} L G_a + t(\lambda_\alpha \bar{\lambda}_{\dot{\alpha}}) + \frac{v}{3} \chi_\alpha \bar{\chi}_{\dot{\alpha}} \\
&+ \frac{1}{3!} \sigma_{d\alpha\dot{\alpha}} \varepsilon^{dcba} \Big\{ 2H_{cba} + 16\tau \left(T_{cb}{}^{\varphi} F_{\varphi a} - T_{cb\dot{\varphi}} F^{\dot{\varphi}}{}_a \right) \\
&+ \frac{32i}{3} \tau \bar{\sigma}_c^{\dot{\gamma}\gamma} F_{\dot{\gamma}b} F_{\gamma a} - 3iv\, G_c \left(2F_{ba} - 3i \mathcal{D}_b G_a \right) \Big\} = 0 .
\end{aligned}
\tag{4.34}
$$

The higher dimensional Bianchi identities, $I_{\delta cba}$, $I^{\dot{\delta}}{}_{cba}$ and I_{dcba} do not contain new information. They are trivially satisfied as a consequence of the results obtained so far.

5. CONCLUSIONS

We have presented the complete superspace geometry for Yang-Mills, chiral U(1) and Lorentz Chern-Simons forms in four-dimensional $N=1$ supergravity. The analysis was carried out in U(1) superspace which has local R and Lorentz transformations in its structure group. For the case of Lorentz Chern-Simons forms consistency of the superspace Bianchi identities requires new additional terms in certain super field strengths of the two form gauge potential (which was introduced to compensate the gauge variations of the superspace Chern-Simons forms). The additional terms are quadratic in the covariant fields which describe all the torsions and curvatures in the supergravitational sector. The superspace geometry exhibited in this paper still can be particularized to old minimal, new minimal or 16-16 supergravity.

The reduction to old minimal supergravity has been explained above, eqs (2.42-44). Old minimal supergravity does not contain an antisymmetric tensor gauge field. The coupling of the linear multiplet L to old minimal supergravity via the invariant superfield action*)

$$\int E f(L) \tag{5.1}$$

would lead to a component field Lagrangian with inconvenient kinetic terms (the same situation arises in coupling the Kähler potential $K(\phi,\phi^+)$ to supergravity[30,31]). Canonical kinetic terms are obtained by performing a Weyl rescaling in superspace and certain covariant redefinitions of the vielbein and the Lorentz connection superfields. Since f(L) is a real function, the Weyl rescaling will not leave the torsion constraints invariant. The superspace geometry in terms of the new torsions and curvatures is then, for suitable f(L), that of 16-16 supergravity[24,25,14]. The corresponding superspace action is simply[17]

$$\mathcal{L}_{16-16} = \int E \tag{5.2}$$

and the same action in terms of component fields has been worked out in ref. 14). It should be clear that a similar procedure, in superspace, holds for the case of the Kähler potential.

For the coupling of Chern-Simons forms to supergravity it is therefore convenient to start right away from the superspace geometry of 16-16 supergravity. This is obtained from U(1) superspace geometry as described in chapter two by

---

*) the integration is over space-time and superspace.

the U(1) symmetry breaking

$$A_\alpha = -\mathcal{D}_\alpha X \quad , \quad A^{\dot\alpha} = \mathcal{D}^{\dot\alpha} X \tag{5.3}$$

$$A_{\alpha\dot\alpha} = \frac{3i}{2} G_{\alpha\dot\alpha} + \frac{i}{2}\left[\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}\right] X \tag{5.4}$$

and by imposing

$$L = e^{bX} \tag{5.5}$$

with some real constant $b \neq -\frac{4}{3}$. In ref. 17) it was shown that in the presence of Yang-Mills Chern-Simons forms in 16-16 superspace (e.g. the superspace geometry described in this paper together with (5.3-5) and $\tau = \sigma = 0$) the superfield action (5.2) describes supersymmetric Yang-Mills theory coupled to 16-16 supergravity via Chern-Simons forms. Again, the corresponding component field action has been worked out before in ref. 14). The general case including Lorentz Chern-Simons forms (and possible non-linear modifications of eq.(5.5)) is discussed in detail in ref. 19).

New minimal supergravity, without Chern-Simons forms, is obtained if one requires

$$L = 1 \tag{5.6}$$

From eqs (4.32-34) one learns that then $R = R^+ = 0$ and the superfield G_a is proportional to the dual of the superfield H_{cba}. In the presence of Chern-Simons forms, as discussed in our paper, eqs (4.32-34) tell us how the superfields R, R^+ and G_a are expressed in terms of other geometric quantities:

$$-8R^+ = \frac{t}{2}\left(\bar\lambda_{\dot\alpha}\bar\lambda^{\dot\alpha}\right) + \frac{v}{6}\bar X_{\dot\alpha}\bar X^{\dot\alpha} - 8\tau\, \bar W_{\dot\gamma\dot\beta\dot\alpha}\bar W^{\dot\gamma\dot\beta\dot\alpha} \tag{5.7}$$

$$-8R = \frac{t}{2}\left(\lambda^\alpha\lambda_\alpha\right) + \frac{v}{6} X^\alpha X_\alpha - 8\tau\, W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \tag{5.8}$$

$$4G_{\alpha\dot\alpha} = t(\lambda_\alpha\bar\lambda_{\dot\alpha}) + \frac{\nu}{3}\chi_\alpha\bar\chi_{\dot\alpha}$$
$$+\frac{1}{3!}\sigma_{d\alpha\dot\alpha}\varepsilon^{dcba}\Big\{2H_{cba} + 16\tau(T_{cb}{}^{\varphi}F_{\varphi a} - T_{cb\dot\varphi}F^{\dot\varphi}{}_a) \tag{5.9}$$
$$+\frac{32i}{3}\tau\bar\sigma_c^{\dot\gamma\gamma}F_{\dot\gamma b}F_{\gamma a} - 3i\nu G_c(2F_{ba} - 3i\mathcal{D}_b G_a)\Big\}$$

For $\nu \neq 0$ eq.(5.9) is a complicated implicit superfield equation for G_a. On the other hand $\nu = 0$ does not mean that new minimal supergravity prefers a particular combination of Lorentz and chiral $U(1)$ Chern-Simons forms. In fact, one still may add a Chern-Simons form (with arbitrary coefficient λ) of the shifted $U(1)$ gauge potential A_1 defined as

$$A_1 = A - \frac{3i}{2}E^a G_a \tag{5.10}$$

This corresponds, in particular, to

$$F_\beta{}^{\dot\alpha}(A_1) = 0 \tag{5.11}$$

and the analysis of the Bianchi identities shows in a straightforward way that the ugly term in eq.(5.9) is then absent.

In terms of component fields the super-covariant field strengths in the presence of Chern-Simons forms are slightly more complicated. Consider, for instance $H_{cba}|$, the lowest component of the super field strength of the antisymmetric tensor gauge field. Employing standard techniques[32,18] one sees that it contains a term linear in τ of the form

$$e_c{}^m\,\psi_m{}^{\underline{\gamma}}\,H_{\underline{\gamma}\,ba}| + \text{cycl. perm. }(cba) \tag{5.12}$$

In turn, by using eqs (4.29-30) and (5.7-9) for $\nu=0$, one realizes that $H_{\underline{\gamma}ba}|$ contains itself $H_{cba}|$ as well as $T_{cb}{}^{\underline{\alpha}}|$, the lowest component of the super-covariant field strength of the Rarita-Schwinger field. For $T_{cb}{}^{\underline{\alpha}}|$ a similar mechanism takes place but the iteration eventually terminates due to the anticommutativity of the Rarita-Schwinger fields appearing at each step. In closing we would like to emphasize the importance of the knowledge of a complete superspace geometry for the derivation of supersymmetry transformations and for the construction of the BRS differential algebra for component fields[18).

APPENDIX

In order to complete the description of U(1) superspace geometry of chapter two, we still have to discuss the properties of the Riemann curvature tensor $R_{dcb}{}^{a}$. Due to the vanishing torsion, $T_{cb}{}^{a} = 0$, the curvature tensor satisfies $R_{dc\,ba} = R_{ba\,dc}$. For many purposes it is useful to use its spinor decomposition defined as follows:

$$R_{\delta\dot\delta\,\gamma\dot\gamma\,\beta\dot\beta\,\alpha\dot\alpha} = \sigma^{d}_{\delta\dot\delta}\,\sigma^{c}_{\gamma\dot\gamma}\,\sigma^{b}_{\beta\dot\beta}\,\sigma^{a}_{\alpha\dot\alpha}\,R_{dcba} \tag{A.1}$$

$$\begin{aligned} R_{\delta\dot\delta\,\gamma\dot\gamma\,\beta\dot\beta\,\alpha\dot\alpha} = {}& 4\,\epsilon_{\dot\delta\dot\gamma}\epsilon_{\dot\beta\dot\alpha}\,\chi_{\underbrace{\delta\gamma}\,\underbrace{\beta\alpha}} + 4\,\epsilon_{\delta\gamma}\epsilon_{\beta\alpha}\,\bar\chi_{\underbrace{\dot\delta\dot\gamma}\,\underbrace{\dot\beta\dot\alpha}} \\ & - 4\,\epsilon_{\dot\delta\dot\gamma}\epsilon_{\beta\alpha}\,\Psi_{\underbrace{\delta\gamma}\,\underbrace{\dot\beta\dot\alpha}} - 4\,\epsilon_{\delta\gamma}\epsilon_{\dot\beta\dot\alpha}\,\Psi_{\underbrace{\beta\alpha}\,\underbrace{\dot\delta\dot\gamma}} \end{aligned} \tag{A.2}$$

$$\chi_{\underbrace{\delta\gamma}\,\underbrace{\beta\alpha}} = \chi_{\underbrace{\delta\gamma\beta\alpha}} + \left(\epsilon_{\delta\beta}\epsilon_{\gamma\alpha} + \epsilon_{\delta\alpha}\epsilon_{\gamma\beta}\right)\chi \tag{A.3}$$

$$\bar\chi_{\underbrace{\dot\delta\dot\gamma}\,\underbrace{\dot\beta\dot\alpha}} = \bar\chi_{\underbrace{\dot\delta\dot\gamma\dot\beta\dot\alpha}} + \left(\epsilon_{\dot\delta\dot\beta}\epsilon_{\dot\gamma\dot\alpha} + \epsilon_{\dot\delta\dot\alpha}\epsilon_{\dot\gamma\dot\beta}\right)\chi \tag{A.4}$$

Here $\chi_{\underbrace{\delta\gamma\beta\alpha}}$ and $\bar\chi_{\underbrace{\dot\delta\dot\gamma\dot\beta\dot\alpha}}$ describe the Weyl tensor in spinor notation, $\Psi_{\underbrace{\delta\gamma}\,\underbrace{\dot\beta\dot\alpha}}$ and χ correspond to the Ricci tensor and curvature scalar, respectively. They are related to the basic superfields of chapter two in the following way:

$$\chi_{\underbrace{\delta\gamma\beta\alpha}} = \frac{1}{4}\left(\mathcal{D}_\delta W_{\underbrace{\gamma\beta\alpha}} + \mathcal{D}_\gamma W_{\underbrace{\beta\alpha\delta}} + \mathcal{D}_\beta W_{\underbrace{\alpha\delta\gamma}} + \mathcal{D}_\alpha W_{\underbrace{\delta\gamma\beta}}\right) \tag{A.5}$$

$$\bar\chi_{\underbrace{\dot\delta\dot\gamma\dot\beta\dot\alpha}} = \frac{1}{4}\left(\bar{\mathcal{D}}_{\dot\delta}\bar W_{\underbrace{\dot\gamma\dot\beta\dot\alpha}} + \bar{\mathcal{D}}_{\dot\gamma}\bar W_{\underbrace{\dot\beta\dot\alpha\dot\delta}} + \bar{\mathcal{D}}_{\dot\beta}\bar W_{\underbrace{\dot\alpha\dot\delta\dot\gamma}} + \bar{\mathcal{D}}_{\dot\alpha}\bar W_{\underbrace{\dot\delta\dot\gamma\dot\beta}}\right) \tag{A.6}$$

$$\Psi_{\underbrace{\delta\gamma}\,\underbrace{\dot\beta\dot\alpha}} = -\frac{1}{16}\sum_{\delta\gamma}\sum_{\dot\beta\dot\alpha}\left(\left[\mathcal{D}_\delta, \bar{\mathcal{D}}_{\dot\beta}\right]G_{\gamma\dot\alpha} - 2\,G_{\delta\dot\beta}G_{\gamma\dot\alpha}\right) \tag{A.7}$$

$$6\chi = -\frac{1}{2}\left(\mathcal{D}^\alpha\mathcal{D}_\alpha R + \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}R^+\right) + \frac{1}{8}\left[\mathcal{D}^\alpha, \bar{\mathcal{D}}^{\dot\alpha}\right]G_{\alpha\dot\alpha} - \frac{3}{4}G^{\alpha\dot\alpha}G_{\alpha\dot\alpha} + 12\,RR^+ \tag{A.8}$$

Using the information presented in chapter two we present alternative expressions for some of the torsions and field strengths. For instance, from eqs (2.14), (2.15) and (2.31), (2.32), one finds

$$F_{\beta a} = -\frac{3}{4}\left(g_{ae} + \frac{1}{3}\sigma_a \bar{\sigma}_e\right)_\beta{}^{\varphi} \varepsilon^{efcb} \sigma_{f\,\varphi\dot{\varphi}} T_{cb}{}^{\dot{\varphi}} \tag{A.9}$$

$$F^{\dot{\beta}}{}_a = -\frac{3}{4}\left(g_{ae} + \frac{1}{3}\bar{\sigma}_a \sigma_e\right)^{\dot{\beta}}{}_{\dot{\varphi}} \varepsilon^{efcb} \bar{\sigma}_f{}^{\dot{\varphi}\varphi} T_{cb\,\varphi} \tag{A.10}$$

and, conversely,

$$T_{cb\alpha} = -\frac{1}{2}\left(\epsilon\sigma_c\bar{\sigma}_b\right)^{\gamma\beta} W_{\underline{\gamma\beta\alpha}} + \frac{i}{3}\left(\sigma_c\epsilon\right)_\alpha{}^{\dot{\alpha}} F_{\dot{\alpha}b} - \frac{i}{3}\left(\sigma_b\epsilon\right)_\alpha{}^{\dot{\alpha}} F_{\dot{\alpha}c} \tag{A.11}$$

$$T_{cb\dot{\alpha}} = \frac{1}{2}\left(\bar{\sigma}_c\sigma_b\epsilon\right)^{\dot{\gamma}\dot{\beta}} \overline{W}_{\underline{\dot{\gamma}\dot{\beta}\dot{\alpha}}} - \frac{i}{3}\left(\epsilon\bar{\sigma}_c\right)_{\dot{\alpha}}{}^{\alpha} F_{\alpha b} + \frac{i}{3}\left(\epsilon\bar{\sigma}_b\right)_{\dot{\alpha}}{}^{\alpha} F_{\alpha c} \tag{A.12}$$

The chirality conditions (2.37)

$$\mathcal{D}_\alpha \bar{X}_{\dot{\alpha}} = 0 \quad , \quad \mathcal{D}_{\dot{\alpha}} X_\alpha = 0 \tag{A.13}$$

are equivalent to the equations

$$\mathcal{D}^\varphi \mathcal{D}_\varphi G_a = 4i\, \mathcal{D}_a R^{+} \tag{A.14}$$

$$\mathcal{D}_{\dot{\varphi}} \mathcal{D}^{\dot{\varphi}} G_a = -4i\, \mathcal{D}_a R \tag{A.15}$$

On the other hand, eq.(2.39) in spinor notation reads:

$$\mathcal{D}^\varphi W_{\underline{\varphi\beta\alpha}} = -\frac{1}{6}\sum_{\beta\alpha}\left(\mathcal{D}_\beta \mathcal{D}^{\dot{\varphi}} G_{\alpha\dot{\varphi}} + 3i\, \mathcal{D}_\beta{}^{\dot{\varphi}} G_{\alpha\dot{\varphi}}\right) \tag{A.16}$$

$$\mathcal{D}^{\dot{\varphi}} W_{\underline{\dot{\gamma}\dot{\beta}\dot{\alpha}}} = \frac{1}{6}\sum_{\dot{\beta}\dot{\alpha}}\left(\mathcal{D}_{\dot{\beta}} \mathcal{D}^{\varphi} G_{\varphi\dot{\alpha}} + 3i\, \mathcal{D}^{\varphi}{}_{\dot{\beta}} G_{\varphi\dot{\alpha}}\right) \tag{A.17}$$

Finally, we define the spinor decompositions of H_{cba} and its dual,

$$H^d \equiv \frac{1}{3!}\varepsilon^{dcba} H_{cba} \, , \quad H_{cba} = -H^d \varepsilon_{dcba} \tag{A.18}$$

as follows:

$$H_{\gamma\dot{\gamma}\,\beta\dot{\beta}\,\alpha\dot{\alpha}} = \sigma^{c}{}_{\gamma\dot{\gamma}}\,\sigma^{b}{}_{\beta\dot{\beta}}\,\sigma^{a}{}_{\alpha\dot{\alpha}}\,H_{cba} \tag{A.19}$$

$$H_{\delta\dot{\delta}} = \sigma^{d}{}_{\delta\dot{\delta}}\,H_{d} \tag{A.20}$$

In spinor notation the second of eqs (A.18) reads

$$H_{\gamma\dot{\gamma}\,\beta\dot{\beta}\,\alpha\dot{\alpha}} = 2i\,\epsilon_{\dot{\gamma}\dot{\beta}}\,\epsilon_{\gamma\alpha}\,H_{\beta\dot{\alpha}} - 2i\,\epsilon_{\gamma\beta}\,\epsilon_{\dot{\gamma}\dot{\alpha}}\,H_{\alpha\dot{\beta}} \tag{A.21}$$

where we have used the spinor decomposition of the totally antisymmetric tensor ε_{dcba} :

$$\varepsilon_{\delta\dot{\delta}\,\gamma\dot{\gamma}\,\beta\dot{\beta}\,\alpha\dot{\alpha}} = \sigma^{d}{}_{\delta\dot{\delta}}\,\sigma^{c}{}_{\gamma\dot{\gamma}}\,\sigma^{b}{}_{\beta\dot{\beta}}\,\sigma^{a}{}_{\alpha\dot{\alpha}}\,\varepsilon_{dcba} \tag{A.22}$$

As a consequence of this definition one obtains

$$\varepsilon_{\delta\dot{\delta}\,\gamma\dot{\gamma}\,\beta\dot{\beta}\,\alpha\dot{\alpha}} = 4i\left(\epsilon_{\delta\beta}\,\epsilon_{\gamma\alpha}\,\epsilon_{\dot{\delta}\dot{\alpha}}\,\epsilon_{\dot{\gamma}\dot{\beta}} - \epsilon_{\delta\alpha}\,\epsilon_{\gamma\beta}\,\epsilon_{\dot{\delta}\dot{\beta}}\,\epsilon_{\dot{\gamma}\dot{\alpha}}\right) \tag{A.23}$$

That this expression is indeed totally antisymmetric in the four vector indices $\delta\dot{\delta}$, $\gamma\dot{\gamma}$, $\beta\dot{\beta}$, $\alpha\dot{\alpha}$ may be easily verified with the help of the cyclic identity

$$\epsilon_{\gamma\beta}\,\chi_{\alpha} + \epsilon_{\beta\alpha}\,\chi_{\gamma} + \epsilon_{\alpha\gamma}\,\chi_{\beta} = 0 \tag{A.24}$$

Our superspace notations are the same as those of Wess and Bagger[32].

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