IC/87/5

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THE POINCARÉ INEQUALITY FOR FLOW-DOMAINS AND ITS APPLICATIONS

Duong Minh Duc

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1987 MIRAMARE-TRIESTE

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THE POINCARÉ INEQUALITY FOR FLOW-DOMAINS AND ITS APPLICATIONS *

Duong Minh Duc **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We establish a Poincaré inequality for some unbounded domains and apply it to study the Dirichlet problem for these domains.

MIRAMARE – TRIESTE
January 1987

* To be submitted for publication.

** Permanent address: Department of Mathematics, University of Hochiminh City, 227 Nguyen Van Cu, Hochiminh City, S.R. Vietnam.

INTRODUCTION

The Poincaré inequality has been studied by different methods and applied to many problems [1-8] . In this paper we propose to prove an Poincaré inequality for the flow-domain defined below, and apply it to the Laplace equation with the Dirichlet or Neuman boundary conditions. Our results may also be used to generalize other results in [1,3].

This paper consists of two sections. In the first section we prove the Poincaré inequality for unbounded domains in $\mathbb{R}^n$. The Laplace equation will be studied in the second section.

1. POINCARE INEQUALITY FOR FLOW-DOMAIN

In this section let G be a subset of the n-1 dimensional space of real numbers $\mathbb{R}^{n-1}$ and $x \longmapsto b_x$ be an application from G into $[0,\infty]$ and $D = \{ (t,x) : x \in G , 0 \leq t < b_x \}$.

Definition 1. Let h be a continuous differentiable one-to-one mapping from D into R^n such that the Jacobian determinant $J_h(t,x)$ is nonnull for every $t > 0$. Then h is called a flow defined on D

Let h be a flow defined on D and let w be a continuous positive function on h(D) and $p \in (1,\infty)$. For each x in G and t in $(0,b_x)$ we put

$$s(h,p,x,t) = \int_0^t \left|\frac{\partial h}{\partial y}(y,x)\right|^{\frac{p}{p-1}} dy$$

$$d_p(h,w) = \sup_{(y,x)\in D} \int_y^{b_x} \frac{s(h,p,x,t)^{p-1} w(h(t,x))\,|J_h(t,x)|}{|J_h(y,x)|}\, dt$$

Definition 2. Let h be a flow defined on D, we say $V = h(D)$ is a flow-domain parametrized by (h,D). We put

$$C_o^{1,p}(h,D,w) = \Big\{ u \in C^1(V) : \int_V (|u(y)|^p w(y) + |\nabla u(y)|^p) dy < \infty \text{ and } u(h(o,x)) = 0 \text{ for every } x \text{ in } G \Big\}.$$

We have the following Poincaré inequality.

Theorem 1. *Let* V *be a flow-domain parametrized by* (h,D) *and* p *in* (o,∞). *Let* u *in* $C_o^{1,p}(h,D,w)$, *then*

$$\int_V |u(y)|^p w(y) dy \leq d_p(h,w) \int_V |\nabla u(x)|^p dx$$

Proof.

For every $(t,x) \in D$ we have :

$$u(h(t,x)) = \int_o^t \frac{\partial}{\partial y}(u(h(y,x)))dy = \int_o^t \nabla u(h(y,x)) . \frac{\partial h}{\partial y}(y,x)dy$$

By Holder's inequality we have

$$|u(h(t,x))|^p \leq \int_o^t |\nabla u(h(y,x))|^p dy \Big(\int_o^t \Big| \frac{\partial h}{\partial y}(y,x) \Big|^{p/p-1} dy \Big)^{p-1}$$

$$= s(h,p,x,t)^{p-1} \int_o^t |\nabla u(h(y,x))|^p dy \qquad (1)$$

Then by Fubini's theorem, theorem of change of variables and (1) we have

$$\int_V |u(y)|^p w(y) dy = \int_D |u(h(y))|^p w(h(y)) |J_h(y)| dy$$

$$= \int_G \int_o^{b_x} |u(h(t,x))|^p w(h(t,x)) |J_h(t,x)| dt\, dx$$

$$\leq \int_G \int_o^{b_x} \Big(\int_o^t |\nabla u(h(y,x))|^p dy \Big) s(h,p,x,t)^{p-1} w(h(t,x)) |J_h(t,x)| dt\, dx$$

$$= \int_G \int_o^{b_x} \Big(\int_y^{b_x} \frac{s(h,p,x,t)^{p-1} w(h(h(t,x)) |J_h(t,x)|}{|J_h(y,x)|} dt \Big) |\nabla u(h(y,x))|^p |J_h(y,x)| dy dx$$

$$\leq d_p(h,w) \int_G \int_o^{b_x} |\nabla u(h(y,x))|^p |J_h(y,x)| dy dx =$$

$$= d_p(h,w) \int_D |\nabla u(h(y))|^p |J_h(y)| dy = d_p(h,w) \int_V |\nabla u(y)|^p dy$$

////////

Definition 3. Let h be a flow defined on D. If there exists an i

in $\{1,\dots,n\}$ such that $h(y) = (o,\dots,h_i(y),o,\dots,o)$, we say h is a *flow in* i^{th} *direction*, and have the following result.

Theorem 2. *Let* V *be a flow-domain parametrized by* (h,D). *We have*

(i) *If* h *is a flow in* i^{th} *direction*, *then for every* u *in* $C_o^{1,p}(h,w)$ *we have*

$$\int_V |u(y)|^p w(y) dy = d_p(h,w) \int_V \Big| \frac{\partial u}{\partial x_i}(x) \Big|^p dx$$

(ii) *If* V *is parametrized by the flow in* $i_k{}^{th}$ *direction* (h^k,D_k), *where* $1 \leq i_1 < i_2 < \dots < i_m \leq n$. *Then for every* u *in* $\bigcap_{k=1}^m C_o^{1,p}(h^k,D_k,w)$, *we have*

$$\int_V |u(y)|^p w(y) dy \leq \frac{1}{m} \max(d_p(h^1,w),\dots,d_p(h^m,w)) \sum_{k=1}^m \int_V \Big| \frac{\partial u}{\partial x_i}(y) \Big|^p dy$$

Proof.

It is clear that

$$\nabla u(h(y,x)) . \frac{\partial h}{\partial x_i}(y,x) = \frac{\partial u}{\partial x_i}(h(y,x)) . \frac{\partial h_i}{\partial y}(y,x)$$

By the proof of Theorem 1 we have (i). From (i) we get (ii).

//////////

We shall consider a general domain V in $\mathbb{R}^n$, that is V may not be a flow-domain. Let w be a positive continuous function on V and $p \in (1,\infty)$. We denote by $W^p(V,w)$ the space of all real functions u such that

$$\int_V \Big(|u(y)|^p w(y) + \sum_1^n \Big| \frac{\partial u}{\partial x_i}(y) \Big|^p \Big) dy)^{1/p}$$

Let $W_o^p(V,w)$ be the closure in $W^p(V,w)$ of the set of all u in $C^1(\overline{V})$ with $u|_{\partial V} = 0$. Applying the foregoing theorems we have

Theorem 3. *Let* $p \in (1,\infty)$ *and* V *be a domain contained in the half space* $\{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}$. *Assume for every* $(x_1,\dots,x_n)$ *in* V, $w(x_1,\dots,x_n) \leq x_n^{-p}$.

Then for every $u \in W_o^p(V,w)$ we have

$$\int_V |u(y)|^p w(y)dy \leq A_p \int_V |\frac{\partial u}{\partial x_n}(y)|^p dy$$

where $A_p = \inf \left\{ \frac{a^p}{1-a}(\frac{p-1}{ap-1})^{p-1} ; 1/p < a < 1 \right\}$

Proof.

We can suppose $V = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n > o\}$ and

$$w(x_1,\ldots,x_n) = x_n^{-p}$$

Fix an a in $(1/p, 1)$, let $G = \mathbb{R}^{n-1}$, $D = [o,\infty)xG$ and

$$h : D \longrightarrow \mathbb{R}^n$$

$$h(t,(x_1,\ldots,x_{n-1})) = (x_1,\ldots,x_{n-1},t^a)$$

By calculation we have $d_p(h,w) = \frac{a^p}{1-a}(\frac{n-1}{ap-1})^{p-1}$. Applying Theorem 1 we have the desired result.

////////////

Remark 1. If $p = 2$, $a = \frac{2}{3}$, we have $\frac{a^p}{1-a}(\frac{p-1}{ap-1})^{p-1} = 4$. We have just refound the Lemma 2.1 in Chapter II of [4], as in [4], we see that this estimate is the best one.

It is clear that we have from Theorem 2 and Theorem 3

Corollary 1. Let V be a domain contained in the set $\{(x_1,\ldots,x_n) \in \mathbb{R}^n ; x_1,\ldots,x_m \geq 0\}$ where $1 \leq m \leq n$. Assume $w(x_1,\ldots,x_n) \leq (x_1^2+\ldots+x_m^2)^{\frac{-p}{2}}$ for every $(x_1,\ldots,x_n)$ in V.
Then for every u in $W_o^p(V,w)$ we have

$$\int_V |u(y)|^p w(y)dy \leq \frac{A_p}{m} \sum_{i=1}^{m} \int_V |\frac{\partial u}{\partial x_i}(y)|^p dy$$

Remark 2. If $p = 2$, we can use the Cauchy-Schwartz inequality to get better estimates as in [7].

Let's consider the section-domain :

Theorem 4. Let V be a domain contained in the following set $\{(x\cos s, x\sin s, x_3,\ldots,x_n) ; x > 0, 0 < s < 2a \leq 2\pi, (x_3,\ldots,x_n) \in \mathbb{R}^{n-2}\}$

Assume $w(x_1,\ldots,x_n) \leq (x_1^2 + x_2^2)^{-p/2}$. Then for every u in $W_o^p(V,w)$ we have

$$\int_V |u(y)|^p w(y)dy = \frac{1}{p} a^p \int_V (\sum_1^2 |\frac{\partial u}{\partial y_i}(y)|^p)^{p/2} dy$$

Proof.

We can suppose

$$V = \{(x\cos s, x\sin s, x_3,\ldots,x_n) : x > o, o < s < 2a < 2\pi, x_i \in \mathbb{R}\}$$

and $w(x_1,\ldots,x_n) = (x_1^2 + x_2^2)^{-p/2}$

Put $G = (o,\infty)x\mathbb{R}^{n-1}$ and

$$h : [0,a)xG \longrightarrow V' = \{(x\cos s, x\sin s, x_3,\ldots,x_n) \in V : s < a\}$$

$$h(s,(x,z)) = (x\cos s, x\sin s, z)$$

By calculation we have

$$d_p(h,w) = \frac{a^p}{p}$$

Then we have as in Corollary 1

$$\int_{V'} |u(y)|^p w(y)dy = \frac{1}{p} a^p \int_{V'} (\sum_1^2 |\frac{\partial u}{\partial y_i}(y)|^2)^{p/2} dy$$

By the similar procedure for the other half of V, we have the theorem .

/////////

Remark 3. If $a = \pi$, we have the interesting case, because V become $\mathbb{R}^n \setminus ([o,\infty)x\{0\}x\mathbb{R}^{n-2})$

Let's consider the exterior domain :

Theorem 5. Let V be a domain contained in $\{x \in \mathbb{R}^n : |x| > A > 0\}$, where $|x| = (x_1^2+\ldots+x_n^2)^{1/2}$. Assume $w(x) \leq |x|^{-b}$ with $b > \frac{n}{p}$.
Then for every u in $W_o^p(V,w)$, we have :

$$\int_V |u(y)|^p w(y)dy \leq C_{p,A,b} \int_V |\nabla u(y)|^p dy$$

where $C_{p,A,b} = \inf\left\{ \frac{a^p A^{ap-b}}{1+b-ap-an}\left(\frac{p-1}{ap-1}\right)^{p-1} ; a < \frac{b}{p}, \frac{1}{p} < a < \frac{1+b}{p+n} \right\}$

Proof.

We can suppose $V = \{x \in \mathbb{R}^n : |x| > A\}$ and $w(x) = |x|^{-b}$. Let k be the homeomorphism from $[0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]^{n-2}$ into the following set $\{x \in \mathbb{R}^n : \|x\| = 1\}$, which is used in the change of variables into the polar coordinates. Let a be such that $a < \frac{b}{p}$ and $\frac{1}{p} < a < \frac{1+b}{p+n}$, we put

$$h : [A,\infty) \times \left([0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]^{n-2}\right) \longmapsto V$$

$$h(t,x) = t^a k(x)$$

Then by calculation we have $d_p = C_{p,A,b}$, this implies the theorem.

////////////

2. APPLICATIONS.

In this section we assume V satisfies the conditions of one of the theorem 3,4 and 5. Let E be a domain in $\mathbb{R}^n$ and $C_o^2(E)$ be the space of all real functions u in $C^2(E)$ having compact support contained in E. We denote by H(V) and $\overset{o}{H}(V)$ the closures in $W^2(V,w)$ of the sets $C_o^2(\mathbb{R}^n) \cap W^2(V,w)$ and $C_o^2(V) \cap W^2(V,w)$ respectively. We denote $L^2(V,w)$ the set $\{ f : \int_V f^2(y) w^{-1}(y) dy < \infty \}$.

For each g in H(V) and f in $L^2(V,w)$, let's consider the following Dirichlet problem

$$\begin{cases} \Delta u = f \\ u|_{\partial V} = g|_{\partial V} \end{cases} \tag{2.1}$$

where the derivatives are in the distributional sense.

Theorem 6. The problem (2.1) has an unique solution u in H(V).

Proof.

Resolving the problem (2.1) is equivalent to finding a v in $\overset{o}{H}(V)$ such that

$$\Delta v = f - \Delta g \tag{2.2}$$

And it is clear that (2.2) is equivalent to finding a v in $\overset{o}{H}(V)$ such that

$$(\Delta v, h) = (f,h) - (\Delta g, h) \quad \text{for every h in } C_o^2(V)$$

or

$$-(\nabla v, \nabla h) = (f,h) + (\nabla g, \nabla h) \text{ for every h in } C_o^2(V) \tag{2.3}$$

where $(f,h) = \int_V f(y)h(y)dy$ and $(\nabla v, \nabla h) = \sum_1^n \left(\frac{\partial v}{\partial x_i}, \frac{\partial h}{\partial x_i}\right)$.

For every u,v in $\overset{o}{H}(V)$ we put

$$((u,v)) = \int_V u(y)v(y)w(y)dy + (\nabla u, \nabla v)$$

and $[u,v] = (\nabla u, \nabla v)$

By the foregoing theorems, these inner products are equivalent in $\overset{o}{H}(V)$. Let's return to the problem (2.3). Since f is in $L^2(V,w)$ and g is in H(V), we see that $h \longmapsto -(f,g) - (\nabla g, \nabla h)$ defines a continuous linear functional on $\overset{o}{H}(V)$ for every given f and g.

Therefore by the Riesz Theorem there exists a unique v in $\overset{o}{H}(V)$ such that

$$(\nabla u, \nabla h) = [v,h] = -(f,h) - (\nabla g, \nabla h)$$

This implies the theorem.

////////////

Remark 4. If $w(x) = (1+|x|^2)^{-1}$, the foregoing theorem is proved in [3]. In a similar way, we can extend the other results in [3] for the general function w, by e.g. for the Neuman problem.

ACKNOWLEDGMENT.

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, TRIESTE.

REFERENCES

[1] C.J. Amick, "Some remarks on Rellich's theorem and the Poincaré inequality", J. London Math. Soc.(2) 18, No.1 (1978) 81-93.

[2] H. Brezis and E.H. Lieb, "Sobolev inequalities with remainder terms", J. Funct. Anal. 62, (1985) 73-86.

[3] D.M. Duo and L. Dung, "Compactness of embeddings of Sobolev spaces" (to appear).

[4] G.P. Galdi and S. Rionero, "Weighted energy methods in fluid dynamics and elasticity", Lecture Notes in Math. 1134, Springer, Berlin (1985).

[5] D.D. Joseph, Stability of Fluid Motions, Vol.1, Springer Tracts in Natural Philosophy (27), 1976.

[6] J. Maulen, "Losungen der Poissongleichung und harmonische Vectorfelder in unbeschrankten Gebieten", Math. Meth. in the Appl. Sci. 5, (1983) 233-255.

[7] S. Rionero and F. Salemi, "On some weighted Poincaré inequalities", Fisica Matematica, Suppl. Bull. Un. Mat. It. IV5, No.1 (1985) 253-260.

[8] E.T. Sawyer, "A weighted inequality and eigenvalue estimates for Schrödinger operators", Indiana Univ.Math. J. 35, No.1 (1986) 1-28.