IC/87/189

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1987 MIRAMARE-TRIESTE

IC/87/189

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

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ABSTRACT

Two examples are presented in this paper, the first is unfavorable to the c-boundary construction given by Geroch, Kronheimer and Penrose but in favor of that given by Budic and Sachs, while the second plays an opposite role. The second example is also an example of a causally continuous spacetime with a "really big gap", contrary to what was believed in the literature.

MIRAMARE - TRIESTE
July 1987

* To be submitted for publication.

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1. INTRODUCTION

In order to have a better description of spacetime singularities within the framework of classical general relativity, one would like to construct an enlarged topological space $\bar{M}$ interpreted as the spacetime manifold M with some singular boundary ∂ attached. Various constructions have been put forward. The constructions of b-boundary[1] and g-boundary[2] have been known to be unsatisfactory[3,4]. The construction known as the c-boundary (causal-boundary) construction given by Geroch, Kronheimer and Penrose[5] in 1972 makes use only of the causal structure of the spacetime and hence has certain merit from the physical point of view. However, as illustrated by its authors, it fails to construct a Hausdorff topological space $\bar{M}$ which is also a causal space in general. To surmount this difficulty, Budic and Sachs gave an improved definition of the c-boundary construction in 1974[6]. They proved that the resulting Hausdorff topological space $\bar{M}$ is also a causal space with causal structure extended from that of the original spacetime (M,g) itself, provided that (M,g) is causally continuous (a causal requirement much stronger than distinguishing required by Ref. 5), thus it makes good sense to ask whether signals with speed less than or equal to that of light can be sent between a regular point and an ideal point. We will refer to the c-boundary construction given in Ref.5 as the GKP construction and that given in Ref.6 as the BS construction. In a recent paper by Kuang, Li and Liang[7], it was shown that for some singular exact solutions to Einstein equations the c-boundary of the GKP construction is unsatisfactory, for example, the "singular portion" of the c-boundary of Taub's plane-symmetric vacuum solution turned out to be a single point, suggesting that it might not be fruitful describing the structure of singularities using the notion of c-boundary defined by GKP. Besides, as will be shown in the next section, there is something else that is also unfavorable to the GKP construction. The fact that these two deficiencies do not exist in the BS construction suggests that the BS construction might be more acceptable. Nevertheless, we will give an example in section 3 showing that there is also something unfavorable to it, a drawback which is not shared by the GKP construction. Therefore it seems still an open question whether one can construct some improved c-boundary which is free of deficiencies.

2. A SECOND EXAMPLE UNFAVORABLE TO THE GKP CONSTRUCTION

Assuming the reader is familiar with the GKP construction, we present the example as follows.

Let $(\tilde{M},\eta)$ be a three-dimensional Minkowski spacetime with Cartesian coordinates (t,x,y) and (M,η) a subspacetime where $M=\{(t,x,y):y>0\}$. Consider a future directed timelike curve $\gamma\subset M$ with the origin $(0,0,0)$ as its future endpoint in $\tilde{M}$ and a past directed timelike curve $\lambda\subset M$ with $(0,0,0)$ as past endpoint in $\tilde{M}$. γ (resp. λ) is future (resp. past) inextendible in M. It is reasonable to require that the TIP, $I^-(\gamma,M)$, and the TIF, $I^+(\lambda,M)$, be identified in $\overline{M}$, and this is exactly the case according to the BS identification rule. It is however not true in the GKP construction. Indeed, there exist two open sets O_1 and O_2 with $I^-(\gamma,M)^*\in O_1$, $I^+(\lambda,M)^*\in O_2$ and $O_1\cap O_2=\emptyset$. To see this, consider the following two subsets of M:

$$A = \{(t,x,y):t>x,\ y>0\},$$

$$B = \{(t,x,y):t<x,\ y>0\}.$$

They are, respectively, a TIF and a TIP in M, since there exist some past (resp. future) inextendible timelike curves α (resp. β) in M such that $A = I^+(\alpha,M)$ and $B = I^-(\beta,M)$. For instance, one can take the following curve to be β: $t=t,\ x=t+1/t,\ y=1/t\ \ (t>1)$, and dually for α. According to the GKP construction, the following two subsets of the intermediate space $\mathcal{M}^\#$ are open:

$$A^{ext} = \{\ P^*\in\mathcal{M}^\#:P\in\hat{\mathcal{M}} \text{ and } P=I^-(S) \Rightarrow I^+(S)\not\subseteq A \text{ for all } S\subseteq M\ \},$$

$$B^{ext} = \{\ F^*\in\mathcal{M}^\#:F\in\check{\mathcal{M}} \text{ and } F=I^+(S) \Rightarrow I^-(S)\not\subseteq B \text{ for all } S\subseteq M\ \}.$$

It is straightforward to check that $I^+(\lambda,M)^*\in B^{ext}$ and $I^-(\gamma,M)^*\in A^{ext}$ by showing that any $S\subseteq M$ with $I^+(S,M)=I^+(\lambda,M)$ [resp. $I^-(S,M)=I^-(\gamma,M)$] satisfies $I^-(S,M)\not\subseteq B$ [resp. $I^+(S,\ M)\ \not\subseteq A$]. Consequently A^{ext} and B^{ext} can be taken to be the desired O_1 and O_2 respectively. Note, however, that this is not true if we are dealing with $\tilde{M}$ instead of M since the origin $(0,0,0)$ can then be taken as S violating the requirement in the definitions of A^{ext} and B^{ext}.

3. AN EXAMPLE UNFAVORABLE TO THE BS CONSTRUCTION

We first give a brief outline of the essential contents of the BS construction relevant to this paper as follows.

Define binary relations $\geq$ and $\gg$ on a time-orientable spacetime (M,g) as usual. Define concepts IP and IF as in the GKP construction. Denote the power set, the topology, the collections of past sets, future sets, IP's and IF's of (M,g) as $\mathcal{S}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{F}$, $\hat{\mathcal{M}}$, and $\check{\mathcal{M}}$ respectively. Define a map $I^-:\ \mathcal{S}\rightarrow\mathcal{P}$ by $I^-S=\{x\in M:x\ll s \text{ for some } s\in S\}\ \forall\ S\in\mathcal{S}$. Define a map $\hat{I}:M\rightarrow\mathcal{P}$ by $\hat{I}x=I^-\{x\}\forall\ x\in M$. Define a map $\Downarrow:\mathcal{T}\rightarrow\mathcal{P}$ by $\Downarrow U=I^-\{x\in M:x\ll u\ \forall\ u\in U\}=interior\{x\in M : x\ll u\ \ \forall\ u\in U\ \}\ \forall\ U\in\mathcal{T}$. The maps I^+, $\check{I}$ and $\Uparrow$ are defined dually. Define $\geq$ and $\gg$ on $\hat{\mathcal{M}}\cup\check{\mathcal{M}}$ by table 2.2 in Ref.6. For example, if $P,Q\in\hat{\mathcal{M}}$, then $P\gg Q$ iff $P\cap\Uparrow Q\neq\emptyset$. Define an equivalence relation $\sim$ on $\hat{\mathcal{M}}\cup\check{\mathcal{M}}$ as follows: for $A,B\in\hat{\mathcal{M}}$ (or $\check{\mathcal{M}}$), $A\sim B$ iff $A=B$; for $A\in\hat{\mathcal{M}}$, $B\in\check{\mathcal{M}}$, $A\sim B$ iff $A=\Downarrow B$ and $B=\Uparrow A$. Define the causal completion of (M,g) as $\overline{M}=\hat{\mathcal{M}}\cup\check{\mathcal{M}}/\sim$, then $\geq$ and $\gg$ are well defined on $\overline{M}$. Define the extended Alexandrov topology $\mathcal{T}$ on $\overline{M}$ as the smallest topology on $\overline{M}$ such that for all $c\in\overline{M}$, each of the following four subcollections is open: $I^+\{c\}$, $I^-\{c\}$, $K^+\{c\}\equiv\overline{M}-J^-\{c\}$, $K^-\{c\}\equiv\overline{M}-J^+\{c\}$, where $I^+\{c\}\equiv\{a\in\overline{M}:a\gg c\}$, and $J^+\{c\}\equiv\{a\in\overline{M}:a\geq c\}$. It was shown that $(\overline{M},\geq,\gg,\mathcal{T})$ is a causal space with Hausdorff topology and $\hat{I}:M\rightarrow\overline{M}$ has all the desired properties (e.g., it is a dense imbedding) provided that (M,g) is causally continuous, thus the boundary ϑ is naturally interpreted as the causal-boundary of (M,g).

An essential requirement for constructing a causal completion $\overline{M}$ which is both a Hausdorff topological space and a causal space is the causal continuity of (M,g). A spacetime (M,g) is said to be causally continuous iff it is both distinguishing and reflective. (M,g) is said to be reflective iff $\Uparrow\hat{I}x=\check{I}x$ and $\Downarrow\check{I}x=\hat{I}x\ \ \forall\ \ x\in M$. The causal continuity of spacetimes has been investigated in detail by some authors[6,8,9,10]. It was pointed out in Ref.8 that "roughly, a causally continuous spacetime ...... has no really big gaps (gaps of 'dimension' more than 2)" and some statements similar in spirit to it can also be found in the other references quoted. However, we have found a four-dimensional spacetime (artificial though) with a "really big gap", i.e., a "gap" of four-dimensions which is causally continuous. It is also this spacetime to which the application of the BS construction gives some unfavorable result, as will be illustrated shortly.

Although the motivation of the BS construction was to overcome the non-cooperation between the Hausdorff topology and the causal structure of the resulting space $\bar{M}$, it turns out that the two defects of the GKP construction mentioned in sections 1 and 2 are also surmounted. Nonetheless, the following example illustrates that it might have its own drawback.

Consider an (n+1)-dimensional Minkowski spacetime $(\tilde{M},\eta)$. Denote the Cartesian coordinates of $\tilde{M}$ by $(t,x^1,\dots,x^n)$. Let $a=(-1,0,\dots,0)\in\tilde{M}$ and $b=(1,0,\dots,0)\in\tilde{M}$. By removing a closed subset $R=closure[I^+(a,\tilde{M})\cap I^-(b,\tilde{M})]$ of the same dimension from $\tilde{M}$ we get a submanifold $M=\tilde{M}-R$ and a subspacetime (M,η). Since two spacetimes $(\tilde{M},\eta)$ and (M,η) will be alternatively dealt with, we will, whenever necessary, add subscripts "$\tilde{M}$" or "M" to the symbols for the relation $\gg$ and maps I^-, I^+, $\hat{I}$, $\check{I}$, $\downarrow$ and $\uparrow$ to clarify the spacetime involved. We will also write $\check{I}_{\tilde{M}}a\cap\hat{I}_{\tilde{M}}b$ instead of $I^+(a,\tilde{M})\cap I^-(b,\tilde{M})$ to be in accordance with the BS notation. It will be proved in the next section that the subspacetime (M,η) is causally continuous provided that $n>1$, thus the BS construction is applicable. Let $\bar{M}$ be the causal completion of (M,η). In addition to the infinity portion ∂_i of the c-boundary ∂, there is also some "singular portion" ∂_s. Obviously, there is a natural correspondence between ∂R and ∂_s, hence one would, intuitively, expect that near ∂_s the topological structure of $\bar{M}$ should be the same as that of $\tilde{M}$, i.e., the way of "gluing" ∂_s to M should be the same as that of "gluing" ∂R to M. However, the following shows that it is not the case, thus suggesting that there might be something unsatisfactory about the BS construction.

Choose a point $e=(-1/2,-1,0,\dots,0)\in M$, then $\check{I}_M e\sim\hat{I}_M e$ is a regular point in $\bar{M}$. Let $\gamma\subset M$ be a past inextendible timelike curve which, viewed as a curve in $\tilde{M}$, has b as its past endpoint,

then $I^+_M\gamma$ is an ideal point in $\bar{M}$. Since $\downarrow I^+_M\gamma\cap\check{I}_M e\neq\emptyset$, we have, according to the BS construction, $I^+_M\gamma\gg\check{I}_M e$ or equivalently $I^+_M\gamma\in I^+(\check{I}_M e)$. Consider a point sequence $\{f_i\}$ in M defined by $f_i=(1,1/i,0,\dots,0)$, then one has a corresponding point sequence $\{F_i\}$ in $\bar{M}$ defined by $F_i=\check{I}_M f_i\in M$. Since $\downarrow\check{I}_M f_i\cap\check{I}_M e=\emptyset$, we have $F_i\notin I^+(\check{I}_M e)$ for any i. This, together with the fact that $I^+(\check{I}_M e)$ is an open set in the extended Alexandrov topology, implies that $\{F_i\}$ does not converge to $I^+_M\gamma$ in $\bar{M}$. It is however obvious that $\{f_i\}$ converges to b in $\tilde{M}$, therefore we conclude that the topology of $\bar{M}$ near $I^+_M\gamma$ is different from that of $\tilde{M}$.

4. PROOF OF THE CAUSAL CONTINUITY OF $(\tilde{M}-R,\eta)$

Throughout the proof we will use the following notation : for $x\in M$ (resp. $x\in\tilde{M}$) and $S\subseteq M$ (resp. $S\subseteq\tilde{M}$), we write $x\ll_M S$ (resp. $x\ll_{\tilde{M}} S$) iff $x\ll_M s$ (resp. $x\ll_{\tilde{M}} s$) $\forall$ $s\in S$. Dual statements (if any) to those in the following lemmas are taken for granted and are not written.

LEMMA 1. Let x, $y\in M$ and $\{u_i\}$ be a sequence in M satisfying

(1) $\{u_i\}\subset\check{I}x$,

(2) x is a limit point of $\{u_i\}$,

then $y\ll\check{I}x$ iff $y\ll\{u_i\}$.

This lemma is true for all chronological spacetimes, the proof is trivial and is omitted.

To prove the causal continuity of (M,η) is to prove $\uparrow_M\hat{I}_M c=\check{I}c$ and $\downarrow_M\check{I}_M c=\hat{I}_M c$ for all $c\in M$. Since $\check{I}_{\tilde{M}}c\cap R\neq\emptyset$ and $\hat{I}_{\tilde{M}}c\cap R\neq\emptyset$ would imply $c\in R$, we have only three possible cases:

(1) $\check{I}_{\tilde{M}}c\cap R=\emptyset$, $\hat{I}_{\tilde{M}}c\cap R=\emptyset$;

(2) $\check{I}_{\tilde{M}}c\cap R\neq\emptyset$, $\hat{I}_{\tilde{M}}c\cap R=\emptyset$;

(3) $\check{I}_{\tilde{M}}c\cap R=\emptyset$, $\hat{I}_{\tilde{M}}c\cap R\neq\emptyset$.

LEMMA 2. $\check{I}_{\tilde{M}}c\cap R=\emptyset$ implies $\check{I}_M c=\check{I}_{\tilde{M}}c$.

PROOF. It suffices to show $\check{I}_{\tilde{M}}c\subseteq\check{I}_M c$. For any $x\in\check{I}_{\tilde{M}}c$, the timelike curve connecting c to x must not intersect R or there would be $y\in\check{I}_{\tilde M}c\cap R$. Hence $x\in\check{I}_M c$. ■

LEMMA 3. $\Uparrow_M\hat I_M c\subseteq\Uparrow_{\tilde M}\hat I_{\tilde M}c$.

PROOF. For any $x\in\Uparrow_M\hat I_M c$, there exists $y\in M$, $x\gg_M y\gg_M\hat I_M c\subset\hat I_{\tilde M}c$. Let $\{u_i\}$ be a sequence in $\hat I_M c$ with c as its limit point, then $y\gg_M\{u_i\}$ which implies $y\gg_{\tilde M}\{u_i\}$, hence $y\gg_{\tilde M}\check I_{\tilde M}c$ and $x\in\Uparrow_{\tilde M}\hat I_{\tilde M}c$. Note that lemma 1 has been used twice. ■

Since we always have $\check I_M c\subseteq\Uparrow_M\hat I_M c$ and $\hat I_M c\subseteq\Downarrow_M\check I_M c$, what remains to be shown is $\Downarrow_M\check I_M c\subseteq\hat I_M c$ and $\Uparrow_M\hat I_M c\subseteq\check I_M c$. On account of lemmas 2 and 3 as well as the causal continuity of $(\tilde M,\eta)$, $\Uparrow_M\hat I_M c\subseteq\check I_M c$ is true for cases (1) and (3), while $\Downarrow_M\check I_M c\subseteq\hat I_M c$ is true for cases (1) and (2). Therefore the essential issue is to prove $\Uparrow_M\hat I_M c\subseteq\check I_M c$ for case (2) since $\Downarrow_M\check I_M c\subseteq\hat I_M c$ for case (3) will then follow dually.

Let $c=(t_c,x_c^1,\dots,x_c^n)$, then $\check I_{\tilde M}c\cap R\neq\emptyset$ implies $t_c<0$. Define

$$S_1\equiv\Uparrow_M\hat I_M c\cap\{(t,x^1,\dots,x^n):t\le 0\},$$

$$\tilde S_1\equiv\Uparrow_{\tilde M}\hat I_{\tilde M}c\cap\{(t,x^1,\dots,x^n):t\le 0\}=\check I_{\tilde M}c\cap\{(t,x^1,\dots,x^n):t\le 0\},$$

$$S_2\equiv\Uparrow_M\hat I_M c\cap\{(t,x^1,\dots,x^n):t>0\},$$

$$\tilde S_2\equiv\Uparrow_{\tilde M}\hat I_{\tilde M}c\cap\{(t,x^1,\dots,x^n):t>0\}=\check I_{\tilde M}c\cap\{(t,x^1,\dots,x^n):t>0\},$$

then $\Uparrow_M\hat I_M c=S_1\cup S_2$, $S_1\subseteq\tilde S_1-R$, $S_2\subseteq\tilde S_2-R$. We want to show $S_1\subseteq\check I_M c$ and $S_2\subseteq\check I_M c$.

Let $p\in S_1\subseteq\tilde S_1-R$, then $p\in\check I_{\tilde M}c$. The timelike curve connecting c to p must not intersect R or there would be $q\in R\cap\hat I_{\tilde M}p$ which implies $p\in\check I_{\tilde M}a\cap\{(t,x^1,\dots,x^n):t\le 0\}\subset R$. thus $p\in\check I_M c$.

Let $p=(t_p,x_p^1,\dots,x_p^1)\in S_2$, then $p\in interior\{y\in M:y\gg_M\hat I_M c\}$ and one can choose $\alpha<t_p$ such that $p'=(t_p-\alpha,x_p^1,\dots,x_p^n)\in interior\{y\in M:y\gg_M\hat I_M c\}$, i.e., $p'\gg_M\hat I_M c$. Let $v_i=(t_c-1/i,x_c^1,\dots,x_c^n)$, then $\{v_i\}\subset\hat I_M c$ and $\{v_i\}$ converges to c. By lemma 1 we have $p'\gg_M\{v_i\}$, hence there exists timelike curves γ_i in M connecting v_i to p'. Since $t_p-\alpha>0$ and $t_c-1/i<0$, each γ_i much intersect the plane

$E=\{(0,x^1,\dots,x^n)\}$ at some point $q_i\in E\cap M$. The timelike property of γ_i gives

$$(t_c-1/i)^2>(x^1_{q_i}-x^1_c)^2+\dots+(x^n_{q_i}-x^n_c)^2.$$

while $q_i\in\gamma_i\cap E$ and $\gamma_i\cap R=\emptyset$ yield

$$1<(x^1_{q_i})^2+\dots+(x^n_{q_i})^2.$$

On the other hand, $\lim(t_c-1/i)=t_c$ implies that all q_i's with sufficiently large i are within a compact region of the n-dimensional Euclidean space E, hence there exists a subsequence $\{q_i'\}$ of $\{q_i\}$ such that $\{q_i'\}$ converges to a point $q=(0,x^1_q,\dots,x^n_q)\in E$ satisfying

$$t_c^2\ge(x^1_q-x^1_c)^2+(x^n_q-x^n_c)^2, \qquad (1)$$

$$1\le(x^1_q)^2+\dots+(x^n_q)^2. \qquad (2)$$

And $q_i'\in\hat I_M p'\subset\hat I_{\tilde M}p'$ implies $q\in closure(\hat I_{\tilde M}p')\subset\hat I_{\tilde M}p$.

LEMMA 4. If there exists $r\in E$ satisfying

(a) r is sufficiently close to q so that $r\in\hat I_{\tilde M}p$,

(b) $r\in(\check I_{\tilde M}c-R)\cap E=(\check I_{\tilde M}c\cap E)-(R\cap E)$,

then $p\in\check I_M c$.

PROOF. Since $(\tilde M,\eta)$ is a Minkowski spacetime, $r\in\hat I_{\tilde M}p$ implies that there exists a timelike geodesic γ connecting r to p. But $r\in E-R$ implies $\gamma\cap R=\emptyset$, hence $p\in\check I_M r$. On the other hand, requirement (b) leads to $r\in\tilde S_1-R=S_1\subseteq\check I_M c$, therefore $p\in\check I_M c$. ■

Let B_o, $B_c\subset E$ be open balls centered at $(0,0,\dots,0)$ and $(0,x^1_c,\dots,x^n_c)$ with radii 1 and $|t_c|$ respectively, then the requirement $r\in(\check I_{\tilde M}c-R)\cap E$ and inequalities (1), (2) are equivalently to $r\in B_c-closure(B_o)$ and $q\in closure(B_c)-B_o$ respectively. Since $B_c\not\subseteq B_o$ or c would be in R, it is clear that one can always find such an r for any q unless $q\in\partial B_c\cap\partial B_o$ and n=1. Therefore we conclude that the spacetime (M,η) with n>1 is causally continuous.

ACKNOWLEDGEMENTS

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where part of this work was done.

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