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A GEOMETRIC GAUGE THEORY OF METRIC DEFECTS *

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ABSTRACT

The mechanical behaviour of a material manifold with dislocations and disclinations is explored by applying nonriemannian geometry and gauge field theory. A geometric gauge theory of metric defects is introduced by local Lorentz invariance. As a result, we give the connection coefficients with the affine and the gauge connection. Taking the displacement field, the frame field and the gauge field as basic parameters, we obtain the constitutive equations and the governing equations based on a variational principle with respect to the groups of a coordinate transformation and a gauge transformation.

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INTRODUCTION

Generalized continuum mechanics is an important phase of current development in modern continuum mechanics. This field, initially studied by K. Kondo [1,2], E. Kröner[3,4] and B. A. Bilby et al.[5,6], is closely related to the theory of nonriemannian geometry. In continuous distribution theory of defects, it has been discovered that the reference configuration, in the constructs of nonriemannian space, such as metric, torsion and curvature tensors, is an Euclidean space with Euclidean metric structure and topological structure. According to the breaking of different structures of Euclidean space, defects are called metric or topological defects, respectively.

Since 1954 when the Yang-Mills theory[7] was established, one recognized that Riemannian geometry itself essentially belongs to a kind of gauge field theory[8,9]. Furthermore, recently it was learned from the study of supergravity that the geometry of nonriemannian space with nonvanishing torsion also belongs to a kind of non-Abelian gauge theory.

It is known that non-Abelian gauge theory can be naturally applied to any field in theoretical physics, provided that the field is related to Riemannian or nonriemannian geometry. Based on this point of view, some work has been done in using the gauge theory to study generalized continuum. A.G. Herrmann [10)] and A.G. Herrmann and D.G.B. Edelen [11)] first used Abelian gauge theory to discuss the gauge invariance of the governing equations with electro-magnetic field theory. Her works lead to further study by D.G.B. Edelen [12)] and A. Kadic and D.G.B. Edelen [13)] in the Yang-Mills minimal coupling theory for materials with dislocations and disclinations. In the field of geometric gauge theory, Y.S. Duan and Z.P. Duan [14] and Z.P. Duan [15] discussed the geometric representation of the gauge theory of defects.

For a complete theory of generalized continuum mechanics, we have to deal with not only the geometric aspects of the material manifold but some process of physics. In this paper, we establish a geometric gauge theory of metric defects based on Lorentz invariance and continuous distribution theory of defects.

I. THE GAUGE POTENTIALS

The mathematical theory of non-Abelian gauge theory, which is introduced by C. N. Yang and R. L. Mills[7], takes the transformation of gauge potentials as

$$B'_\mu = S^{-1} B_\mu S + \frac{i}{\epsilon} S^{-1} \partial_\mu S \tag{1.1}$$

where S is the spin gauge transformation.

E. Bortorotti[16], K. Kondo and Ishizuka [17] gave the transformation of the connection as follows:

$$\Gamma^\alpha_{\beta\lambda} = A^\alpha_i B^j_\beta B^k_\lambda \Gamma^i_{jk} + A^\alpha_i \partial_\beta B^i_\lambda \tag{1.2}$$

which is symmetric[16] or non-symmetric[17] with respect to indices β and λ under the coordinate transformation

$$B^j_\beta = \frac{\partial x^j}{\partial x^\beta}, \qquad B^{-1} = (A^\alpha_i). \tag{1.3}$$

From the expressions (1.1) and (1.2), we obtain some information as follows:

(1) These transformations have non-homogeneous terms $\frac{i}{\epsilon} S^{-1}\partial_\mu S$ and $A^\alpha_i \partial_\beta B^i_\lambda$ under corresponding transformation. Therefore, (1.1) and (1.2) mean that the symmetry is broken.

It is well known that the transformation operator B^j_β may be written as

$$B^j_\beta = \delta^j_\beta + \nabla_\beta W^j \tag{1.4}$$

where

$$\nabla_\beta W^j = \partial_\beta W^j + \Gamma^j_{ki} W^k \partial_\beta W^i \tag{1.5}$$

are covariant derivatives with respect to the connections Γ^j_{ki}. Under the first order approximation, covariant derivatives $\nabla_\beta W^j$ may be rewritten as

$$\nabla_\beta W^j \overset{(\beta)}{=\!=} \partial_\beta W^j \tag{1.6}$$

then we have

$$B^j_\beta = \delta^j_\beta + \partial_\beta W^j = \delta^j_\beta + \pi^j_\beta \tag{1.7}$$

where π^j_β are called differential extensions.

For fixed index j, π^j_β are covariant components of a vector in (β) system and it is written as the sum of a gradient and a rotation

$$\pi^j_\beta = a^j_\beta + c^j_\beta$$

based on the principle of decomposition, where

$$a^j_\beta = (\text{grad.}\varphi^j)_\beta, \qquad c^j_\beta = (\text{rot.}d^j)_\beta.$$

Therefore

$$A_i^\alpha \partial_\beta B_\lambda^j = A_i^\alpha (\partial_\beta a_\lambda^j + \partial_\beta c_\lambda^j) \quad (1.8)$$

In a vector field, rot.grad ≡ 0, the Ricci coefficients,

$$\Omega_{\beta\lambda}^\alpha = 2A_i^\alpha \partial_{[\beta} c_{\lambda]}^i . \quad (1.9)$$

It means that antisymmetric field c_β^j plays a leading role and symmetric field a_β^j plays an indirect role. The non-symmetry of the connection gives an antisymmetric field which is not only inducted by a stress field, but also can be generalized to be inducted by other physical effects. Therefore, a symmetry breaking will act as a role of an antisymmetric field.

(2) Since Lorentz group $\mathcal{L}$ is a linear transformation group depending on some parameters, the gauge symmetry of a rotation field will be broken under a local Lorentz transformation group and the role of a rotation field may be determined by physical effect of a gauge field. Therefore, gauge potentials $B_\mu^{a'b'}$ must be antisymmetric for contravariant indices a' and b', i.e.

$$B_\mu^{a'b'} + B_\mu^{b'a'} = 0. \quad (1.10)$$

It means that gauge potentials $B_\mu^{a'b'}$ take values in Lie algebra of Lorentz group $\mathcal{L}$. The coordinate components of gauge potentials are $B_{\beta\lambda}^\alpha$ which satisfy the following transformation laws:

$$B_{\beta\lambda}^\alpha = A_i^\alpha B_\beta^j B_\lambda^k B_{jk}^i + A_i^\alpha \partial_\beta B_\lambda^i \quad (1.11)$$

and their antisymmetric parts are

$$B_{[\beta\lambda]}^\alpha = A_i^\alpha B_\beta^j B_\lambda^k B_{[jk]}^i + \Omega_{\beta\lambda}^\alpha$$

or

$$B_{[\beta\lambda]}^\alpha - \Omega_{\beta\lambda}^\alpha = A_i^\alpha B_\beta^j B_\lambda^k B_{[jk]}^i .$$

Let

$$S_{\beta\lambda}^{\cdot\cdot\alpha} = B_{[\beta\lambda]}^\alpha - \Omega_{\beta\lambda}^\alpha \quad (1.12)$$

then

$$S_{\beta\lambda}^{\cdot\cdot\alpha} = A_i^\alpha B_\beta^j B_\lambda^k S_{jk}^{\cdot\cdot i} \quad (1.13)$$

where

$$S_{jk}^{\cdot\cdot i} = B_{[jk]}^i , \qquad \Omega_{jk}^i \equiv 0. \quad (1.14)$$

The expression (1.13) means that gauge potentials $B_{\beta\lambda}^\alpha$ are gauge-invariant and the role of an antisymmetric field is determined by both the antisymmetric part $B_{[\beta\lambda]}^\alpha$ of gauge potentials and Ricci coefficients $\Omega_{\beta\lambda}^\alpha$.

(3) In the mathematical theory of a gauge field, we choose a torsion tensor $S_{\beta\lambda}^{\cdot\cdot\alpha}$ and a spin

$$S_{\beta\lambda}^\alpha = S_{\beta\lambda}^{\cdot\cdot\alpha} - S_{\beta.\lambda}^{\cdot\alpha} - S_{\lambda.\beta}^{\cdot\alpha} \quad (1.15)$$

as a gauge-invariant physical variable which is independent on the choice of a coordinate system, where

$$S_{\beta.\lambda}^{\cdot\alpha} = g_\lambda^\mu B_{[\mu\beta]}^\alpha + g_\lambda^\mu \Omega_{\mu\beta}^\alpha$$
$$S_{\lambda.\beta}^{\cdot\alpha} = g_\beta^\mu B_{[\lambda\mu]}^\alpha + g_\beta^\mu \Omega_{\lambda\mu}^\alpha . \quad (1.16)$$

From (1.13) and (1.15), it is obvious that we must choose gauge potentials $B_{\beta\lambda}^\alpha$ as basic variables of a field.

II. THE PHYSICAL MODEL

In the classical theory of a continuum with defects, the dynamics of dislocations satisfied global Lorentz invariance[18,19], and defects in generalized solids satisfied only local Lorentz invariance in coupled physical fields. From the viewpoint of fields, just as the physical substance of topological defect introduced an antisymmetric field, there are other physical fields having antisymmetry. Therefore, we treat it with the role of antisymmetric fields by gauge potentials having antisymmetry.

To give the physical model of this paper, we introduce two basic assumptions as follows[9]:

(1) The dynamics of continuum dislocations satisfies a local Lorentz invariance in generalized solids;

(2) The first set of integrable conditions of a frame field is broken.

Thus, from the assumption (1), there exists a local Lorentz frame field $(e_\alpha^{a'}(x))$ which transforms on every spacetime point, where a' = 0, 1, 2, 3 is the index of the frame and α = 0, 1, 2, 3 is the index of the coordinate. For a', $e_\alpha^{a'}(x)$ are contravariant components of the local Lorentz frame and for α, are covariant components of the local coordinate system, and they satisfy

$$e_\alpha^{a'} e_{a'}^{\beta} = \delta_\alpha^\beta; \qquad e_\alpha^{a'} e_{b'}^{\alpha} = \delta_{b'}^{a'}.$$

$$\alpha,\ \beta = 0,\ 1,\ 2,\ 3;\quad a',\ b' = 0,\ 1,\ 2,\ 3.$$

The Lorentz frame field determines the metric of spacetime

$$(dS)^2 = g_{\alpha\beta} dX^\alpha dX^\beta; \qquad g_{\alpha\beta} = \eta_{a'b'} e_\alpha^{a'} e_\beta^{b'}$$

where $\eta_{a'b'}$ = diag(1,-1,-1,-1) is local Minkowski value of the metric and

$$g^{\alpha\beta} = e_0^\alpha e_0^\beta - e_1^\alpha e_1^\beta - e_2^\alpha e_2^\beta - e_3^\alpha e_3^\beta.$$

The matrix of the metric may be written as

$$G = (g_{\alpha\beta})_{0\leqslant\alpha,\beta\leqslant3} = (e_\alpha^{a'}) J (e_\beta^{b'})$$

where $J = (\eta_{a'b'})$. Therefore, the local transformation group of the frame is a Lorentz group $\mathcal{L}$.

From the assumption (2), there are

$$\Omega_{\beta\lambda}^{a'} = \partial_{[\beta} e_{\lambda]}^{a'} = \gamma_{b'c'}^{a'} e_{[\beta}^{b'} e_{\lambda]}^{c'} = 0,$$

where $\gamma_{b'c'}^{a'}$ is a connection of the frame field $e_\alpha^{a'}(x)$, and

$$\gamma_{b'c'}^{a'} = e_{b';\beta}^{\alpha} e_{c'}^{\beta} e_\alpha^{a'} = [\partial_\beta e_{b'}^\alpha + \{{}_{\beta}{}^{\alpha}{}_{\lambda}\} e_{b'}^\lambda] e_{c'}^\beta e_\alpha^{a'} \neq 0$$

where $\partial_\beta = \dfrac{\partial}{\partial x^\beta}$.

We introduce a differential operator

$$X_{a'} = e_{a'}^\alpha \partial_\alpha$$

for the covariant components of the frame, there are

$$2X_{[c'} e_{b']}^\beta = X_{c'} e_{b'}^\beta - X_{b'} e_{c'}^\beta = C_{b'c'}^{a'} e_{a'}^\beta$$

where

$$C_{b'c'}^{a'} = \gamma_{b'c'}^{a'} - \gamma_{c'b'}^{a'} = 2\gamma_{[b'c']}^{a'}$$

and

$$2X_{[c'} \gamma_{|b'|d']}^{a'} = X_{c'} \gamma_{b'd'}^{a'} - X_{d'} \gamma_{b'c'}^{a'}$$

$$= \gamma_{f'[d'}^{a'} \gamma_{|b'|c']}^{f'} - \gamma_{b'f'}^{a'} C_{b'c'}^{f'} - R_{b'c'd'}{}^{a'}$$

where $R_{b'c'd'}{}^{a'}$ is the frame components of a curvature tensor.

III. THE EUCLIDEAN CONNECTION

Let $(e_\alpha^{a'}(x))$ be a Lorentz frame field in spacetime of four dimensions and $V_\beta^{a'}$, $V_\beta^{a'} + dV_\beta^{a'}$ be two vectors at point $P(x^\beta)$ and its adjacent point $Q(x^\beta + dx^\beta)$, respectively. The Lorentz frame field introduces a change in $V_\beta^{a'}$

as

$$\delta V^{a'}_{\beta} = dV^{a'}_{\beta} - \bar{d}V^{a'}_{\beta} \tag{3.1}$$

where $dV^{a'}_{\beta}$ is the change introduced by vector $V^{a'}_{\beta}$ itself, and $\bar{d}V^{a'}_{\beta}$ is the change introduced by Lorentz frame field at points P and Q.

A matrix element of Lorentz group $\mathcal{L}$ is $L^{a'}_{a}$ at point P and $L^{a'}_{a} + dL^{a'}_{a}$ at point Q, then

$$V^{a'}_{\beta} + dV^{a'}_{\beta} = V^{a'}_{\beta} + L^{a'}_{a} dV^{a}_{\beta} + V^{a}_{\beta} dL^{a'}_{a}$$

and

$$V^{a'}_{\beta} + \bar{d}V^{a'}_{\beta} = V^{a'}_{\beta} + L^{a'}_{a} \bar{d}V^{a}_{\beta} + V^{a}_{\beta} dL^{a'}_{a}.$$

Substituting (3.1) into the above two equations and substracting them, we obtain

$$\delta V^{a'}_{\beta} = L^{a'}_{a} \ \delta V^{a}_{\beta} \tag{3.2}$$

It means that $\delta V^{a'}_{\beta}$ *is a vector and it is independent on the choice of the* Lorentz frame.

For every vector $V^{a'}_{\beta}$, if

$$- \bar{d}V^{a'}_{\beta} = \delta V^{a'}_{\beta} = 0 \tag{3.3}$$

holds, we say that the Lorentz frames at points P and Q are quasi-parallel.

Let $\Gamma^{\alpha}_{\beta\lambda}$ be the spacetime connection of the local coordinate system. For contravariant and covariant vectors $W^{\beta}_{a'}$ and $V^{a'}_{\beta}$, there are

$$\delta W^{\beta}_{a'} = dW^{\beta}_{a'} + \Gamma^{\beta}_{\lambda\alpha} W^{\alpha}_{a'} dx^{\lambda}$$

$$\delta V^{a'}_{\beta} = dV^{a'}_{\beta} - \Gamma^{\alpha}_{\lambda\beta} V^{a'}_{\alpha} dx^{\lambda}$$

then their covariant derivatives are

$$\begin{aligned} \nabla_{\lambda} W^{\beta}_{a'} &= W^{\beta}_{a'|\lambda} = W^{\beta}_{a',\lambda} + \Gamma^{\beta}_{\lambda\alpha} W^{\alpha}_{a'} \\ \nabla_{\lambda} V^{a'}_{\beta} &= V^{a'}_{\beta|\lambda} = V^{a'}_{\beta,\lambda} - \Gamma^{\alpha}_{\lambda\beta} V^{a'}_{\alpha} \end{aligned} \tag{3.4}$$

where the symbol "," means a partial derivative with respect to coordinates and the symbol "|" means a covariant derivative with respect to the connection $\Gamma^{\alpha}_{\lambda\beta}$.

We can obtain a transformation of the connection $\Gamma^{\alpha}_{\lambda\beta}$ *with respect to non-*holonomic transformation of the coordinates. On the one hand

$$\partial_{\lambda} e^{a'}_{\beta} = \Gamma^{\alpha}_{\lambda\beta} e^{a'}_{\alpha} = \Gamma^{\alpha}_{\lambda\beta} B^{i}_{\alpha} e^{a'}_{i}$$

and on the other hand

$$\begin{aligned} \partial_{\lambda} e^{a'}_{\beta} &= \partial_{\lambda}[B^{j}_{\beta} e^{a'}_{j}] \\ &= (\partial_{\lambda} B^{j}_{\beta}) e^{a'}_{j} + B^{j}_{\beta} (\partial_{\lambda} e^{a'}_{j}) \\ &= (\partial_{\lambda} B^{j}_{\beta}) e^{a'}_{j} + B^{j}_{\beta} B^{k}_{\lambda} (\partial_{k} e^{a'}_{j}). \end{aligned}$$

Thus, we have

$$\Gamma^{\alpha}_{\lambda\beta} = A^{\alpha}_{i} B^{j}_{\beta} B^{k}_{\lambda} \Gamma^{i}_{kj} + A^{\alpha}_{j} \partial_{\lambda} B^{j}_{\beta}. \tag{3.5}$$

The frame field $(e^{a'}_{\alpha}(x))$ determines the metric and the geometric structure of the spacetime. Therefore, indices of the frame describe all geometric objects which must be Lorentz invariant. Obviously, there is

$$\Gamma^{a'}_{b'\beta} = \Gamma^{\alpha}_{\lambda\beta} e^{a'}_{\alpha} . e^{\lambda}_{b'} = \Gamma^{\alpha}_{\lambda\beta} L^{a'}_{a} e^{a}_{\alpha} 1^{b}_{b'} . e^{\lambda}_{b} = \Gamma^{\alpha}_{\lambda\beta} e^{a}_{\alpha} e^{\lambda}_{b} L^{a'}_{a} 1^{b}_{b'} = \Gamma^{a}_{b\beta} L^{a'}_{a} 1^{b}_{b'},$$

we obtain

$$\Gamma^{\alpha}_{\lambda\beta} = \Gamma^{a'}_{b'\beta} e^{\alpha}_{a'} . e^{b'}_{\lambda} \quad \text{and} \quad \Gamma^{\alpha}_{\lambda\beta} = \Gamma^{a}_{b\beta} e^{\alpha}_{a} . e^{b}_{\lambda} \tag{3.6}$$

Thus, on the one hand

$$\partial_{\lambda} e^{b'}_{\beta} = \Gamma^{\alpha}_{\lambda\beta} e^{b'}_{\alpha} = \Gamma^{\alpha}_{\lambda\beta} L^{b'}_{b} e^{b}_{\alpha} \tag{3.7}$$

and on the other hand

$$\begin{aligned} \partial_{\lambda} e^{b'}_{\beta} &= \partial_{\lambda}[L^{b'}_{b} e^{b}_{\beta}] \\ &= (\partial_{\lambda} L^{b'}_{b}) e^{b}_{\beta} + L^{b'}_{b} (\partial_{\lambda} e^{b}_{\beta}) \end{aligned} \tag{3.8}$$

in local coordinate system. This means that $S_{\lambda\beta}^{\cdot\cdot\alpha}$ is a tensor and is gauge-invariant. $S_{\lambda\beta}^{\cdot\cdot\alpha}$ is called a torsion tensor.

From the isometric principle

$$-Q_{\mu\beta\lambda} = \nabla_\mu g_{\beta\lambda} = 0 \quad \text{or} \quad Q_\mu^{\cdot\beta\lambda} = \nabla_\mu g^{\beta\lambda} = 0, \tag{3.9}$$

we obtain the symmetric part of Euclidean connection as

$$\Gamma^\alpha_{(\mu\lambda)} = \{^{\ \alpha}_{\mu\ \lambda}\} - g^\beta_\lambda \Gamma^\alpha_{[\mu\beta]} - g^\beta_\mu \Gamma^\alpha_{[\lambda\beta]}, \tag{3.10}$$

then the Euclidean connection is

$$\Gamma^\alpha_{\mu\lambda} = \{^{\ \alpha}_{\mu\ \lambda}\} + S_{\mu\lambda}^{\cdot\cdot\alpha} - S^{\cdot\alpha}_{\mu.\lambda} - S^{\cdot\alpha}_{\lambda.\mu} - Q^\alpha_{\mu\lambda} + Q^\alpha_{\mu.\lambda} + Q^\alpha_{\lambda.\mu}. \tag{3.11}$$

Let us now consider the gauge theory of metric defects under local Lorentz invariance. $B^\alpha_{\mu\lambda}$ is the coordinate component of the gauge potential and substituting (1.12) into (3.11) we obtain

$$\Gamma^\alpha_{\mu\lambda} = \{^{\ \alpha}_{\mu\ \lambda}\} + B^\alpha_{\mu\lambda} - B^\alpha_{\mu.\lambda} - B^\alpha_{\lambda.\mu} \tag{3.12}$$

where

$$B^\alpha_{\mu.\lambda} = g^\beta_\lambda B^\alpha_{\mu\beta} \quad \text{and} \quad B^\alpha_{\lambda.\mu} = g^\beta_\mu B^\alpha_{\lambda\beta} \tag{3.13}$$

(3.12) shows that the frame field $\{e^{a'}_\alpha(x)\}$ and the gauge potential $B^\alpha_{\beta\lambda}$ *together determine the geometrical structure of the continuum with metric* defects and the gauge field.

IV. EQUATIONS OF THE FIELD

First, we consider the structure of a Lagrangian function describing metric defects and a gauge field.

Obviously, $e^{a'}_\alpha$ and $B^{a'b'}_\mu$ are independent variables of the field describing the properties of a continuum with metric defects and a gauge field.

Setting

$$e^{\mu\nu} = (e^{a\mu\nu}_b)_{0\leqslant a,b\leqslant 3} \tag{4.1}$$

where

$$e^{a\mu\nu}_b = \eta_{bc}\, e^a_{[\mu} e^c_{\nu]}, \tag{4.2}$$

we obtain the strength of a gauge field as

$$F^a_{b\mu\nu} = \partial_\nu B^a_{b\mu} - \partial_\mu B^a_{b\nu} + B^a_{c\nu} B^c_{b\mu} - B^a_{c\mu} B^c_{b\mu}, \tag{4.3}$$

then

$$F_{\mu\nu} = (F^a_{b\mu\nu})_{0\leqslant a,b\leqslant 3}. \tag{4.4}$$

The strength of a gauge field satisfies the Bianchi equality

$$F^a_{b\mu\nu\|\lambda} + F^a_{b\nu\lambda\|\mu} + F^a_{b\lambda\mu\|\nu} = 0, \tag{4.5}$$

where the symbol "‖" means the covariant derivative with respect to the coordinate and the strength of a gauge field, i.e.

$$F^a_{b\mu\nu\|\lambda} = \partial_\lambda F^a_{b\mu\nu} - \{^{\ \alpha}_{\lambda\ \mu}\} F^a_{b\alpha\nu} - \{^{\ \alpha}_{\lambda\ \nu}\} F^a_{b\mu\alpha} + B^a_{c\lambda} F^c_{b\mu\nu} - B^c_{b\lambda} F^a_{c\mu\nu}. \tag{4.6}$$

Now the total Lagrangian function of the system is

$$L = L_e + L_g + L_{int}, \tag{4.7}$$

where L_e is the Lagrangian function of an elastic field, L_g, the Lagrangian function of a gauge field and L_{int}, the Lagrangian function describing the interaction between the frame field $e^{ab}_{\mu\nu}$ and the gauge field B^{ab}_μ, and there are

$$L_e = L_e(u_\mu, u_{\mu\|\nu}) \tag{4.8}$$

$$L_g = tr(F_{\mu\nu}F^{\mu\nu}) \tag{4.9}$$

and

$$L_{int} = tr(F_{\mu\nu}e^{\mu\nu}) = KR \tag{4.10}$$

where

$$R = R^\mu_a e^a_\mu \tag{4.11}$$

and

$$R^\mu_a = F^{\mu'\nu}_{a'b} e^b_\nu e_{a\mu'} e^{a'\mu}. \tag{4.12}$$

Obviously, L_e, L_g and L_{int} are invariant under the group-couple $\mathcal{L} \times \mathcal{A}$, where the group $\mathcal{A}$ is the group of a non-homonomic transformation of the coordinates.

(4.7) represents that the gauge field (B^{ab}_μ) and the frame field (e^a_α) commonly describe the field of metric defects.

The action functional is

$$S = \int[L_e - \frac{\eta}{4} tr(F_{\mu\nu}F^{\mu\nu}) - Ktr(F_{\mu\nu}e^{\mu\nu})]\varepsilon d^4x, \tag{4.13}$$

where

$$\varepsilon = \det(e^a_\mu) = [-\det(g_{\mu\nu})]^{1/2}$$

and η is a new coupled constant.

Suppose that the Hamilton principle holds, and take the variation of the action functional with respect to e^a_μ and B^{ab}_μ, we can obtain the new equations of the elastic field and the metric defect field as follows:

$$\frac{\partial L_e}{\partial u_\mu} - \frac{1}{\varepsilon}\partial_\nu[\varepsilon\frac{\partial L_e}{\partial u_{\mu\|\nu}}] = 0, \tag{4.14}$$

$$R^\mu_a - \frac{1}{2}Re^\mu_a = -\frac{1}{2K}(\eta t^\mu_a + T^\mu_a) \tag{4.15}$$

and

$$\eta F^{\mu\nu}_{ab\|\nu} = 2K\hat{t}^\mu_{ab} + \hat{T}^\mu_{ab} \tag{4.16}$$

where

$$t^\mu_a = -[tr(F_{\sigma\lambda}F^{\sigma\mu})e^\lambda_a - \frac{1}{4}tr(F_{\lambda\sigma}F^{\lambda\sigma})e^\mu_a] \tag{4.17}$$

is an energy-momentum tensor of metric defects, and

$$\hat{t}^\mu_{ab} = -\frac{1}{2}[S^\mu_{ab} - S^\lambda_{ac}e^c_\lambda e^\mu_b - S^\lambda_{cb}e^c_\lambda e^\mu_a] = -e^{\mu\nu}_{ab\|\nu} \tag{4.18}$$

is the density of the spin flow in the field of metric defects, and

$$S^\mu_{ab} = S^\lambda_{cd}e^a_\lambda e^c_\mu e^d_\nu = -2\partial_{[\mu}e^a_{\nu]} - 2B^a_{c[\mu}e^c_{\nu]}. \tag{4.19}$$

is the torsion tensor of the spacetime. T^μ_a is an energy-momentum tensor of the elastic field, and

$$\hat{T}^\mu_{ab} = \frac{1}{\varepsilon}\cdot\frac{\partial(\varepsilon L_e)}{\partial B^{ab}_\mu} \tag{4.20}$$

is the density of the spin flow in the elastic field.

Therefore, the action of metric defects represents not only the geometric effect of the metric, but the physical effect of the gauge potential. Thus, this theory is different from the dynamic theory of the continuous distribution[18,19].

V. DISCUSSION

1. In our treatment, the space-time is a nonriemannian space of four dimensions, the geometry of the space-time is connected with metric defects, and the gauge field is nonriemannian geometry. This space-time does not

only have the curvature, but also the torsion. The connection of the space-time has two parts, namely the affine and the gauge. In particular, the gauge potential describes both the defects and the effect of the physical field. Thus, our procedure is different from the three state theory of the gauge field of a continuum with dislocations and disclinations[14].

2. Since

$$u_{\mu\|\nu} = \partial_\nu u_\mu - \left\{ {}^{\sigma}_{\mu\ \nu} \right\} u_\sigma - B^{\sigma}_{\mu\nu} u_\sigma ,$$

then the field equation (4.14) contains a coupled effect of the metric defect and the gauge potential. Therefore, (4.14) is different from the representation of Cauchy strain tensor[13,14].

3. Eq. (4.15) is a new field equation. It is the generalization of Einstein equation[22] (see also Refs. 23 and 24), and it describes the gauge theory of the metric defect containing the elastic field. Based on continuous distribution theory of metric defects similar to gravity theory, and setting $t^{\mu}_{a} = 0$, we obtain

$$K = \frac{1}{16\pi k}$$

where k is the so-called Newton constant of gravity. When the gauge potential $B^{\mu}_{ab} \equiv 0$, we obtain the results of continuous distribution of the dynamic metric defect[18,19].

K. Kondo[20] has analogously discussed the general relativity and H. Y. Guo[21] has analogously discussed the gauge theory of gravitational field. The correspondence given in this paper is the generalization of the correspondence between continuous distribution theory of defects and gravity field theory or electromagnetic field theory.

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