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FRACTALS AND MULTIFRACTALS IN PHYSICS

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Abstract

We present a general introduction to the world of fractals. The attention is mainly devoted to stress how fractals do indeed appear in the real world and to find quantitative methods for characterizing their properties. The idea of multifractality is also introduced and it is presented in more details within the framework of the percolation problem.

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1. Introduction

The physical world contains an amazing variety of beautiful objects with intricate and complex geometrical properties: the structure of a fern leaf, the irregular surface of a mountain or the ramified pattern of an electric discharge in a dielectric material (Fig.1) are only few examples. On the other hand, the physical laws into play are often a set of rather simply-stated mathematical rules. How does then a simple law give rise to such complex outcomes ?

A class of objects with interesting geometrical properties are called fractals [1]. These are defined by the statement that a fractal contains within itself an infinite number of little copies of itself. Fractal is a fern leaf, which has branches that look almost like the whole object, and fractal can be considered the peaks and valleys structure of the surface of a mountain, whose irregular shape can be simulated by a simple computer algorithm.

How would then a mathematician, a computer scientist or a physicist construct an object with such a fractal structure ?

We'll try to answer this question, showing how these beautiful structures that appear in the real world can be actually described by rather simple models, and we will address the problem of defining a measure which could allow us to distinguish between different fractal objects. We will see that this is not a trivial problem and that, to describe a fractal, a single outcome of a measure does not give sufficient information but that an infinite set of numbers is usually necessary. This property is known as "multifractality".

The outline of this paper is the following: in section 2 some examples of fractal objects are given, pointing out how they can actually arise in natural situations. Some quantitative methods for characterizing their properties are also discussed. This section mainly consists of our free interpretation of the lecture given by L.P. Kadanoff at this school.

In section 3 we will focus our attention on the percolation problem. The percolation cluster is in fact an example of a statistical fractal and, in the attempt of finding the right quantities to characterize its structure, we show that it is, indeed, a multifractal ensemble of fractal sets, within a mathematical approach analogous to the one introduced in the previous section.

Finally, we point out how this theoretical approach represents a powerful tool to characterize a variety of fractal objects and some general conclusions are given.

2. Fractals and their measure

How can we construct a fractal ? Let's take a triangle and, within this triangle, let's draw three little triangles identical in shape to the whole one. Each little triangle can then be partitioned, in the same way, three smaller triangles and the process can be carried on to have a complicated structure of triangles nested within triangles to the finest level. This fractal is known as the Sierpinski gasket (Fig.2).

But we can think of a more natural process to obtain a fractal structure. The Diffusion Limited Aggregate (DLA) is constructed by computer in the following way [2]: we start with an aggregate of nearest-neighbor occupied sites on a grid. We send a random walker from a distance very large compared to the dimension of the cluster and we let it walk on the lattice until it hits a site neighboring the aggregate, which then grows by this one unit. The process starts again by sending more walkers. The DLA aggregate so obtained has a very ramified structure, tree-like and with no loops (Fig.3). Objects with this structure do indeed appear in nature: aggregates of uniform gold colloids [3] or Hele-Shaw experiments [4], where a low viscosity fluid is pushed into a high viscosity fluid, show patterns very similar to DLA.

Why an aggregate grown with such an algorithm is a fractal, why does it have this ramified structure ?

Let's assign for example a different color to walkers which stick to the aggregate in different intervals of time. We see that the later walkers cannot penetrate over a larger distance than the previous walkers, that is they cannot reach into the deep fjords of the aggregate to fill up the structure but they are screened by the most exposed regions of the tips. This screening effect causes the fractal geometry of the aggregate. And indeed this type of mechanism is responsible for the beautiful dendrites that grow in water when it is cooled below the freezing point, or that electrons trace in a plastic material in a dielectric breakdown experiment [5] (Fig.1).

Is it possible to introduce a measurement able to determine the properties of a fractal and to distinguish among different fractals ? The goal is to associate to each fractal some number which can tell us, for example, if the DLA model, which is rather well understood, does catch the physics of similar fractal objects constructed in a laboratory experiment.

Let's start with the classical way to characterize a fractal by introducing the Hausdorff dimension d_f. If we want to measure a fractal object like a line of length L, we take a small line of unit length u, we superpose to the line L many little lines u and count how many unit lengths we need to completely cover L. The number $N = L/u$ represents a measure of the line L. Analogously, to measure a square (cube) of side L, we take a unit square (cube) of side u and measure the numbers of units $N = L^2/u^2$ ($N = L^3/u^3$) needed to cover up the object. In general, this algebraic process leads to $N = (L/u)^{d_f}$ and to the determination of the exponent d_f, the Hausdorff dimension. For an uniform compact object d_f is simply equal to the Euclidean dimension d.

How does it work for the Sierpinski gasket ?

We start with a triangle of side L partitioned into infinitely many small triangles. By taking a unit triangle of side $u = L/2$, we need $N = 3$ unit triangles to cover up the gasket. If we take instead $u = L/4$, the measure will be $N = 9$. Therefore, in general, for the gasket the measure will be $N = 3^K$, with $u = L/2^K$, and the Hausdorff dimension $d_f = \log N/\log(L/u)$ will be $\log 3/\log 2$.

The same process can be applied to DLA. We can cover the aggregate of linear dimension L with boxes of side u and count the number of boxes N which have a piece of DLA in it. The logarithm of this number, logN, plotted versus log(L/u) will give for large values of L/u a straigth line whose slope is the Hausdorff dimension d_f. Starting from this simple model, we can also change some of the rules, by assigning for example a sticking probability $p < 1$ to the incoming walker, and what we see is that d_f is a number that does not change for a wide variety of models.

This is known as "universality" and is a concept familiar in particle phys:cs or in critical phenomena. One has seen that a quantity to measure is relative insensitive to the microscopic details or to the method used in the measurement. For example, the log-log plot of the magnetic susceptibility versus the temperature gives the same slope for different materials. Of course, this is a very good situation for physicists who, therefore, do not have to deal with microscopic details.

Having defined the Hausdorff dimension as a suitable measure, for example, for DLA, we want now to compare the DLA aggregate with the similar object obtained with a dielectric breakdown or an Hele-Shaw experiment, by measuring d_f and seeing whether it is the same for all three cases.

To do this, there are indeed several problems. To start, the box counting method does not give very accurate results for d_f. Moreover, only one number, d_f, does not provide enough information and it is insufficient for making any incisive comparison. Finally, the Hausdorff dimension probably does not reflect much of the physics which produced the fractal and cannot therefore describe all its properties. Can we define then another measurement to get a more detailed description of a fractal object ?

Let us consider again a DLA aggregate on a lattice and let's send a large number of random walkers from infinity. Without growing the aggregate any further, that is removing the walker when it hits the aggregate. Let's count the number of walkers that hit a given site of the aggregate, for each site. We see that some sites are more likely to have walkers landing than others and, if we look at the sites which have, let's say, more than 10, 100, 1000 walkers landing, we see that the larger the number of walkers, the farther from the center and towards the tips these sites are located.

If ρ_j is the probability that a walker lands at site j, we can define the quantity

$$\alpha_j \equiv \log \rho_j / \log(L/u) \qquad (1)$$

which represents how often a walker arrives at site j; then, if we count the number of sites $n(\alpha)$ characterized by the same value of α, the quantity

$$f(\alpha) \equiv \log n(\alpha)/\log(L/u) \quad (2)$$

is the fractal dimension of the set of sites having the same value of α. We have defined then an ensemble of fractal dimensions $f(\alpha)$ and the Hausdorff dimension d_f will just be the fractal dimension corresponding to the most probable α, that is the maximum value of the function $f(\alpha)$. A fractal object that can be partitioned in sets, each with a different fractal dimension, is said to be a multifractal.

Multifractality was first introduced in the contest of turbulence [6] and sucessively found in percolation [7,8], DLA [9,10] and dynamical systems [11]. With the quantities α and $f(\alpha)$ we have constructed a tool that now can be used to try to answer some crucial questions about the different dynamical processes which give rise to different fractal objects and that can help us to relate models of statistical fractals with fractal patterns observed in physical experiments.

3. Multifractality in percolation

We will now focus our attention on the problem of percolation; we present a new approach to the study of this problem, based on the ideas of multifractality, that will lead to a better understanding of the structure of the percolating cluster and its properties [7,8].

We start by defining the bond percolation problem [12]. Let's consider a lattice and say that a bond is present with probability p and missing with probability 1-p. For small values of p, we will have on the lattice isolated bonds or small clusters, but, as the value of p increases, some of these clusters will grow or coalesce until there will be a connected cluster of bonds which spans the whole system. The value of p at which this spanning cluster first appears in an infinite system is called p_c, the percolation threshold.

A typical example of system which exhibits such percolation transition is the random resistor network. To each bond we now assign a conductance equal to one if the bond is present and zero if the bond is missing. By monitoring then the conductance of the whole system, we see that this quantity is critical and goes to zero as p decreases toward p_c.

What is the structure of this incipient infinite cluster at the percolation threshold ?

To answer this question, we consider a random resistor network at the percolation threshold, we apply a difference of potential $\Delta V = 1$ at the opposite edges of the lattice of size L and then we can partition the bonds in the spanning cluster in three different sets: the bonds that do not carry any current, called dangling ends. The bonds which are simply connected, that is such that, once they are removed, the cluster becames disconnected; they carry the whole current and they are called red bonds or links. Finally, the bonds which are multiply connected, that carry a fraction of the whole current and that are called blue bonds or bonds in a blob.

Therefore, the part of the incipient infinite cluster which carries current consists in the red bonds and the blobs and is called "backbone", whereas the dangling ends do not give any contribution to the transport properties since they don't carry any current. On the other hand, the dangling ends represent almost the total mass of the infinite cluster, having its same fractal dimension $d_f = 91/48$ in two dimensions, whereas the blobs have a fractal dimension $d_{BB} \simeq 1.6$ and the red bonds have a fractal dimension exactly equal to the inverse of the connectedness length critical exponent [13], $d_{RB} = \frac{1}{\nu}$, equal to .75 in two dimensions. Moreover, the bonds in a blob can have a very different role in the network since they can carry a large fraction of the current, or they can be embedded in a very large blob and be almost balanced.

It becomes clear, then, that depending on which property of the infinite cluster we are interested in studying, we have to focus our attention on different sets of bonds. Moreover, this initial partition of the cluster in three sets of bonds is not exhaustive, since does not reflect the very different roles played by the bonds belonging to the blobs.

In order to give a better characterization of the different bonds in the backbone, we assign to each bond a weight equal to the voltage drop across it, V [7]. We can then define the quantity n(V) as the number of bonds characterized by a voltage drop V and the moments of such voltage distribution

$$M(k) = \sum_{V} n(V)\, V^{k} \sim L^{-\frac{p(k)}{\nu}} \qquad (3)$$

By performing the sum in (3) by steepest descent method, and by supposing that the exponents p(k) form an infinite set of independent exponents, we find the critical behaviour of the quantity n(V) is given by

$$n(V) \sim C(\alpha)\, L^{f(\alpha)} \qquad (4)$$

where

$$\alpha = -\frac{\ln V}{\ln L} \quad \text{i.e.} \quad V \sim L^{-\alpha} \qquad (5)$$

Here, the $f(\alpha)$ are the Legendre transform of the critical exponents p(k), that is

$$-f(\alpha) = \frac{p(k)}{\nu} - k\,\alpha \qquad (6)$$

where $\alpha = \frac{1}{\nu}\frac{d\,p(k)}{dk}$

The $f(\alpha)$ represents the fractal dimension of the set of bonds characterized by the value of V given by (5) and α therefore describes how this value of V goes to zero as the system size L goes to infinity.

We see that, if $\alpha(k)$ is a simple constant, the exponents p(k) would have a linear dependence on k and $f(\alpha)$ would be also constant. This simple type of scaling is what is usually found in critical phenomena and it is called gap exponent scaling. Here for the voltage distribution the scaling exponents of the moments are a more complicated function of k and the fractal dimension $f(\alpha)$ have a non trivial dependence on α.

All these results, are verified by computer simulations of a random resistor network at p_c in two dimensions. The numerical results show that the voltage distribution, n(V), as function of ln V is peaked at a most probable value of ln V and has a long tail in the low voltage region (Fig.4). As function of V, instead, the n(V) is not well behaving since it all collapses near the origin.

Starting from the knowledge of the voltage distribution, it is then possible to determine the critical exponents of its moments, p(k), and their Legendre transfrom, f(α). The function f(α), representing the fractal dimensions, is shown in Fig.5. It is a bell-shaped function, whose maximum is the Hausdorff dimension of the backbone ($\simeq$ 1.6 in d = 2), and which recovers the known value of the fractal dimension of the set of red bonds (= .75 in d = 2) in the limit $k \longrightarrow +\infty$, where the moments are dominated by the largest voltage drop V_{max} that occurs on the links.

The anomalous behaviour of the voltage distribution is responsible for the infinite sets of critical exponents p(k). Turns out, in fact, that, in order to perform the sum leading to the determination of M(k) in (3), is much more convenient to integrate over the ln V instead that the V variable. The leading contribution to the integral in (3) will then arises from different "typical" voltages depending on the different k-th moment considered. Only the k = 0 moment will derive its leading contribution from the most probable value of the n(V), whereas positive moments will be dominated by value of V in the high voltage region and the negative moments by the low voltage tail. Of course, if instead we were looking at the moments of ln V, and not V, these would not suffer by this anomalous behaviour; they will be always dominated by the most probable value and they will simply exhibit a gap exponent type of scaling in term of a finite number of independent exponents.

The point is that the physical properties of a random resistor network are related to the moments of V and not of ln V: the second moment represents the conductance of the backbone, the fourth moment is related to the noise [8] and the k = -1 moment is related to the mean square passage time in the hydrodynamic dispersion problem in the pure convective limit [14].

Therefore, the infinite hierarchy of exponents p(k) reflects the richness of the structure of the percolating backbone, which we can consider partitioned into subsets of bonds, characterized by a value of α, each set with its own fractal dimension f(α) and each set playing a dominant role in the determination of different physical properties.

This anomalous behaviour of the voltage ditribution can be verified by exact calculations on a hierarchical model introduced for the percolating backbone in ref. [7]. The model is shown in Fig.6: to each bond we substitute the unit cell made of two links and a blob of two bonds. At the N-th order of iteration, the model will contain therefore 2^N links and, by the scaling relation $2^N = L^{1/\nu}$, the arbitrary parameter N can be eliminated in favor of the size L of the system that we are modeling. This is one of the advantages of the model together with its simplicity and the fact that it presents the self-similar structure of links and blobs, typical of the real percolating backbone.

On this model, we can calculate the voltage distribution n(V) and we find that n(V) is a log-binomial distribution as function of the logarithm of the voltage

$$n(V) = 2^N \binom{N}{j} \tag{7}$$

where $V(j) = 2^j/5^N$ with $j = 0,1,\ldots N$.

Moreover, we find that the scaling law for this distribution $n(V) = L^{f(\alpha)}$ is expressed in term of a set of fractal dimensions, $f(\alpha)$, which have a not trivial dependence on α

$$f(\alpha) = 1 - [(1-y(\alpha)\ \ln\ (1-y(\alpha)) + y(\alpha)\ \ln\ y(\alpha)]/\ln 2 \tag{8}$$

with $y(\alpha) = \ln 5/\ln 2 - \alpha\nu$, which is again a bell-shaped curve as function of α in good agreement with the numerical results.

Therefore, also exact calculations on the hierarchical models show that infinitely many subsets of bonds can be individuated, each with a different fractal dimension $f(\alpha)$. That is the infinite hierarchy of critical exponents p(k) is expression of the multifractal structure of the percolating cluster.

4. Conclusions

The concept of multifractality arises in many different problems. Here we have discussed in particular its implications for DLA and percolation. Although these two problems represent very different physical processes, they can be usefully described within the same formalism. In fact, the voltage V in percolation plays the same role of the growth probability ρ for DLA and the exponent α represents the strenght of the singularity of these fields. The voltage distribution takes the place of the growth site probability distribution, which represents the number of perimeter sites of the DLA aggregate characterized by the same probability of growing. The sites with highest probability, the tips, correspond therefore to the bonds with highest voltage drop, the links, whereas the sets characterized by smaller and smaller values of the growth probability (voltage) correspond to those sites (bonds) situated into the deep fjords (large blobs) of the cluster.

As for these two previous examples, every phenomenon which gives rise to a fractal object can be described by means of the infinite set of fractal dimensions $f(\alpha)$. Fractals which are therefore characterized by the same $f(\alpha)$ curve will supposedly belong to the same universality class. To this extent, the $f(\alpha)$ curve plays for a fractal object the same role as critical exponents in critical phenomena.

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FIGURE CAPTIONS

Figure 1: Time integrated photograph of a surface leader discharge on a 2mm glass plate on SF_6 gas at 3 atmospheres from the dielectric breakdown experiment of ref. [5].

Figure 2: The first three iterations for the Sierpinski gasket.

Figure 3: A 100,00 sites cluster grown on the square lattice using the DLA model in $d = 2$ from ref. [15].

Figure 4: The voltage distribution $n(V)$ of a 130×130 square lattice random resistor network at the percolation threshold (950 configurations). The distribution has been normalized by the number of configurations and by the number of bonds in the backbone.

Figure 5: The fractal dimensions $f(\alpha) = \Phi(x(\alpha))$ plotted as function of the quantity $x = \ln V/\ln V_{max}$, where V_{max} is the maximum voltage drop which occurs on the links.

Figure 6: Few levels of iteration of the hierachical model for the backbone of the incipient infinite cluster.

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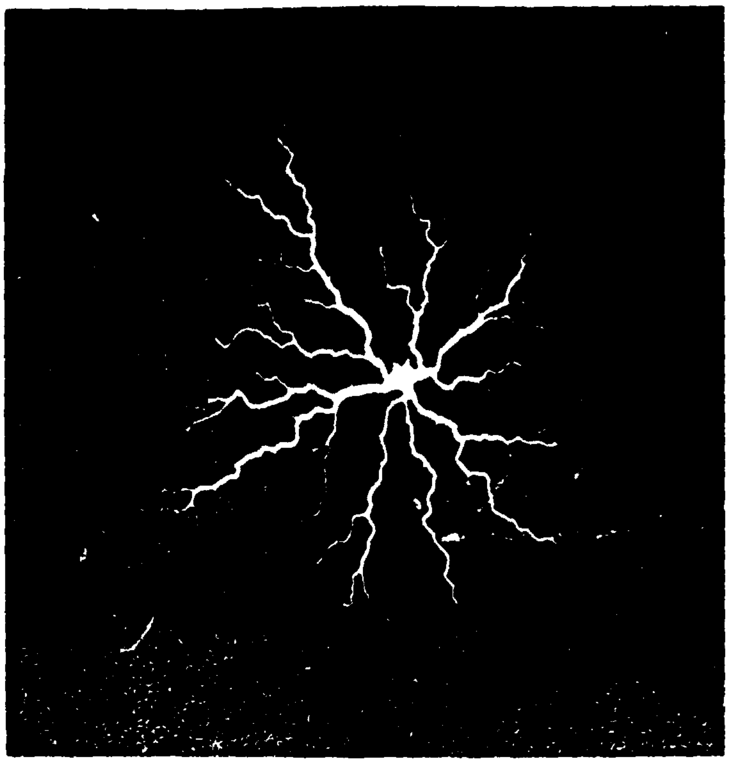

FIGURE 1

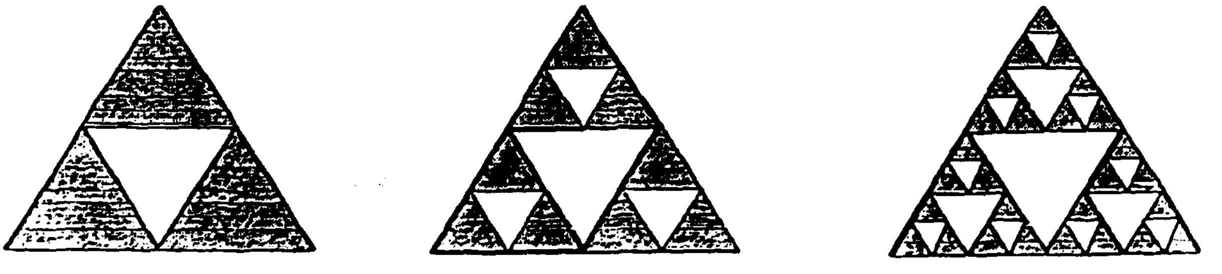

FIGURE 2

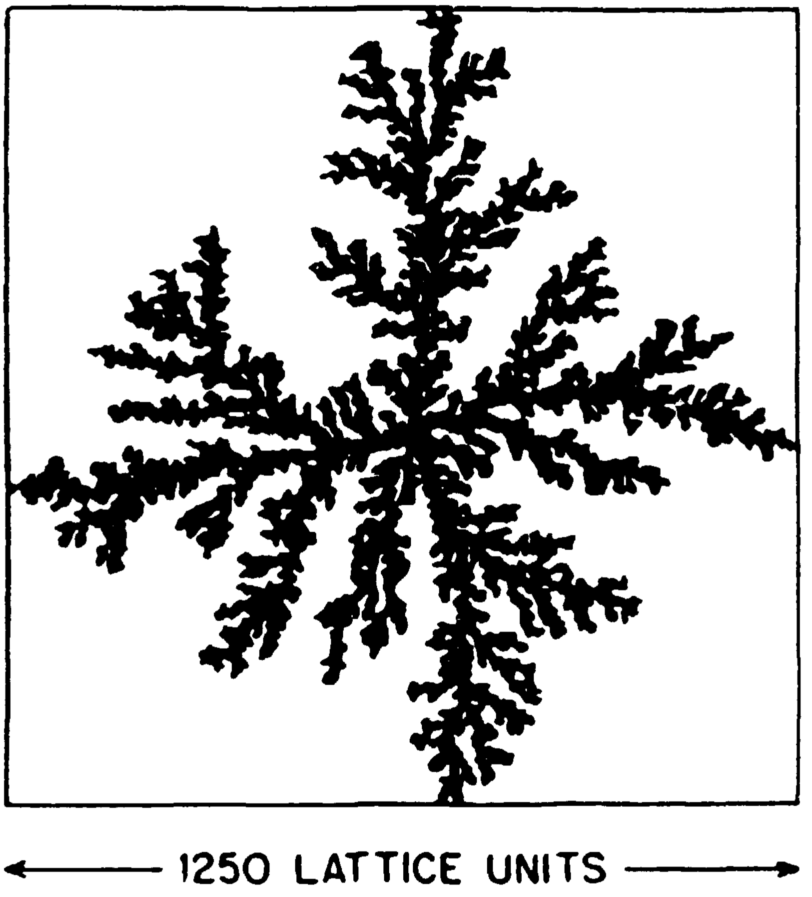

FIGURE 3

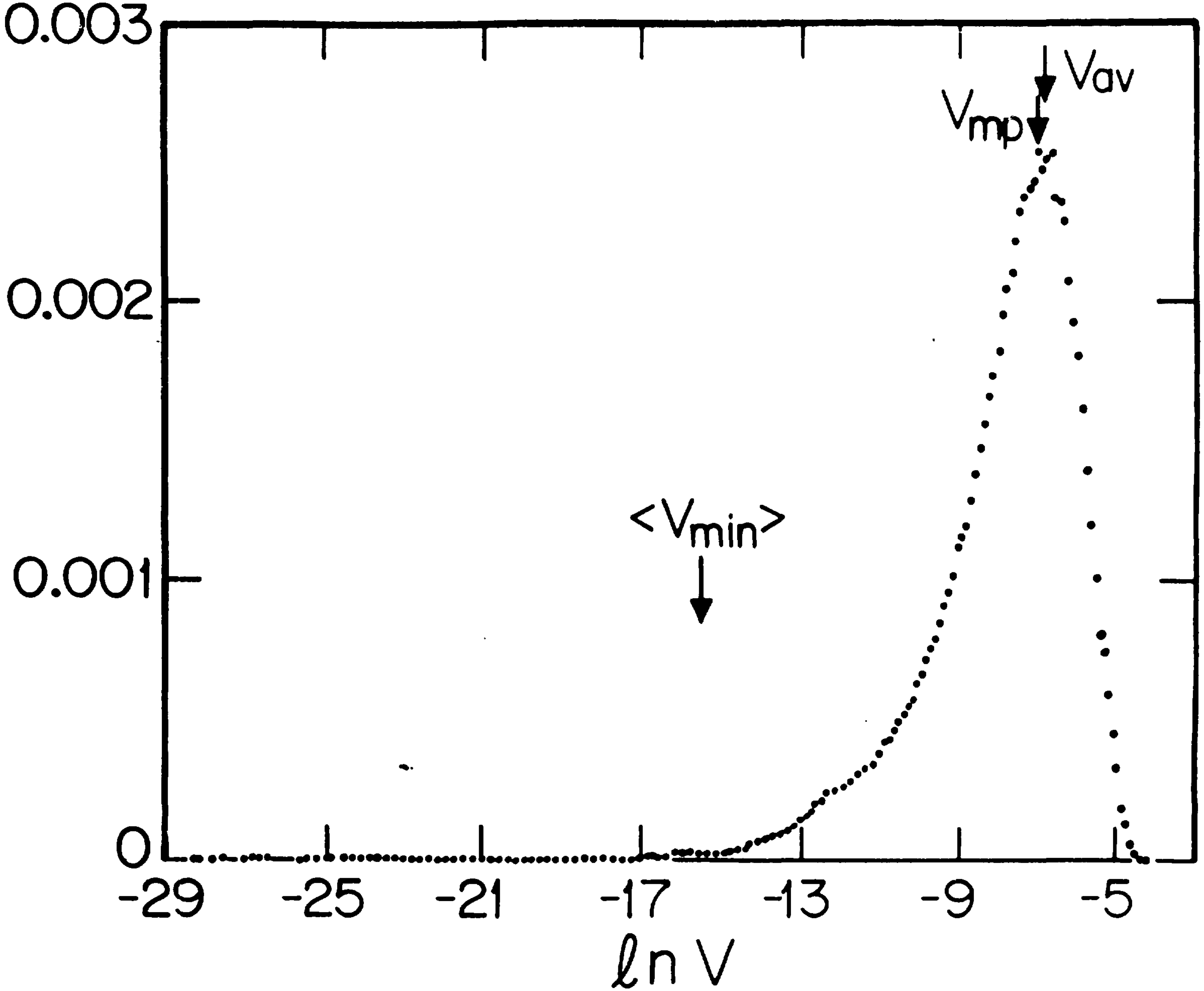

FIGURE 4

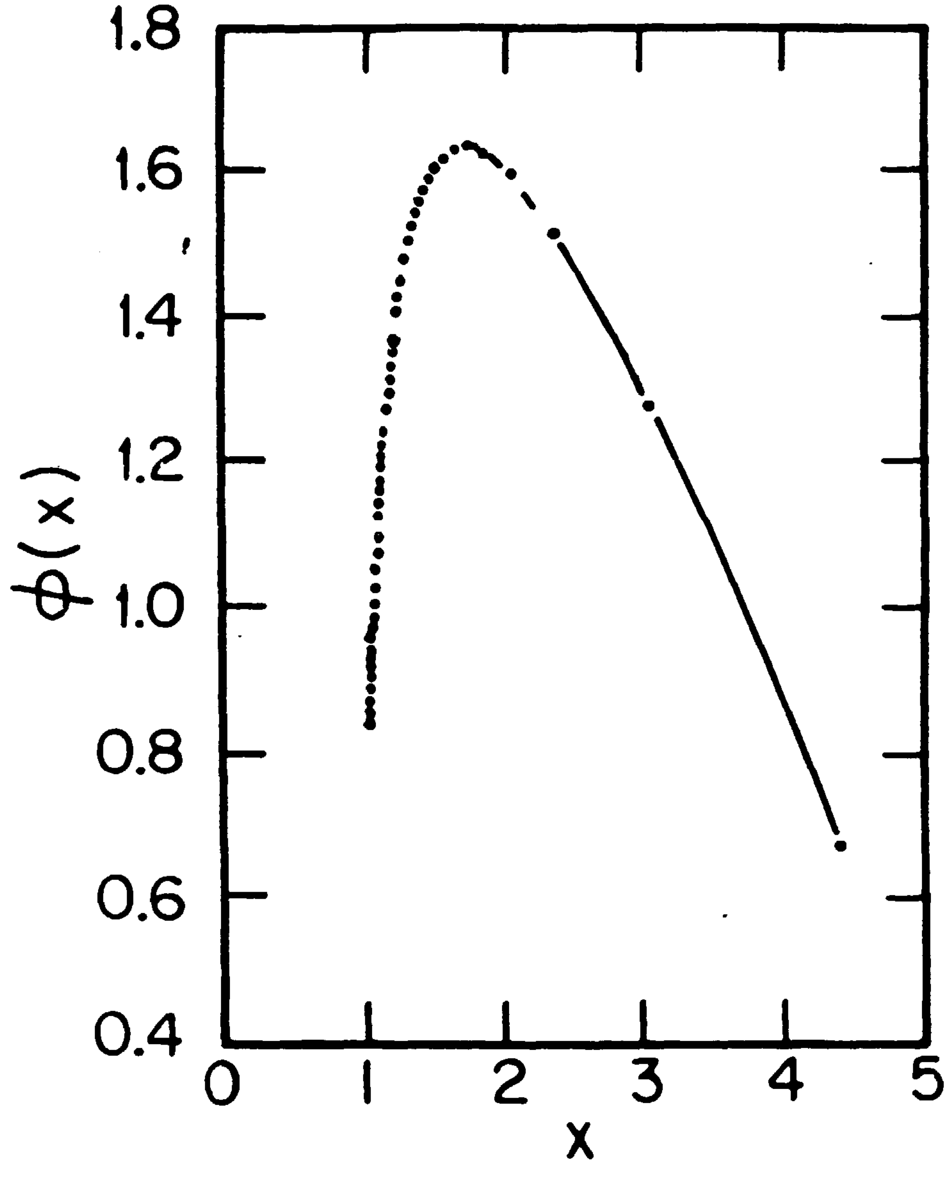

FIGURE 5

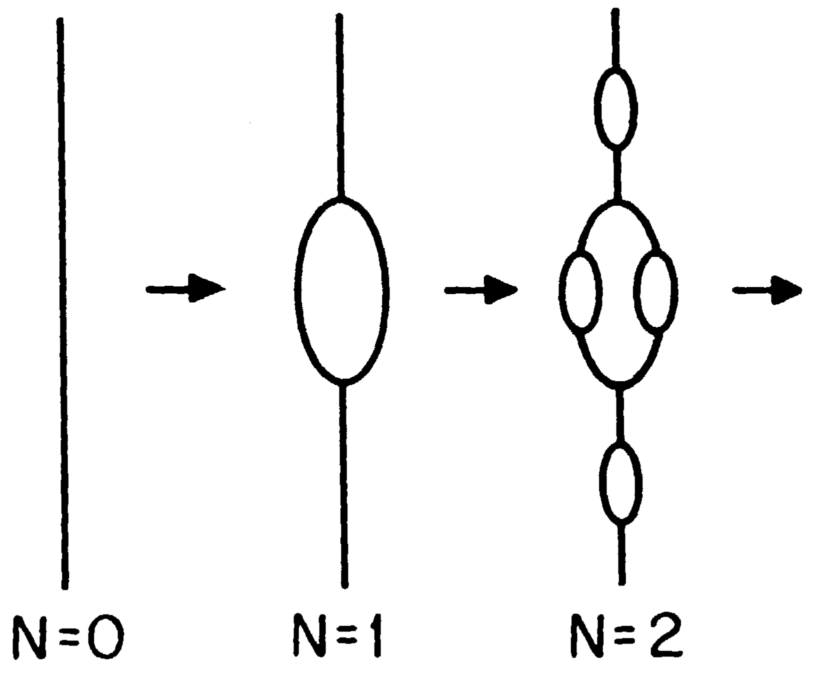

FIGURE 6