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TWO DIMENSIONAL CRITICAL MODELS ON A TORUS

Communication présentée à : Summer school on conformal invariance and string theory

Poiana Brasov (RO)
1-15 Sep 1987

TWO DIMENSIONAL CRITICAL MODELS ON A TORUS

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After the general developments of conformal invariance in two dimensions, it was realized that the study of critical models in finite geometries, in addition to the practical information it could provide through finite size scaling (see Pr. Rittenberg lecture), was also of great conceptual interest. The simplest example is the case of the torus, a genus 1 surface which is thus not conformally equivalent to the plane. This geometry appears quite frequently in lattice calculations for systems with periodic boundary conditions, and is also very natural from the point of view of string theory. We will discuss briefly in these notes the main results obtained so far in this simple case.

The torus can be parametrized by a complex number τ which is defined up to a modular transformation only, and the associated partition function must be invariant under the modular group. This proves to be a very strong constraint, which determines the possible operator content of the theories ie the degeneracies of the primary fields. These were not fixed by considerations in the plane only. We discuss in part 1 the case of the minimal theories with $c < 1$, and reexpose briefly the resulting ADE classification.

From another point of view, the toroïdal geometry imposes as well, through boundary conditions effects, the operator content of theories which are mapped onto a free field (Coulomb gas). Using know information about these mappings, we rederive in part 2 results of part 1 which were obtained so far by more formal considerations, providing in this way a connection between the microscopic models and their continuum limit partition functions.

To determine completely the possible conformal theories, one would need also to obtain the structure constants of the operator algebra. This can be in principle answered in various ways, like calculating four point functions on the plane, or two point functions on the torus. Correlators in toroïdal geometry have also interest on their own, in relation in particular to various lattice calculations. We expose in part 3 the simplest attempts in this direction by calculating multi-energies or spins correlations for the Ising model on a torus.

"Brasov summer school on conformal invariance and string theory" Saclay, SPhT/87-17

Poiana Brasov (Roumanie), 1-15 Septembre 1987

1. Partition Functions on a Torus and Modular Invariance

In statistical mechanics it is quite common to study models on a lattice $L \times L'$ with periodic boundary conditions, ie on a geometry which has the topology of a torus. For a temperature larger than the critical temperature T_c, the correlation lenght ξ is finite in the bulk and the properties of the finite system become independent of the precise boundary conditions as $L, L' \to \infty$, up to $exp(-L'/\xi)$ corrections. For $T = T_c$ however, the bulk correlation length is infinite and finite size scaling theory [1] shows that for the finite system it increases linearly with the dimensions, so one expects now corrections of the kind $exp(-cst\ L'/L)$. If $L, L' \to \infty$ keeping the ratio L'/L fixed, one gets for the partition function $\mathfrak{Z}$ an asymptotic behaviour

$$\mathfrak{Z} \simeq exp\ f\ .\ LL' \times Z\left(\frac{L'}{L}\right) \tag{1.1}$$

where f is the bulk free energy per unit surface, and Z contains the finite size corrections. f is not universal but Z is expected to be [1]. Using the additional property of conformal invariance, one can get a formal expression for Z. Indeed, it is the trace of the L' power of the transfer matrix, whose spectrum is determined (see Pr. Rittenberg Lecture) using the logarithmic mapping $w = L/2\pi\ log\ z$. It is known that the ground state scales as [2]

$$f_L = f + \frac{\pi c}{6L^2} \tag{1.2}$$

where c is the central charge, and the gaps as [2]

$$m_L = 2\pi\ x/L \tag{1.3}$$

associated to eigenstates of moment

$$k_L = 2\pi\ S/L \tag{1.4}$$

x,S being the dimension and spin of the operators of the theory ($x = h + \bar{h}$, $s = h - \bar{h}$). In the continuum limit, it will be useful to generalize the calculation of Z to an arbitrary torus described by two complex numbers ω_1, ω_2 and $\tau = \frac{\omega_1}{\omega_2} = \omega_R + i\omega_I$, $\omega_I > 0$ (fig. 1). Introducing $q = exp\ 2\pi i\ \tau$ one gets [3]

$$Z=(q\bar{q})^{-c/24} \sum_{all\ operators} q^h \bar{q}^{\bar{h}} = (q\bar{q})^{-c/24} \sum_{h\bar{h}} N_{h\bar{h}}\, \chi_h(q) \chi_{\bar{h}}(\bar{q}) \qquad (1.5)$$

In the last expression the sum is restricted to the primary fields, the contribution of their descendants being resummed in the characters (see Pr. Rittenberg Lecture)

$$\chi_h(q) = q^h \sum D_{h,N}\, q^N \qquad (1.6)$$

where D_N is the number of independent descendents at level N.

On the other hand it has been shown that consistent conformal theories ("minimal" ones) can be built which have a finite number of primary fields, for the central charges [4]

$$c = 1 - \frac{6(p-p')^2}{pp'} \qquad (1.7)$$

the dimensions of these fields being given by Kac formula [5]

$$h_{rs} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'} \qquad (1.8)$$

where $1 \leq r \leq p - 1$, $1 \leq s \leq p' - 1$. For such theories, the correlation functions and the whole operator algebra can be calculated [6].However the formalism in the plane does not tell the exact operator content, ie the number $N_{h\bar{h}}$ of fields of dimensions h, $\bar{h}$ being actually present in the theory [4]. This can be answered by turning to the genus 1 surface (ie the torus) and using the expression (1.5) for partition functions. Indeed it's clear that for a given torus, ω_1 and ω_2 are not uniquely defined and one could consider as well any

$$\begin{pmatrix}\omega'_1\\ \omega'_2\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, \quad abcd \in \mathbb{Z}, \quad ad - bc = 1 \tag{1.9}$$

We must thus demand that $Z(\tau)$ is modular invariant, ie invariant under the transformation $\tau \to (a\tau + b)/c\tau + d$, and this will turn out to be a very strong constraint [5].

It is of course sufficient to ensure invariance under the two generators

$$T : \tau \longrightarrow \tau + 1 \qquad\qquad S : \tau \longrightarrow -1/\tau \tag{1.10}$$

Under T, $q \to e^{2i\pi}q$ and by (1.6), $\chi_h(q) \to e^{2i\pi h}\,\chi_h(q)$. Hence invariance under T is satisfied if

$$N_{h\bar{h}} = 0 \quad \textit{for} \quad h - \bar{h} \notin \mathbb{Z} \tag{1.11}$$

ie only integer spin operators contribute to Z. This was expected since in the spectrum of the model with periodic boundary conditions on a strip, only integer momenta (in units of $\frac{2\pi}{L}$) appear. For studying the effect of S one needs the precise expression of the characters χ which has been obtained in [7]. Restricting ourselves for simplicity to the unitary series [8] $p = m+1$, $p' = m$, $m \geq 2$, one has

$$\chi_h = \chi_{rs} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{n=-\infty}^{\infty} \left\{ q^{[2nm(m+1)+(m+1)r-ms]^2/4m(m+1)} - (s \to -s) \right\} \tag{1.12}$$

Then, denoting $\tau' = -1/\tau$, Poisson formula gives

$$q'^{-c/24}\,\chi_h(q') = \sum_{h'} A_{hh'}\, q^{-c/24}\,\chi_{h'}(q) \tag{1.13}$$

A being the $\left(\frac{m(m-1)}{2}\right)^2$ matrix

$$A_{hh'} = A_{rs,r's'} = \left[\frac{8}{m(m+1)}\right]^{1/2} (-)^{(r+s)(r'+s')} \frac{\sin \pi rr'}{m} \frac{\sin \pi ss'}{m+1} \tag{1.14}$$

and the sum in (1.13) runs over $1 \leqslant s' \leqslant r' \leqslant m - 1$. A is symmetric and its square is equal to one (since $S^2 = 1$). Modular invariance translates thus into

$$A \,.\, N = N \,.\, A \tag{1.15}$$

which is a system of diophantine equations [3,9]. One shows easily that the choice $N_{hh} = \delta_{h,h}$ is always a solution. For $c = \frac{1}{2}$ $(m = 3)$ it's the only one and

$$Z = (q\bar{q})^{-1/48} \left(|\chi_0|^2 + |\chi_{1/16}|^2 + |\chi_{1/2}|^2\right) \tag{1.16}$$

which is identified as the Ising model partition function, matching the known exponents $x_\epsilon = 1$, $x_H = 1/8$ or using the exact solution [10].For $c = 4/5$ $(m = 5)$, two solutions are possible, either

$$Z = (q\bar{q})^{-1/30} \left\{|\chi_0|^2 + |\chi_{1/15}|^2 + |\chi_{2/5}|^2 + |\chi_{2/3}|^2 + |\chi_{7/5}|^2 + |\chi_3|^2\right\} \tag{1.17}$$

$$Z = (q\bar{q})^{-1/30} \left\{|\chi_0 + \chi_3|^2 + |\chi_{2/5} + \chi_{7/5}|^2 + 2|\chi_{1/15}|^2 + 2|\chi_{2/3}|^2\right\} \tag{1.18}$$

(1.17) is identified with the 3 state Potts model matching dimensions [8] *and* using the fact that the spin with $\chi_H = \frac{2}{15}$ msut be twice degenerate due to the Z_3 symmetry [3]. (1.18) corresponds to the tetracritical Ising model [3].

Finding all the solutions of (1.12) is a difficult task which has been achieved only recently. Surprisingly it turns out that the modular invariants can be labelled by a pair of simply laced algebra, one of which is always of A type [11]. We refer the reader to the literature and quote only the results in table 1. In addition to two infinite series (A,A) and (D,A) generalyzing (1.17) and (1.18) one notes the appearance of three exceptional cases $(E_6 - E_7 - E_8, A)$. Various physical realizations can been proposed for these partition functions. Generic models are obtained by building Restricted Solid on Solid Models, the heights of which are located on the Dynkin diagram of the underlying algebra [12] (Fig. 2).

It has to be noticed that the invariants of table 1 generalize readily to non unitary theories. A simple example is obtained with $p = 5$, $p' = 2$, $c = -22/5$ corresponding to the Lee Yang singularity

$$Z = (q\bar{q})^{11/60}\left(|\chi_0|^2 + |\chi_{-1/5}|^2\right) \tag{1.19}$$

A *negative* dimension appears then in the Kac table (1.8), $h_{12} = -1/5$ for (1.19), and the formula (1.2)-(1.4) must be corrected. Indeed the small q behaviour of Z is given now by $Z \sim (q\bar{q})^{-c/24}(q\bar{q})^{h_{12}}$ so (1.2) reads instead $f_L = f + Tc'/6L^2$ where

$$c' = c - 24h_{12} = \frac{4}{5} \tag{1.20}$$

Similarly the first gap is $m_L = 2\pi x'/L$ where $x' = |x| - 2/5$. The presence of negative dimensions is general in the non unitary case, all invariants containing [11] the dimension $h = \dfrac{1-(p-p')^2}{4pp'}$. Precise rules for the modification of the transfer matrix spectrum have been given in [13].

Attempts have been made in [14] to construct also invariants associated to central charges (1.7), but with dimensions outside the minimal grid.

2. Coulomb Gas Construction of the Partition Functions on a Torus

The results of the previous section have been obtained using very general methods, but they have the drawback of being rather formal. For more practical applications in statistical mechanics, one would like to have a method for establishing connections between the abstract partition function (1.5) associated to a conformal field theory and models formulated on the lattice. In the simplest cases it is enough to proceed by inspection, like for the Ising model [8] or the Lee & Yang singularity [15] where there is only one modular invariant reproducing results known by other methods. For the three state Potts model, we have seen in section 1 that there are already two candidates, and it is an argument about the symmetry of the model which allows the correct identification [3]. For more complicated models the procedure obviously becomes hopeless, and numerical calculations can then prove very useful.

To derive critical properties of 2D models, a somehow different field theoretic technique had been developed in the past, the so called Coulomb Gas technique [16]. In this approach, one first reformulates the model to be studied (usually as a solid on solid interface model) in such a way that it can be argued to renormalize at criticality onto a free Gaussian bosonic theory with action

$$\mathcal{A} = \frac{g}{4\pi} \int |\vec{\nabla}\varphi|^2 \, d^2x \qquad (2.1)$$

This crucial property cannot in general be rigorously established, but it can nevertheless be checked in various ways, performing for instance approximate renormalization group calculations. The coupling constant of the associated Gaussian model g has also to be determined, usually using some extra information from the eight-vertex model solution [17]. Physical observables can then be translated in Gaussian terms [18], as combination of electric or spin wave (ie exponentials of the field *exp* ie φ) and magnetic or vortex operators (ie operators creating a branch cut with discontinuity of $2\pi m$ for the field φ). Their dimensions are readily calculated [18]

$$x_{em} = \frac{e^2}{2g} + \frac{gm^2}{2} \quad , \quad S_{em} = em \qquad (2.2)$$

This approach which was pioneered by José et al. [18] for the XY model has been successfully used in a number of cases, such as the Q-state Potts or O(n) [19] models providing, under the above mentioned renormalization flow assumptions, an exact determination of the main exponents. It turns out, as demonstrated in [10], that one can in fact use this Coulomb gas mapping to derive also the partition functions on a torus, and then the whole operator content of these models.

The partition function for the free field (2.1) with periodic boundary conditions has been calculated in many different contexts already. A dzeta regularization [9] of the determinant of the Laplacian gives

$$Z_0 = \sqrt{\frac{g}{\tau_I}} \frac{1}{|\eta(q)|^2} \qquad (2.3)$$

where τ, η and q are the same as in section 1. The dependance on g in (2.3) comes from the zero mode, the subtraction of which prevents a rescaling of φ. Z_0 has a small q behaviour corresponding to c = 1 which is known to be the central charge of a free bosonic field. The complete expansion in powers of q, $\bar{q}$ shows also the presence of a marginal operator (with x = 2) identified as $\partial_z \partial_{\bar{z}} \varphi$, but none of the dimensions (2.2) is observed.

In mapping a lattice model onto a free field, some other terms are usually generated due to the boundary conditions. We can discuss for instance the case of the XY model whose lattice action is

$$\mathcal{A} = -\frac{1}{T}\sum_{\langle ij\rangle}[\cos(\varphi_i-\varphi_j)-1] \tag{2.4}$$

At very low temperature all spins are almost parallel, and it is justified to expand (2.4) into

$$\mathcal{A} = \frac{1}{2T}\sum_{\langle ij\rangle}(\varphi_i-\varphi_j)^2 \qquad (T \longrightarrow 0) \tag{2.5}$$

which becomes (2.1) in the continuum limit with

$$g \simeq \frac{2\pi}{T} \qquad (T \longrightarrow 0) \tag{2.6}$$

At higher temperatures, the approximation (2.5) fails, in particular due to the presence of vortices, ie configurations where local angle variations are still small, but globally φ can vary by multiples of 2π around one point [18]. It is well known however that for T smaller than a critical value T_c, these vortices are irrelevant. Under renormalization transformation, vortices of opposite charges come closer and ultimately annihilate each other [18], giving still rise to a fixed point action (2.5), but with a renormalized temperature [21]

$$g \simeq \frac{2\pi}{T}\left(1 - \frac{T}{4} - \frac{T^2}{96} + \ldots\right) \tag{2.7}$$

The end of this critical line is characterized by the marginality of the lowest dimension vortex operator : $g(T_c)\ (m-1)^2/2 = 2$, thus

$$g(T_c) = 4 \tag{2.8}$$

For $T > T_c$, the vertices do not disappear under renormalization, and the model is no more critical.

Now on a torus special care must be taken of boundary conditions. Indeed the periodic geometry allows configurations where φ varies along the generators of the torus and this topological property remains of course unchanged in a renormalization. Thus for $T < T_c$, the XY model in the continuum limit is described by the free field (2.1) where φ is not periodic but can have shifts

$$\begin{cases} \varphi(z+1,\bar{z}+1) = \varphi(z,\bar{z}) + 2\pi m \\ \varphi(z+\tau,\bar{z}+\bar{\tau}) = \varphi(z,\bar{z}) + 2\pi m' \end{cases} \tag{2.9}$$

for any $m, m' \in \mathbb{Z}$. For boundary conditions (2.9), the partition function is readily calculated introducing the classical field such that $\Delta \varphi_{cl} = 0$

$$\varphi_{cl} = \frac{m\bar{\tau} - m'}{\tau_I} z + cc \tag{2.10}$$

and

$$Z_{mm'}(g) = Z_0(g) \; exp - \pi g \frac{|m-m'\tau|^2}{\tau_I} \tag{2.11}$$

It has the modular transformation properties

$$Z_{mm'}\left(\frac{a\tau + b}{c\tau + d}\right) = Z_{am+bm',cm+dm'}(\tau) \tag{2.12}$$

expected from its definition. The complete XY model partition function is obtained by summing over m and m' [20]

$$\mathcal{Z}_{XY} \rightarrow Z_c(g) = \sum_{m,m' \in \mathbb{Z}} Z_{mm'}(g) = \frac{1}{|\eta|^2} \sum_{em \in \mathbb{Z}} q^{\frac{1}{4}\left(\frac{e}{\sqrt{g}}+m\sqrt{g}\right)^2} \bar{q}^{\frac{1}{4}\left(\frac{e}{\sqrt{g}} - m\sqrt{g}\right)^2} \tag{2.13}$$

this last equality being obtained after a Poisson transformation. It has still $c = 1$, and presents now the operator content expected form (2.2). Expression (2.13) will be called in the following a Coulombic partition function; it is instructive to notice that this object appeared also in the past in the context of string theory [22] as the partition function of a free field compactified on a circle of radius $R \sim \sqrt{g}$. Similar formulas can be obtained for describing the F-model, or the Ashkin Teller model critical lines. In the latter case however one must add the contribution of sectors where the field φ has antiperiodic boundary conditions, corresponding to the identification of the spin as the twist operator (creating a branch cut with $\varphi \longrightarrow -\varphi$) with dimension $x_H = \frac{1}{8}$ independent of g, we refer the redear to references [22].

To reproduce the partition functions of models with $c < 1$, an additional ingredient must be added. generalizing somehow the charge at infinity of *ref.*[6] to the torus. We shall discuss that in some detail for the O(n) model on the honeycomb lattice defined by

$$\mathcal{Z}_n = \int \prod_i d\vec{S}_i \prod_{\langle jk \rangle} \left(1 + \frac{1}{T} \vec{S}_j . \vec{S}_k\right) \tag{2.14}$$

$\vec{S}$ being a n-component spin with $|\vec{S}|^2 = n$. (2.14) can be analytically continued to $n \in \mathbb{R}$ using a high temperature expansion

$$\mathcal{Z}_n = \sum_{\text{graphs}} \left(\frac{1}{T}\right)^{\mathcal{N}_B} n^{\mathcal{N}_P} \tag{2.15}$$

In (2.15) the graphs are formed by $\mathcal{N}_P$ non intersecting self avoiding loops (or polygons) of total length $\mathcal{N}_B$. The model is known to be critical for $n \in [-2,2]$. It can be transformed into a SOS model [23] by introducing height variables φ on the centers of the hexagons. An arbitrarily oriented polygon corresponds then to a wall between two regions of constant height, with a step $\pm \varphi_0$, the highest φ being on the left of each arrow. The Boltzmann weight consists of a factor $\frac{1}{T}$ for each bond, times $e^{iv}(e^{-iv})$ for each left (right) turn. Then, since the difference between the numbers of left and right turns for a polygon on the honeycomb lattice in the plane is $n_\ell - n_r = \pm 6$, one has $\mathcal{Z}_n = \mathcal{Z}_{SOS}$ if $n = 2 \cos 6v$. At criticality, this SOS model is argued to renormalize onto the free field (2.1), and for the choice $\varphi_0 = \pi$ one gets [19]

$$n = -2 \cos 4\pi g \qquad\qquad g \in \left[\frac{1}{4},\frac{1}{2}\right] \tag{2 16}$$

On a torus however, $\mathcal{Z}_n \neq \mathcal{Z}_{SOS}$ since polygons which wrap around it have $n_\ell = n_r$. The necessary correction is easy to obtain in the strip limit where one has only loops wraping around w_1. To give to these loops the weight n instead of 2, one introduces charges $+(-)e_0$ at $+(-)\infty$. Then the Boltzmann weight of a configuration has an additional $exp\ i\ e_0 [\varphi_\infty - \varphi_{-\infty}]$, each non contractbile loop contributing to it by $exp \pm 2i\ \pi\ e_0$ depending on its orientation. Summing over all configurations gives then the new weight $2 \cos 2\pi\ e_0$ which is the desired result provided

$$n = 2 \cos 2\pi\, e_0 \tag{2.17}$$

The ground-state is modified by these charges, and accordingly the central charge becomes [13]

$$c = 1 - 24h_{e_c} = 1 - \frac{6(2g - 1/2)^2}{g} \tag{2.18}$$

In the Ising case for instance, $n = 1$ gives $g = \frac{1}{3}$, $e_0 = \frac{1}{6}$ and $c = \frac{1}{2}$ as expected. Result (2.18) was first obtained in [2] using a somehow different language.

The situation in the case of the torus is more difficult since there is no "infinity" anymore where to put the charges. First, it is clear that since the loops-heights correspondence is only local, the presence of non contractible loops leads to shifts of φ similar to (2.9). For a given configuration these read (fig. 3)

$$\begin{cases} \delta\varphi = 2\pi n \sum \epsilon_i \\ \delta'\varphi = 2\pi n' \sum \epsilon_i \end{cases} \tag{2.19}$$

To write (2.19) we have used the property that if two unoriented polygons $\mathcal{P}$ and $\mathcal{P}'$ coexist on the torus, then they are homotopic. Thus there is one species of polygons $\mathcal{P}$ *only* at a given time defining two basic shifts $(2\pi n, 2\pi n')$ and $(\delta\varphi, \delta'\varphi)$ is obtained by summing over all the polygons with their orientation $\epsilon_i = \pm 1$. Now the crucial observation is that n and n' are indeed prime together [20]. Thus $\sum \epsilon_i = \pm \frac{|\delta\varphi|}{2\pi} \wedge \frac{|\delta'\varphi|}{2\pi}$ where $\wedge$ denotes the greatest common divisor, and

$$\cos\left[2\pi\, e_0 \sum \epsilon_i\right] = \cos\left[2\pi\, e_0\, \frac{|\delta\varphi|}{2\pi} \wedge \frac{|\delta'\varphi|}{2\pi}\right] \tag{2.20}$$

Then, exactly as in the case of the strip, summing over all ϵ_i gives to each of these non contractible loop the weight $n - 2\cos 2\pi e_0$. One gets finally [20]

$$\mathfrak{Z}_n \to \sum_{m,m' \in \mathbb{Z}} Z_{mm'}(g) \cos[2\pi e_0 \; m \wedge m'] \tag{2.21}$$

an expression which is clearly modular invariant since $m \wedge m'$ is. (2.21) is interesting for any n, but it presents some pathologies for $n \notin \mathbb{N}$. In the case of the Ising model, the sum (2.21) can be performed decomposing it on classes of congruence of mm' *mod* 6, and one gets in the end

$$\mathfrak{Z}_{n=1} \to \frac{1}{2}\left[Z_c(12) - Z_c\left(\frac{4}{3}\right)\right] \tag{2.22}$$

which can be shown to be equal to (1.16). A completely similar calculation reproduces as well the Q-state Potts model partition functions, both in the critical and tricritical regime.

This shows clearly how the Coulomb gas technique can provide a link between lattice models and their critical properties such as the partition functions. Of course, it has the drawback of being much less general than conformal invariance, since a new mapping has to be established for every new model under consideration before its properties can be calculated. The two approaches must however be more deeply related. Indeed Dotsenko and Fateev [6] have shown for instance that the construction of four point functions in a free field supplemented by a charge at infinity and screening operators led naturally to the kac formula for the dimensions of the fields, and to the Feigen and Fuchs integral representation for the solutions of the BPZ equations. It is also quite puzzling to notice that *all* minimal partition functions can be written [20] as

$$Z(ADE, A_{p-1}) = \sum_{r} \sum_{mm' \in \mathbb{Z}} Z_{mm'}\left(g = \frac{p}{p'}\right) \cos\left[\frac{2\pi}{p'} n \; m \wedge m'\right]$$

where n belongs to the exponent of the associated ADE algebra (2.23). (2.23) is a generalization of (2.21) involving several charges.

3. Correlation Functions on a Torus

To complete the understanding of critical theories on a torus one would like to calculate also the correlation functions. These have first an interest on their own, since their knowledge provides a systematic (perturbative) route to the study of deviation from criticality [23]. Also, correlation functions on a torus can be shown to be related to partition functions on higher genus surfaces after a proper "pinching" [14].

In principle, the correlators on the torus can be calculated using the structure constants of the theory and the logarithmic mapping which give the necessary transfer matrix elements. The general formula however seem quite untractable. Even the two point function which has the simplest form in the plane

$$\left\langle A(z,\bar{z})\ A(0,0)\right\rangle = \frac{1}{z^{2h}\bar{z}^{2\bar{h}}} \tag{3.1}$$

must have a rather complicated expression on the torus. If the short distance expansion reads

$$A(z,\bar{z})\ A(0,0) = \sum_{\Delta,\bar{\Delta}} z^{\Delta-2h}\ \bar{z}^{\bar{\Delta}-2\bar{h}}\ C_{AAB}\ B_{\Delta\bar{\Delta}}(0,0) \tag{3.2}$$

one must have now

$$\left\langle A(z,\bar{z})\ A(0,0)\right\rangle = \sum_{\Delta,\bar{\Delta}} z^{\Delta-2h}\ \bar{z}^{\bar{\Delta}-2\bar{h}}\ C_{AAB}\ \left\langle B_{\Delta\bar{\Delta}}\right\rangle \tag{3.3}$$

all the mean values $\langle B\rangle$ being a priori non zero.

One would like to find a more compact method of calculation, generalizing for instance the approaches of [6], but it seems difficult to implement the notion of charge at infinity and screening operators here. One can write nevertheless the ward identity [15]

$$\begin{aligned}\langle T(z)\ A_1(1)\ldots A_n(n)\rangle - \langle T\rangle\langle A_1(1) - A_n(n)\rangle \\ = \sum_{i=1}^{n}\left[-\ h_i\ \left(\frac{\theta_1'}{\theta_1}\right)'\ (z-z_i) + \frac{\theta_1'}{\theta_1}\cdot (z-z_i)\ \partial_{z_i}\right] \\ \langle A_1(1)\ldots A_n(n)\rangle \\ +\ 2i\pi\ \partial_\tau\ \langle A_1(1)\ldots A_n(n)\rangle\end{aligned} \tag{3.4}$$

where

$$\langle T \rangle = 2i\pi\, \partial_\tau\, Log\, Z \tag{3.5}$$

and θ_1 is the Jacobi theta function (see Appendix A), generalizing the expression in the plane [4] to the doubly periodic case, with the appearance of a new term associated to the variation of modular ratio in the change of coordinates. For degenerate fields [4], differential equations follow. In the simplest case of degenerescence at level 2 one has

$$\left\{ \frac{3}{2(2h+1)} \partial_z^2 - 2h\, \eta_1 - \sum_{i=1}^{n} \left[-h_i \left(\frac{\theta_1'}{\theta_1} \right)' (z-z_i) + \frac{\theta_1'}{\theta_1} (z-z_i)\, \partial_{z_i} \right] -2i\pi\, \partial\tau \right\} Z \left\langle A(z,\bar{z})\; A_1(1) \ldots A_n(n) \right\rangle = 0 \tag{3.6}$$

where

$$\eta_1 = -\frac{1}{6} \frac{\theta_1'''(0)}{\theta_1'(0)}$$

These equations seem unfortunately quite difficult to solve in general.

Up to now, progress has been made for the free bosonic theories like the Ashkin Teller model [26], and for the Ising model which is equivalent to a free Majarana fermion theory [27] with action

$$\mathcal{A} = \frac{1}{2\pi} \int (\Psi\, \bar{\partial}\, \Psi + \tilde{\Psi}\, \partial\, \tilde{\Psi})\, d^2x \tag{3.7}$$

We shall now discuss this case in some detail.

As a consequence of the Jordan-Wigner transformation [10] the fermion $(\Psi,\tilde{\Psi})$ must be assigned periodic (P) or antiperiodic (AP) boundary conditions along the periods ω_1 and ω_2 of the torus. This gives rise to four sectors labelled by $\nu = 1,2,3,4$ for PP, PA, AA, AP. The partition function has been evaluated by dzeta-regularization [9] and reads:

$$Z = \sum_{\nu=2}^{4} \frac{|\theta_\nu(0|\tau)|}{2|\eta(\tau)|} = \sum_{\nu}{}' Z_\nu \qquad (3.8)$$

where θ_ν are Jacobi theta functions (see Appendix A), $\eta = \left(\frac{\theta_1'(0)}{2\pi}\right)^{1/3}$ is Dedekind's function. The $\nu = 1$ contribution vanishes in (3.8), due to zero mode of the Laplacian with doubly periodic boundary conditions. The expression (3.8) reproduces (1.16).

One can easily write the fermion propagator $\langle\Psi(z)\ \Psi(w)\rangle_\nu$ in each sector ν. Using the short distance expansion:

$$\Psi(z)\ \Psi(w) = \frac{1}{z-w} + O(1) \qquad (3.9)$$

and the quasi-periodicity properties of the theta functions (see Appendix A) we conclude that for $\nu \neq 1$: $\langle\Psi(z)\ \Psi(w)\rangle_\nu \times \frac{\theta_1(z-w)}{\theta_\nu(z-w)}$ is doubly periodic and has *at most* one pole, the unique zero of θ_ν on the torus (see Appendix A). By a well-known theorem on elliptic functions, it must be a constant, fixed by the short distance limit.

$$\langle\Psi(z)\ \Psi(w)\rangle_\nu = \frac{\theta_\nu(z-w)}{\theta_\nu(0)} \cdot \frac{\theta_1'(0)}{\theta_1(z-w)}, \qquad \nu \neq 1 \qquad (3.10)$$

so the fermion propagator reads:

$$\langle\Psi(z)\ \Psi(w)\rangle = \frac{\sum_{\nu}' Z_\nu \langle\Psi(z)\ \Psi(w)\rangle_\nu}{\sum_{\nu}' Z_\nu} \qquad (3.11)$$

Using (3.10) we deduce the two point energy correlation

$$\left\langle \varepsilon(z,\bar{z})\ \varepsilon(0)\right\rangle = \frac{\sum_{\nu}' |\theta_\nu(z)|^2/|\theta_\nu(0)|}{\sum_{\nu}' |\theta_\nu(0)|} \times \left|\frac{\theta_1'(0)}{\theta_1(z)}\right|^2 \tag{3.12}$$

It should be noted that in (3.11), (3.12) there is no contribution of the $\nu = 1$ sector. This can be justified [27] but is not obvious a priori, even if Z_1 itself vanishes. Indeed one can show for instance that $\langle\varepsilon\rangle \neq 0$, and that *only* the $\nu = 1$ sector contributes in this case [13]

$$\langle\varepsilon\rangle = \frac{Z_1\langle\varepsilon\rangle_1}{Z} = \frac{2\pi|\eta|^3}{\sum_{\nu}' |\theta_\nu(0)|} \tag{3.13}$$

2n points correlation functions can be evaluated easily using (3.11) and Wick's theorem: they are simply modulus squares of Pfaffians of the propagator in each sector [27]. The case of 2n+1 points is more difficult because of the role of the $\nu = 1$ sector.

The calculation of spin correlators is different since due to the Jordan Wigner transformation, the spin σ is not local in terms of Ψ and $\bar{\Psi}$ and one has

$$\begin{aligned} \Psi(z)\ \sigma(w,\bar{w}) &\sim \frac{1}{(z-w)^{1/2}}\,\mu(w,\bar{w}) \\ \bar{\Psi}(\bar{z})\ \sigma(w,\bar{w}) &\sim \frac{1}{(\bar{z}-\bar{w})^{1/2}}\,\mu(w,\bar{w}) \end{aligned} \tag{3.14}$$

where μ is the disorder operator dual to the original spin σ. The non-integer power $\frac{1}{2}$ expresses the monodromy property [4] of insertions of spins in Ψ or $\bar{\Psi}$ correlators, that makes them change sign when the σ -argument describes a loop around Ψ's. Expressions (3.9) and (3.14) and periodicity conditions can be used to determine completely the auxiliairy function

$$G_\nu = \frac{\left\langle \Psi(z)\ \Psi(w)\ \sigma\left(z_1,\bar{z}_1\right)\ \sigma\left(z_2,\bar{z}_2\right)\right\rangle_\nu}{\left\langle \sigma\left(z_1,\bar{z}_1\right)\ \sigma\left(z_2,\bar{z}_2\right)\right\rangle_\nu} \tag{3.15}$$

Using again properties of θ -functions, we can exhibit the following candidate for G_ν:

$$H_\nu = \frac{1}{2}\left[\frac{\theta_1'(0)}{\theta_1(z-w)}\right]^2 \times \left[\left(\frac{\theta_1(z-z_1)\theta(w-z_2)}{\theta_1(z-z_2)\theta(w-z_1)}\right)^{1/2} \frac{\theta_\nu\left(z-w+\frac{z_1-z_2}{2}\right)}{\theta_\nu\left(\frac{z_1-z_2}{2}\right)} + (z\leftrightarrow w)\right] \tag{3.16}$$

Then, for $\alpha_\nu(z) = \dfrac{\theta_\nu\left(z-\frac{z_1+z_2}{2}\right)}{[\theta_1(z-z_1)\ \theta_1(z-z_2)]^{1/2}}$ the function $(G_\nu-H_\nu)/\alpha_\nu(z)\ \alpha_\nu(w)$ is again elliptic in z and w and has at most one pole in z and w, so it is a constant, vanishing by antisymmetry in the exchange of z and w; thus $G_\nu = H_\nu$.

From the expression relating the stress tensor T(z) and the fermion:

$$T(z) = \lim_{z\to w} -\frac{1}{2}\left[\Psi(z)\ \partial_w\ \Psi(w) - \partial_z\ \Psi(z)\ \Psi(\omega)\right] + \frac{1}{(z-w)^2} \tag{3.17}$$

we get now the insertion of T in the spin correlator, generating at short distances first the conformal dimension $h = 1/16$ of operator σ and second a derivative of $\langle\sigma\sigma\rangle$, for which we have an explicit value:

$$\frac{\left\langle T(z)\sigma\left(z_1,\bar{z}_1\right)\sigma\left(z_2,\bar{z}_2\right)\right\rangle_\nu}{\left\langle \sigma\left(z_1,\bar{z}_1\right)\sigma\left(z_2,\bar{z}_2\right)\right\rangle_\nu} = \frac{1}{(z-z_1)^2}\,h_\sigma + \frac{1}{(z-z_1)}\,\partial_{z_1} \mathit{Log}\left\langle \sigma\left(z_1,\bar{z}_1\right)\sigma\left(z_2,\bar{z}_2\right)\right\rangle_\nu + \mathit{reg\ terms} \tag{3.18}$$

We get a similar differential equation for the antianalytic part of $\langle\sigma\sigma\rangle_\nu$ by replacing Ψ with $\tilde{\Psi}$ in (3.15). (3.18) can be integrated to give

$$Z_\nu \left\langle \sigma\left(z_1, \bar{z}_1\right) \sigma\left(z_2, \bar{z}_2\right)\right\rangle_\nu = \frac{(2\pi)^{1/3}}{2} \left|\theta_1'(0)\right|^{-1/2} \frac{\left|\theta_\nu\left(\frac{z_1 - z_2}{2}\right)\right|}{|\theta_1(z_1 - z_2)|^{1/4}} \quad (3.19)$$

Periodicity conditions impose to sum (3.9) over the four sectors, and finally

$$\left\langle\sigma(z,\bar{z})\sigma(0,0)\right\rangle = \frac{\sum_{\nu=1}^{4} Z_\nu \left\langle\sigma(z,\bar{z})\sigma(0,0)\right\rangle_\nu}{\sum_\nu{}' Z_\nu} = \frac{\sum_{\nu=1}^{4} |\theta_\nu(z/2)|}{\sum_\nu{}' |\theta_\nu(0)|} \times \left|\frac{\theta_1'(0)}{\theta_1(z)}\right|^{1/4} \quad (3.20)$$

Physically the insertion of σ's may be viewed as creating a branch cut joining z and 0 along which Ψ has to be antiperiodic. A translation $z \rightarrow z+1$ or $z \rightarrow z+\tau$ lets the cut wrap around the torus and changes the boundary conditions for the fermion, exchanging the contributions of different ν sectors, including $\nu = 1$ (fig.4). (3.12) as well as (3.20) satisfy modular covariance in the following sense:

$$\langle A(z)\ A(w)\rangle(\tau+1) = \langle A(z)\ A(w)\rangle(\tau)$$
$$\langle A(z/\tau)\ A(w/\tau)\rangle(-1/\tau) = z^{2h}\ \bar{z}^{2\bar{h}}\ \langle A(z)\ A(w)\rangle(\tau) \quad (3.21)$$

One checks the short distance expansion of (3.20) against (3.3) and (3.13)

$$\left\langle\sigma(z,\bar{z})\ \sigma(0,0)\right\rangle = \frac{1}{|z|^{1/4}} + C_{\sigma\sigma\varepsilon}\ |z|^{3/4}\ \langle\varepsilon\rangle + \ldots \quad (3.22)$$

where [4] $C_{\sigma\sigma\varepsilon} = 1/2$. Finally (3.12), (3.20) satisfy the expected differential equation (3.6).

The preceding approach was used also in the context of string theory on orbifolds [28]. For higher spin correlators it turns out to be more convenient to use a bosonization technique where one computes squares $\langle\sigma \ldots \sigma\rangle_\nu^2$ of correlation functions in each fermionic sector via a free bosonic model [27]. The general result is

$$Z_\nu^2\langle \sigma(1)\ldots\sigma(2n)\rangle = \frac{1}{|\eta|^2}\sum_{\varepsilon_i=\pm1,\sum\varepsilon_i=0}\left|\theta_\nu\left(\frac{\sum\varepsilon_i z_i}{2}\right)\right|^2\prod_{i<j}\left|\frac{\theta_1(z_i-z_j)}{\theta_1'(0)}\right|^{\frac{\varepsilon_i\varepsilon_j}{2}} \qquad (3.23)$$

(3.23) can be checked numerically for $n = 2,4$. Indeed in [29] Burkhardt and Derrida have considered a lattice Ising model on squares $N\times N$ and calculated using a transfer matrix technique the first moments of the magnetization $M_{2n} = \left\langle\left(\sum\sigma_i\right)^{2n}\right\rangle$. They have obtained in particular the renormalized coupling constant $V(N) = M_4/M_2^2$ for $N \leqslant 14$. In the limit $N \to \infty$ the values converge to a universal constant evaluated as $V = 1.1670\pm0.0015$. Now one can give an expression of M_2, M_4 using (3.23) since $M_{2n} = \int_{square} d^2x_1\ldots d^2x_{2n}\,\langle\sigma(1)\ldots\sigma(2n)\rangle$. The integrals cannot be performed analytically but estimated numerically, giving $V = 1.168\pm0.005$ in agreement with the above value.

Naturally these calculations generalize to mixed correlators of the most general type involving σ, μ, ε [27].

APPENDIX A

Jacobi theta functions

ν =	1	2	3	4
(α,β) =	$(1/2,1/2)$	$(1/2,0)$	$(0,0)$	$(0,1/2)$

$$q = e^{2i\pi\tau}$$

$$\theta_\nu(z|\tau) = \sum_{n\in z} q^{\frac{1}{2}(n+\alpha)^2} e^{2i\pi(n+\alpha)(z+\beta)}$$

Quasi-periodicity relations

	θ_1	θ_2	θ_3	θ_4
$z \to z+1$	$-\theta_1$	$-\theta_2$	θ_3	θ_4
$z \to z+\tau$	$-A\theta_1$	$A\theta_2$	$A\theta_3$	$-A\theta_4$

$$A(z|\tau) = e^{-2i\pi\left(z+\frac{\tau}{2}\right)}$$

Zeroes

	θ_1	θ_2	θ_3	θ_4
z	$m+n\tau$	$m+\frac{1}{2}+n\tau$	$m+\frac{1}{2}+\left(n+\frac{1}{2}\right)\tau$	$m+\left(n+\frac{1}{2}\right)\tau$

Modular transformations

	θ_1	θ_2	θ_3	θ_4
$\tau \to \tau+1$	$e^{i\frac{\pi}{4}}\theta_1$	$e^{i\frac{\pi}{4}}\theta_2$	θ_4	θ_3
$\tau \to -\frac{1}{\tau};\ z \to \frac{z}{\tau}$	$-iB\theta_1$	$B\theta_4$	$B\theta_3$	$B\theta_2$

$$B(z|\tau) = \sqrt{-i\tau}\, e^{\frac{i\pi z^2}{\tau}}$$

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FIGURE CAPTIONS

Figure 1: In the continuum limit the torus will be described by its modular ratio $\tau = \frac{\omega_2}{\omega_1}$. Without loss of generality ω_1 can be taken equal to 1.

Figure 2: The unitary models of the ADE classification can be interpreted as RSOS models the heights of which take values on the associated Dynkin diagram.

Figure 3: A typical configuration with two polygons ont he torus $\varepsilon_1 = \varepsilon_2 = 1$; $n_1 = n_2 = 1$.

Figure 4: Translating the argument in the $\sigma\sigma$ correlation maps a frustration line around the torus and changes thus the fermionic sector.

TABLE 1

List of known partition functions in terms of conformal characters. The unitary series corresponds to $p' = m+1$, $p = m$ or $p = m+1$, $p' = m$, $m = 3,4,\ldots$.

	$\frac{1}{2}\sum_{r=1}^{p'-1}\sum_{s=1}^{p-1}\lvert\chi_{rs}\rvert^2$	$(A_{p'-1}, A_{p-1})$
$p' = 4\rho+2$, $\rho\geq 1$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{\substack{r\ \mathrm{odd}=1\\ r\neq 2\rho+1}}^{4\rho+1}\lvert\chi_{rs}\rvert^2 + 2\lvert\chi_{2\rho+1\,s}\rvert^2 + \sum_{r\ \mathrm{odd}=1}^{2\rho-1}(\chi_{rs}\chi^*_{p'-r\,s}+c.c.)\right\}$	$(D_{2\rho+2}, A_{p-1})$
$p' = 4\rho$, $\rho\geq 2$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\sum_{r\ \mathrm{odd}=1}^{4\rho-1}\lvert\chi_{rs}\rvert^2 + \lvert\chi_{2\rho\,s}\rvert^2 + \sum_{r\ \mathrm{even}=1}^{2\rho-2}(\chi_{rs}\chi^*_{p'-r\,s}+c.c.)\right\}$	$(D_{2\rho+1}, A_{p-1})$
$p' = 12$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\lvert\chi_{1s}+\chi_{7s}\rvert^2 + \lvert\chi_{4s}+\chi_{8s}\rvert^2 + \lvert\chi_{5s}+\chi_{11s}\rvert^2\right\}$	(E_6, A_{p-1})
$p' = 18$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\lvert\chi_{1s}+\chi_{17s}\rvert^2 + \lvert\chi_{5s}+\chi_{13s}\rvert^2 + \lvert\chi_{7s}+\chi_{11s}\rvert^2 + \lvert\chi_{9s}\rvert^2 + [(\chi_{3s}+\chi_{15s})\chi^*_{9s}+c.c.]\right\}$	(E_7, A_{p-1})
$p' = 30$	$\frac{1}{2}\sum_{s=1}^{p-1}\left\{\lvert\chi_{1s}+\chi_{11s}+\chi_{19s}+\chi_{29s}\rvert^2 + \lvert\chi_{7s}+\chi_{13s}+\chi_{17s}+\chi_{23s}\rvert^2\right\}$	(E_8, A_{p-1})

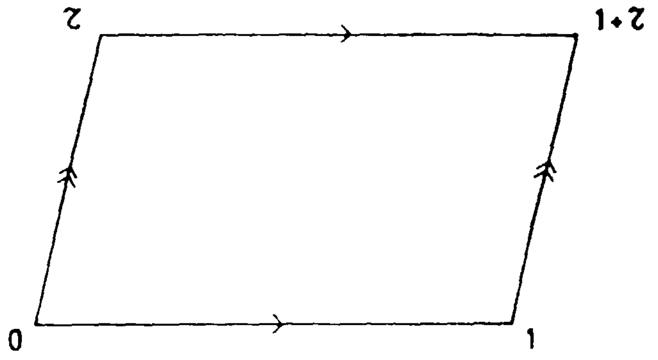


Figure 1

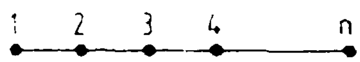
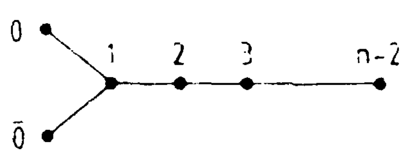
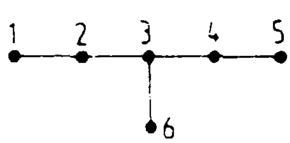
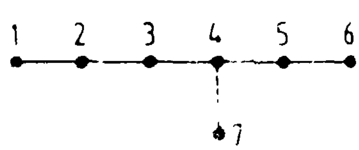
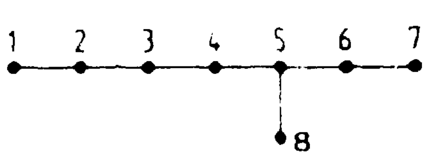
Name of the algebra	Diagram	Coexeter number	Exponent
A_n		n+1	1, 2, ... n
D_n		2(n-1)	1, 3, ..., 2n-3, n-1
E_6		12	1, 4, 5, 7, 8, 11
E_7		18	1, 5, 7, 9, 11, 13, 17
E_8		30	1, 7, 11, 13, 17, 19, 23, 29

Figure 2

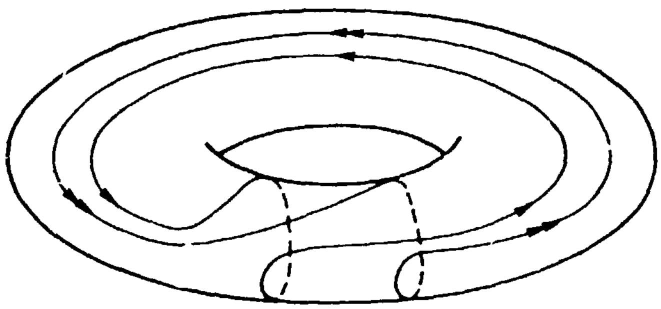


Figure 2

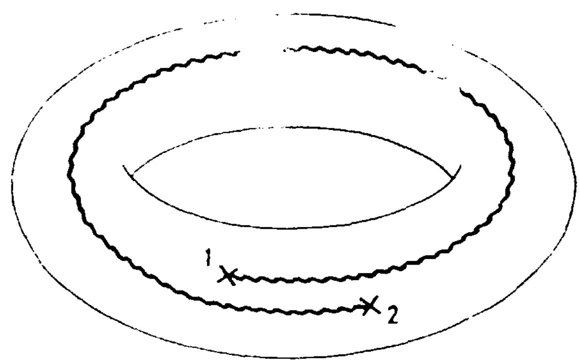


Figure 4