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THE COULOMB SYSTEMS DISTORTED AT SHORT DISTANCES IN ATOMIC AND NUCLEAR PHYSICS

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In systems bound by the Coulomb interaction distorted at short distances there may appear, under certain conditions, a rearrangement of atomic spectrum (or the Zel'dovich effect). Specific features of this effect are discussed for states with an arbitrary angular momentum ℓ (both with and without the absorption). The equation is studied which connects nuclear level shifts with the low-energy scattering parameters a_ℓ, r_ℓ. The conditions have been found under which the rearrangement of spectrum is replaced by oscillations of atomic levels. The Coulomb renormalization of scattering lengths and that of effective ranges is discussed. Some manifestations of the Zel'dovich effect in the physics of hadronic atoms and mesomolecules are considered.

Fig. - 9, ref. -49

1. In the recent years there has been a growing interest both experimental and theoretical towards research into systems bound by the Coulomb potential distorted at short ($r \sim r_0 \ll a_B$) distances[1-11]. The question of the structure of the energy spectrum in such systems arises in many physical applications [1]. The disturbing potential $V_s(r)$ is not assumed to be weak and, as a rule, is deeper than the Coulomb potential $V_c(r) = z_1 z_2 e^2/r$. In particular, $V_s(r)$ can possess its own bound states, or virtual and quasistationary states close to zero energy. We suppose only that the potential $V_s(r)$ to be short-ranged, i.e. $r_0 \ll a_B$, where r_0 is a characteristic radius of $V_s(r)$ and a_B is the Bohr radius. In this case the energy spectrum can be expressed only through the low-energy parameters, i.e. the Coulomb modified nuclear scattering length $a_l^{(cs)}$ and effective range $r_l^{(cs)}$, and is defined by the equation[2-6]

$$\left\{\lambda + 2\zeta\left[\psi\left(1-\frac{\zeta}{\lambda}\right) + \ln\frac{\lambda}{|\zeta|}\right]\right\}\cdot\prod_{j=1}^{l}\left(\frac{\zeta^2}{j^2}-\lambda^2\right) = \frac{1}{a_l^{(cs)}} + \frac{\frac{1}{2}r_l^{(cs)}\lambda^2}{(1)}$$

which is applicable when

$$\lambda r_0 \ll 1, \quad r_0 \ll a_B \qquad (1')$$

Here $\lambda = \left(-2E/E_c\right)^{1/2}$, E is the energy of the level, $\zeta = -z_1 z_2$, l is the angular momentum [2], $\psi(z) = \Gamma'(z)/\Gamma(z)$ is digamma function and atomic units are used, $\hbar = m = e = 1$, m is the reduced mass (the Bohr radius being $a_B = \hbar^2/me^2|z_1 z_2| =$

= $|\zeta|^{-1}$ and the energy unit $E_c = me^4/\hbar^2$).

Note that the case of $\zeta > 0$ corresponds to systems with the Coulomb attraction, while $\zeta < 0$ to the Coulomb repulsion (In the latter case there are evidently no Coulomb levels which condense to E=0). The former case is realized in hadronic atoms ($\bar{p}p$, $\Sigma^- p$, $K^{-4}He$ etc.) and the latter, for example, in pp-scattering. Let us give for orientation the values of the characteristic parameters for pp and $\bar{p}p$ systems: the reduced mass $m = m_p/2$, $a_B = \hbar^2/me^2 = 57.6$ fm, which is $2m_e/m_p \approx 1000$ times less than the Bohr radius in a hydrogen atom equal to $\hbar^2/m_e e^2 = 0.529 \cdot 10^{-8}$cm, and $E_c = 25.0$ keV. (For some other examples see Table I). Thus, in the problem at hand there is, indeed, a small parameter $r_0/a_B \sim 0.03$ which was exploited[3] in deriving the basic equation (1).

The graphical solution of this equation for the case $\ell = 0$ is shown in Fig.1 taken from ref.[3a]. The variable $\nu = \zeta/\lambda$ which is analogous to the principal quantum number n is plotted along the abscissa axis. (Note that at $a_{cs} = 0$, i.e. for the nonshifted Coulomb levels, $\nu = n = 1,2,3, \ldots$). Fig.1 clearly shows that eq.(1) determines not only the positions of atomic ns-levels shifted due to the short-range potential $V_s(r)$ but also the position of a loosely bound (in nuclear scale) s-state if it exists at all (i.e., if $a_{cs} > 0$ and $1 > \nu_N \gg r_0^2$). In Fig.1 this quansinuclear level is denoted as $\nu = \nu_N$.

When applying to pp-scattering ($\zeta = -1$) eq.(1) determines the position of the S-matrix pole nearest to the threshold. With the Coulomb interaction taken into account, the virtual

pole $^1S_0$ ("singlet deuteron") is shifted from an imaginary axis, $k^2 < 0$, to the complex k-plane[6].

When the scattering length is small, function $\psi(1-s/\lambda)$ in eq.(1) is near the pole. In this case eq.(1) directly yields the well-known formula of perturbation theory in terms of the scattering length[7-9]

$$\Delta E_{n\ell} = E_{n\ell} - E_{n\ell}^{(0)} = \frac{2}{(\ell!)^2 n^3} \zeta^{2\ell+3} a_\ell^{(cs)} \prod_{j=1}^{\ell} \left(1 - \frac{j^2}{n^2}\right) \quad (2)$$

Here $\Delta E_{n\ell}$ is the shift of the atomic $n\ell$-level (generally speaking, a complex one, i.e. $Im\, \Delta E_{n\ell} = -\frac{1}{2}\Gamma_{n\ell}$, where $\Gamma_{n\ell}$ is the level width) and $E_{n\ell}^{(0)} = -\zeta^2/2n^2$. This formula is widely used in processing of experimental data (see also eq.(16) below).

2. Rearrangement of atomic spectrum. Of particular interest is the case when potential $V_s(r)$ is attractive and deep enough for the shallow nuclear level to exist (in nuclear scale, i.e. $|\lambda| r_0 \ll 1$). In this case eq.(1) describes the effect of the rearrangement in atomic spectrum[3,4].

Fig.2a represents the behaviour of the lower Coulomb s-levels depending on the coupling constant g (i.e. $V_s(r) = -g\, v(r/r_0)$ where function $v(r/r_0)$ defines the shape of the strong potential and g is a dimensionless coupling constant). At $g = g_0$ there appears a bound s-state in the potential V_s and all the atomic s-levels undergo a drastic rearrangement. The width of the rearrangement region is[1] $\Delta g = |g - g_0| \sim r_0/a_B$ (at $\ell = 0$). Note that in this region all

the atomic ns-levels are considerably displaced relative to its nonperturbed Coulomb energies $E_{ns}^{(0)} = -\zeta^2/2n^2$, while the quasinuclear s-level disturbing the Coulomb spectrum is pulled out of the atomic region (i.e., always $\nu_N < 1$). At $\zeta > \zeta_c$ the energy spectrum is again close to the unperturbed one. However, the atomic wave functions $\chi_{ns}(z)$ acquire an additional zero at distances $z \sim z_0$ which provides their orthogonality to the wave function of quasinuclear state. Note that at $\ell = 0$ the term $\frac{1}{2} z_\ell^{(\ell+1)} \lambda^2$ in eq.(1) is but a minor correction in atomic region, i.e. at $\nu \gtrsim 1$.

Now let us make some comments on references. The rearrangement of atomic spectrum was originally considered by Ya.B.Zel'dovich for the square well and for $\ell = 0$[1]. In ref.[1] all the above qualitative features of this effect have been elucidated. In ref.[10] this effect was independently investigated for $\ell = 0$ and for some model potentials $V_s(r)$ without absorption. It was also marked[10] that the rearrangement of atomic spectrum can manifest itself in hadronic atoms. The model-independent approach to description of the Zel'dovich effect based on eq.(1) was proposed in refs.[3,4]. Applications of the theory to the lightest hadronic atoms ($\bar{p}p$, $\Sigma^- p$, $K^{-4}He$ etc.) are considered in refs.[3,4,25,27,28], see also below, Sec.7.

3. The Zel'dovich effect at $\ell \neq 0$. It has been recently noted[11] that for the nonzero angular momentum states the centrifugal barrier qualitatively changes the picture of level movement as $V_s(r)$ gradually becomes deeper. If $\ell \neq 0$, then at a moment of appearance of a bound ℓ-state we get

$$z_\ell^{(s)} < 0, \quad |z_\ell^{(s)}| \sim z_0^{1-2\ell} \gg 1 \qquad (3)$$

(see ref.[12] as well as Appendix A.). Therefore, in the right-hand side of eq.(1) the term with $z_\ell^{(cs)}$ becomes essential. With this in view, it follows from eq.(1) that there are only a slightly shifted nuclear level, $\nu = \nu_{N\ell}$, and the Coulomb $n\ell$-levels in the system:

$$\nu_{N\ell} = \left(-\frac{1}{2} z^2 a_\ell^{(cs)} z_\ell^{(cs)}\right)^{1/2},$$

$$\nu_{n\ell} = n + \alpha_{n\ell}^2 \cdot \frac{(n\nu_N)^2}{n^2 - \nu_N^2} \qquad (4)$$

Here $\alpha_{n\ell}$ is a dimensionless parameter rapidly decreasing as ℓ grows:

$$\alpha_{n\ell} = 2z^{\ell-1/2}\left(p_{n\ell}/|z_\ell^{(cs)}|\right)^{1/2},$$

$$p_{n\ell} = \frac{1}{\ell!^2}\left(1-\frac{1^2}{n^2}\right)\left(1-\frac{2^2}{n^2}\right)\cdots\left(1-\frac{\ell^2}{n^2}\right), \quad p_{n0} \equiv 1 \qquad (5)$$

($\alpha_{n\ell}^2 \sim (z_0/a_B)^{2\ell-1} \ll 1$ at $\ell \gg 1$, see eq.(A.8) in Appendix A). Eqs.(4) become inapplicable when ν_N approaches the integer number $n \geq \ell+1$ (the values of $\nu = \ell+1, \ell+2, \ldots$ correspond to the nonshifted Coulomb levels). In the narrow region $|\nu_N - n| \sim (z_0/a_B)^{\ell-1/2}$ there occurs an interaction between the nuclear $(N\ell)$ and atomic $n\ell$-levels, their energy being equal to

$$E^{(1,2)} = E_{n\ell}^{(0)} + \frac{1}{2}\left\{\delta E_{n\ell} \pm \left[(\delta E_{n\ell})^2 + 2\lambda^4 \frac{\alpha_{n\ell}^2}{n^3}\right]^{1/2}\right\} \quad (6)$$

where $\delta E_{n\ell}$ is "resonance detuning" commonly used in atomic physics:

$$\delta E_{n\ell} = \frac{\lambda^2}{2}\left(\frac{1}{n^2} - \frac{1}{\nu_N^2}\right) = \left[a_\ell^{(cs)} z_\ell^{(cs)}\right]^{-1} - E_{n\ell}^{(0)} \quad (6')$$

Thus, there is "term quasicrossing" which is well known in quantum mechanics. Unlike s-states, the Zel'dovich effect at $\ell \geq 1$ consists in the following: the nuclear $N\ell$-level which lowers with a gradual deepening of the strong potential $V_s(r)$, successively collides with each of the atomic $n\ell$-levels (see Fig.2b). The atomic $n\ell$-levels with $n > \nu_N$ are shifted upwards, while those with $n < \nu_N$ - downwards.

The rearrangement of atomic spectrum arises when $\ell+1 < \nu_N < \infty$. In terms of the dimensionless coupling constant $g \sim |V_s| z_0^2$ this corresponds to

$$\Delta g = |g - g_c| \sim \frac{1}{\ell^2}\left(\frac{z_0}{a_B}\right)^2, \quad \ell \geq 1 \quad (7)$$

This estimate determines the interval of g values where the nuclear $N\ell$-level passes throughout the whole region of the Coulomb spectrum, see Fig.2b. The term crossing region is much more narrow,

$$\delta g \sim \left(z_0/a_B\right)^{\ell+3/2} \quad (7')$$

According to eq.(1) we obtain at $a_\ell^{(cs)}=0$: $\nu=n$, $E_{n\ell}=E_{n\ell}^{(0)}$. When $a_{cs}=\infty$ and $\ell=0$, the atomic spectrum differs from the Coulomb one most of all (see Fig.3, which refers to $p\bar{p}$-atom but can easily be recalculated for any other hadronic system (see Table 1). The position of the nuclear and atomic ℓ-levels when $\ell\neq 0$ is shown in Fig.4. Figs 2-4 clearly illustrate the difference in the Zel'dovich effect for the states with $\ell=0$ and $\ell\geq 1$.

4. So far we considered the potential $V_s(r)$ to be real. However, eq.(1) remains valid for complex values of a_{cs} as well, that corresponds to the complex optical potential $V_s=U-iW$ and shows the presence of open channels in the problem [4]. In this case the atomic levels are not only shifted but also acquire widths due to strong interaction at small distances.

To elucidate this point it is convenient to consider the properties of eq.(1) in the complex plane of $\nu=\zeta(-2E)^{-1/2}$. In the case of slight absorption, when the depth of the real part of $V_s(r)$ changes, there occurs, as before, rearrangement of the atomic spectrum, the levels being only slightly shifted into the complex plane of energy E. For high absorption the atomic levels move along closed trajectories in the E-plane [13,14]. It seems convenient to examine the movement of levels in the complex plane of variable ν (see Fig.5, which shows the lines of the level [5] of the function $Im\left(a_B/a_{cs}(\nu)\right)$, found by eq.(1) at $\ell=0$ and $a_{cs}=0$). The magnitude of absorption in a hadronic atom can be characterized by a dimensionless quantity ξ ,

$$\xi = \frac{1}{2\pi} Im\left(a_B/a_{cs}\right), \quad \ell = 0 \qquad (8)$$

At $\xi = \xi_1 = 0.991$ the trajectories of the poles near $\nu = 1$ and $\nu = 2$ in the case of weak absorption intersect at the saddle point S_1. One of the poles then returns to $\nu = 1$, the other moves to $\nu = 0$, i.e. it becomes nuclear level $\left(|E| \gg E_c\right)$. At $\xi > \xi_1$ the trajectories of the poles near the point $\nu = 1$ are closed, i.e. rearrangement of the spectrum is replaced by oscillatory movement of levels.

Fig.5 shows an important role of the saddle points S_n: the traversing lines of the level separate the regions of these two regimes. The values $\xi = \xi_n$ correspond to the saddle points S_n:

$$\xi_1 = 0.991; \quad \xi_2 = 0.997, \ldots, \quad \xi_n \underset{(n\gg 1)}{=} 1 - \frac{\log n}{4\pi^2 n^3} + \ldots \qquad (9)$$

At $\xi_n < \xi < \xi_{n+1}$, the 1s, 2s,... ns-levels are in the oscillation regime, while the (n+1)s, (n+2)s, ... levels still remain in the spectrum rearrangement state. Since the values ξ_n are very close to unity, the intermediate region is narrow and it may be accepted that $\xi < 1$ corresponds to spectrum rearrangement, while $\xi > 1$ to the oscillatory state. Note that for $\ell = 0$ the curves in Fig.5 and the values ξ_n practically are independent on z_{cs} [15].

If $\ell \neq 0$, the trajectories of the ℓ-levels and the positions of the saddle points $S_n^{(\ell)}$ depend, to a considerable extent, on the value of $z_\ell^{(cs)}$. Similarly to eq.(8),

define the quantity

$$\xi^{(\ell)} = \frac{(\ell!)^2}{2\pi} \operatorname{Im}\left(a_B^{2\ell+1} / a_\ell^{(cs)}\right) \qquad (8')$$

which characterizes absorption in the system. Denoting its value at the saddle point S_n^{ℓ} by $\xi_n^{(\ell)}$, we find by virtue of eq.(1) that $\lim_{n\to\infty} \xi_n^{(\ell)} = 1$, just as for $\ell = 0$. However, numerical calculations show that at $n \sim \ell+1$ the values $\xi_n^{(\ell)}$ considerably exceed 1, and, unlike the case of $\ell = 0$, the sequence $\{\xi_n^{(\ell)}\}$ is a decreasing one with n increasing. This can be already seen from the asymptotics:

$$\xi_n^{(\ell)} \simeq \begin{cases} 0.254\, \rho_\ell / n, & 1 \ll n \ll \rho_\ell \\ 1 + 6\left(\frac{\rho_\ell}{n}\right)^3 \ln\frac{n}{\rho_\ell}, & n \gg \rho_\ell \end{cases} \qquad (10)$$

where $\rho_\ell = \left\{\frac{\ell!^2}{8\pi^2}\, \zeta^{1-2\ell} \left|z_\ell^{(cs)}\right|\right\}^{1/3} \gg 1$.
The lines of level of the function $Im\{a_B^{2\ell+1}/a_\ell^{(cs)}(\nu)\}$ corresponding to $\xi = \xi_n^{(\ell)}$ remain separatrices, i.e. they separate the region of spectrum rearrangement and that of oscillation of the levels.

For further details concerning analytical properties of the basic eq.(1) in the complex ν -plane, we refer to [4,15]. Let us also note that Kok [16] studied the model problem "Coulomb + + separable potential" (ℓ =0) and emphasized the importance of saddle points S_n . The movement of the S-matrix poles and the positions of the saddle points at $\ell \neq 0$ were studied

for the "Coulomb + delta-potential" model [17] and in ref. [15] on the basis of the model independent eq.(1).

5. Coulomb corrections in the low-energy scattering. In eq.(1) the values $a_\ell^{(cs)}$ and $r_\ell^{(cs)}$ can be connected with the parameters $a_\ell^{(s)}$ and $r_\ell^{(s)}$ which correspond to the strong (short-range) potential $V_s(r)$. At $\ell = 0$ the Coulomb renormalization of the scattering length proves to be logarithmic, $\sim \rho \ln|\rho|$ [18] and that of the effective range is of the power-like type, $\sim \rho$, where $\rho = \zeta r_0$ $(|\rho| = r_0/a_B)$. At $\ell \neq 0$ the relationship of the low-energy parameters is normally of the power-like type though the ℓ-th term of the so-called "effective range expansion" [8,49] is renormalized logarithmically [19]. In particular, the Coulomb renormalization of the p-wave effective range contains the large logarithm $\ln|\rho|$. Let us give formulae of the most essential Coulomb corrections:

$$\frac{1}{a_0^{(cs)}} - \frac{1}{a_0^{(s)}} = -2\zeta \ln|\rho| + \cdots, \quad \ell = 0$$
$$r_1^{(cs)} - r_1^{(s)} = 4\zeta \ln|\rho| + \cdots, \quad \ell = 1 \qquad (11)$$

(for the next terms of these expansions see refs. [4,19,20] and also Appendix B).

When $\ell \geq 1$, there is also a simple formula connecting the scattering lengths $a_\ell^{(cs)}$ and $a_\ell^{(s)}$. The Coulomb correction proves to be of a power-like type and is as follows [19, 20]

$$\frac{1}{a_\ell^{(cs)}} - \frac{1}{a_\ell^{(s)}} = .25 \left[\frac{(2\ell)!}{2^\ell \ell!}\right]^2 \cdot \int_0^\infty \chi_\ell^2(z) \frac{dz}{z} \qquad (12)$$

where $\chi_\ell(z)$ is a radial wave function for the strong potential $V_s(r)$ at $\zeta = 0$ (when the bound ℓ-state appears, i.e. with the boundary conditions $\chi_\ell(z) \sim z^{\ell+1}$ at $z \to 0$ and $\lim_{z\to\infty} z^\ell \chi_\ell(z) = 1$).

6. Using the results of the preceding Section it is easy to obtain an equation directly connecting nuclear level shifts of a hadronic atom with the parameters of strong interaction. Here it is given for the case $\ell = 0$[4]:

$$\frac{\lambda}{2} + \psi(1-\lambda^{-1}) + \ln \lambda z_s + C_0 + C_1 z_s - \frac{1}{4}\lambda^2 z_s = \frac{1}{a_s}\left(\frac{1}{2} + b_1 z_s + \cdots\right) \qquad (13)$$

where $a_s \equiv a_0^{(s)}$, $z_s \equiv z_0^{(s)}$ is the effective range when the bound s-state appears, z_c is the Coulomb radius of the system considered [18-20], $C_0 = 2C + \ln(2z_c/z_s)$, $C = 0.5772\ldots$ is the Euler constant and we put $\zeta = a_B = 1$ which corresponds, for example, to $\bar{p}p$-atom. In eq.(13) the corrections of order $\sim z_0/a_B$ are taken into account, while the terms of the higher order in z_s/a_B are neglected. The coefficients C_0, C_1 and b_1 depend on the shape of the strong potential. They can be easily calculated numerically for an arbitrary potential $V_s(r)$, the method of the calculation is given in refs. [4, 20].

The values of these parameters for some model potentials

$$V_s(z) = -\frac{g}{g z_0^2}\, v(x), \qquad x = z/z_0, \qquad (14)$$

as well as for the OBEP potential, are given in Table 2. It can be seen that c_0 and b_1 are but slightly dependdent on the particular shape of $V_s(r)$. Therefore the calculations of the scattering length a_s with the help of experimental shifts of atomic ns-levels (on the basis of eq.(13)) are not sensitive to the choice of $V_s(r)$. An example of such a calculation for $\bar{p}p$-atom can be found in ref.[3b].

7. Applications to hadronic atoms (the parameters m, a_B, E_c etc. see in Table 1).

a) $\bar{p}p$-atom was considered in refs.[3,4] in connection with the indications[6] to a large shift of its ground state. However, at present the LEAR experiment yields the following values:

$\Delta E_{1s} = (0.5 \pm 0.3)$KeV, $\Gamma_{1s} \leq 1$ KeV[22]

$\Delta E_{1s} = (0.73 \pm 0.15)$KeV, $\Gamma_{1s} = (0.85 \pm 0.39)$KeV[23]

In this case the shifts of s-levels are determined by perturbation theory in terms of the scattering length, see eq.(2), the nuclear level disturbing the spectrum is too far from the atomic spectrum region to be calculated by eq.(1).

Some experimental data on $\bar{p}p$-scattering have appeared recently (at $k_{c.m.} = 90 \div 150$ MeV/c, $k z_0 \sim 0.5$). At these energies, together with s-wave, the p-wave is important. Since there are 4 states with $l = 1$ ($^1P_1$ and $^3P_{0,1,2}$) the results of the analysis of experimental data are still ambiguous[24].

Using different solutions for partial scattering amplitudes given in [24] and the formula

$$f_\ell(k) = k^{2\ell}\left(-\frac{1}{a_\ell} + \frac{1}{2} r_\ell k^2 - i k^{2\ell+1}\right)^{-1}, \quad \ell = 1,$$

we obtain for the p-scattering length the values of the same order as the hard sphere scattering length: $\tilde{a}_1 = r_0^3/3 \sim 1\,fm^3$. Hence one may conclude that the $\bar{p}p$-system in the p-wave is evidently far from rearrangement.

b) $\Sigma^- p$ - atom. This system is of interest due to the fact that studying the data on the shift and width of the atomic $^3S_1$-level one can obtain the ΣN -scattering length for $\ell = 0$, I=1/2 and elucidate the nature of the 2129 MeV peak in the Λp system [25].

c) $K^-{}^4He$-atom. The data on the shift of the 2p-state are [26] $\Delta E_{2p} = (43 \pm 8)$KeV, $\Gamma_{2p} = (55 \pm 34)$eV. Calculations using eq.(1) show that [27,28] in p-wave there may exist a loosely bound state of K^- with α -particle. Its binding energy is $\varepsilon \sim \gamma \approx 0.5$ MeV, see Fig.6.

d) The basic eq.(1) can be applied to description of the shifts and widths in heavier hadronic atoms, such as $\bar{p}^4He$, $\bar{p}^6Li$, $\Sigma^- {}^7Li$, $\Sigma^- {}^{11}B$, ... In this connection it is useful to have a general criterion of existence of a shallow nuclear state. Let $\delta_{n\ell} = |\Delta E_{n\ell}|/(E^{(0)}_{n+1,\ell} - E^{(0)}_{n\ell})$. Using eq.(2) and taking $a_\ell^{(es)} \approx a_\ell^{(s)}$ to be equal to the hard sphere scattering length $\tilde{a}_\ell$ (A.3) we obtain the "critical" value of $\delta_{n\ell}$.

$$\delta_{n\ell}^{(cr)} = \frac{(n+\ell)!}{(2\ell)!\,(2\ell+1)!\,(n-\ell-1)!}\left(\frac{2r_0}{n a_B}\right)^{2\ell+1} \qquad (15)$$

If $\delta \gg \delta^{(cr)}$, the perturbation of the atomic spectrum is strong and one can expect the existence of a shallow nuclear state. Note that $\delta_{n\ell}^{(cr)}$ rapidly decrease with increasing ℓ. Therefore, at $\ell \neq 0$ a relatively small shift of atomic level can indicate the spectrum rearrangement. For example, for $K^{-4}He$ atom at $r_0/a_B \sim 1/20$ and $\ell = 1$ we have $\delta_{2p}^{(cr)} \sim 10^{-4}$, while the above experimental shift of the 2p-state gives $\varepsilon_{2p} \approx 7\cdot 10^{-3} \gg \gg \delta_{2p}^{(cr)}$.

The condition $\delta > \delta^{(cr)}$ allows one to quickly determine a possible existence of shallow nuclear states. As a whole, the research of nuclear shifts and level widths in hadronic atoms will make it possible to obtain useful information on strong interaction parameters at low energies.

e) To calculate the scattering length by the shift and width of atomic level the formula by Deser et al.[7] (see eq.(2)) is commonly used which can be rewritten as

$$\Delta E_{n\ell} - \frac{i}{2}\Gamma_{n\ell} = \frac{2p_{n\ell}}{n^3}\,\xi^{2\ell+3}\,a_\ell^{(PT)}, \qquad (2')$$

for $p_{n\ell}$ see eq.(5). We denote these values of a_ℓ as PT (perturbation theory). Formula (2') can be precised[11]:

$$a_\ell^{(cs)} = a_\ell^{(PT)}\cdot\left\{1 + \sum_{k\geq 1} c_k \alpha^k\right\}, \quad \alpha = \xi^{2\ell+1} a_\ell^{PT} \qquad (16)$$

($\alpha \sim (r_0/a_B)^{2l+1} \ll 1$). Coefficients c_k depend on the quantum numbers n, l as well as on $r_l^{(cs)}$ and are tabulated [7]. Eq.(16) enables one to calculate $a_l^{(cs)}$ with a good accuracy without solving eq.(1). As a rule, it is sufficient to leave in eq.(16) the first two or three terms.

Consider, for example, the $K^{-4}He$-atom. Only two terms in expansion (16) give $a_1^{(cs)}$ = 72.38 fm^3 at ΔE_{2p} = 43 eV, Γ_{2p} = 0 and $a_1^{(cs)}$ = (72.89 - i 45.49) fm^3 at E_{2p} = 43 eV, Γ_{2p} = 55 eV. At the same time, the calculation with eq.(1) gives $a_1^{(cs)}$ = 72.4 fm^3 and $a_1^{(cs)}$ = (72.9 - i 45.5) fm^3, respectively. It is seen that the accuracy of this approximation ("improved PT", see [11]) is more than sufficient.

f) One of the experimental methods of measuring nuclear shifts is a study of radiative spectra for $np \to 1s$, $nd \to 2p$, and similar transitions. It should be emphasized in this connection that the formula for the probabilities of radiative transitions $np \to \nu s$ and $nd \to \nu p$ to a shifted level ($E = -\zeta^2/2\nu^2$, ν is arbitrary) were obtained and tabulated in refs. [3,28].

8. As the last example, let us consider the application of the above approach to the problem [29,30] of the μ-catalyzed nuclear fusion. Ponomarev et al. [31] have evaluated the energies of $dd\mu$ and $dt\mu$ mesic molecules and predicted the existence of loosely bound molecular states (for $dt\mu$-molecule ε_{01} = 34.9 eV, ε_{11} = 0.64 eV, where ε_{Jv} is binding energy, J is rotational and v is vibrational quantum number). These calculations were made for point-like Coulomb centres. A possible effect of strong potential $V_s(r)$ was taken into account in refs.

[32] where simple models for $V_s(r)$ were used whose parameters were fitted according to experimental data on dt-elastic scattering and $d+t \to n+\alpha$ reaction (for energies up to 200 keV). An important result of these calculations is the smallness of nuclear shifts in molecular terms induced by the nuclear dt-interaction.

On the other hand, it is claimed in ref.[33] that the $3/2^+$ resonance level in dt system (E_r = 64 keV, $\Gamma_r/2$ = 70 keV) can greatly affect the position of mesic molecular term due to the spectrum rearrangement effect.

To discuss this point we applied the WKB method to the Coulomb potential $V_c(z) = -\zeta/z$ modified at small distances. This provides the formula

$$E - E_{n\ell}^{(0)} = \frac{[(2\ell+1)!!]^2}{2} \cdot \frac{c_{n\ell}^2}{a_B^{2\ell+3}} \left\{ \frac{1}{a_\ell^{(cs)}} + \frac{k^2}{2} \left[\frac{1}{3(\ell!)^2 a_B^{2\ell-1}} - z_{\ell(17)}^{(cs)} \right] \right\}^{-1}$$

where $c_{n\ell}$ is a dimensionless coefficient determining the behaviour of the normalized wave function (without strong potential V_s) at small distances [8] :

$$R_{n\ell}(z) \underset{(z\to 0)}{=} c_{n\ell}\, a_B^{-(\ell+3/2)}\, z^\ell + \dots, \qquad \int_0^\infty R_{n\ell}^2\, z^2\, dz = 1 \qquad (17')$$

$(a_B = |\zeta|^{-1})$. At $\ell = 0$ and $\zeta = 1$, we find that

$$E - E_{1s}^{(0)} = \frac{1}{2} c_{1s}^2 \left[\frac{1}{a_{cs}} + \left(\frac{1}{3} a_B - z_{cs} \right) E \right]^{-1}, \qquad (18)$$

where the energy $E^{(0)}_{J\nu}$ and $C_{J\nu}$ are calculated, the strong interaction being neglected. Eq.(18) is model-independent. Using the low energy scattering parameters $a_{cs} = -(3.88+i\ 1.27)\cdot a_B$ and $r_{cs} = (0.24+ i\ 0.016)\ a_B$, derived from $d(t,\alpha)n$ reaction and the values $C_{00} = 1.02.10^{-4}$, $C_{01} = 0.927.10^{-4}$ taken from numerical calculations of mesic molecular wave functions [31], we have

$$\Delta E_{00} = -1.25.10^{-3}\text{eV}, \qquad \Gamma_{00} = 9.0.10^{-4}\ \text{eV}$$

$$\Delta E_{01} = -1.04.10^{-3}\text{eV}, \qquad \Gamma_{01} = 7.4.10^{-4}\text{eV}$$

which is in an agreement with the calculations [32] for model potentials $V_s(r)$.

Note that even if the nuclear level N would hit the region of mesic molecule levels, the nuclear shift would have remained small as well. Indeed, solving eq.(17) we find that

$$\Delta E^{(1,2)}_{J\nu} = \frac{1}{2}\left\{\delta E \pm \left[(\delta E)^2 + \frac{6 C^2_{J\nu}}{(a_B - 3r_{cs})a_B^3}\right]^{1/2}\right\} \qquad (19)$$

where $E^{(1,2)}_{J\nu}$ are the energies of two interacting states, $\Delta E = E^{(1,2)} - E^{(0)}_{J\nu}$, $\delta E = E_N - E^{(0)}_{J\nu}$ is resonance detuning and $E_N = -\frac{3}{a_{cs}(a_B - 3r_{cs})}$. Though $\ell = 0$ here formula (19) is analogous to eq.(6). The smallness of the shifts $\Delta E^{(1,2)}_{J\nu}$ is provided by low penetreability of the Coulomb barrier which acts in the same way as the centrifugal barrier for $\ell \neq 0$ states:

$$C^2_{J\nu} \sim \exp(-2\pi/k a_B) \sim 10^{-7} \div 10^{-8}$$

Thus, the influence of $V_s(r)$ cannot have any noticeable

effect on the energies of mesomolecular terms calculated with strong interaction neglected.

9. Exactly soluble models. So far our approach has been based on the model-independent eq.(1). The boundaries of the applicability region for this approximation can be established more accurately by comparing it with analitical solutions of the Coulomb plus short-range potential problem. Such solutions were obtained for the Yamaguchi potential, $\ell = 0$:

$$V_s(\vec{p},\vec{p}') = -\frac{g\mu^3}{\pi^2(p^2+\mu^2)(p'^2+\mu^2)}, \qquad (20)$$

as well as for the delta-shell potential at any ℓ,

$$V_s(r) = -\frac{g}{4\pi r_0}\,\delta(r-r_0) \qquad (21)$$

In x-representation the potential (20) corresponds to nonlocal interaction. In this case the bound state emerges at $g = g_0 = 1$, when $\chi_0(r) = 1 - \exp(-\mu r)$, $r_s = 3\mu^{-1}$, $r_c/r_s = 0.3743$ and $c_0 = 0.865$, $b_1 = \frac{1}{2}c_1 = 2/3$. The discrete spectrum is defined from the equation [34,35]

$$g \cdot {}_2F_1\left(1, -\lambda^{-1}; 2-\lambda^{-1}; z\right) = \left(1-\lambda^{-1}\right)\left(1+\frac{\lambda}{\mu}\right)^2, \qquad (22)$$

where $\zeta = 1$, $z = \left[(\lambda-\mu)/(\lambda+\mu)\right]^2$ and $F = {}_2F_1$ is hypergeometric function. When performing numerical calculations it is convenient to seek for the coupling constant g as a function of $\lambda = (-2E)^{1/2}$. We have made these calculations for different values of μ and compared the results with eq.(13). Using $a_s = 2g/(g-g_0)\mu = \frac{2}{3} r_s \left(1 - g_0/g\right)^{-1}$, $g_0 = 1$

for the Yamaguchi potential, the energies of 1s and 2s-states were calculated with the help of eq.(13) and of the exact eq. (22), see Fig.7. We conclude that eq.(13) is actually valid up to $r_s \sim 0.3 a_B$. It is seen from Fig.7 that rearrangement of the atomic spectrum gradually disappears as $r_s/a_B = 3\mu^{-1}$ increases. At $r_s/a_B \gtrsim 0.1$ the atomic s-levels smoothly move with q increasing.

A similar study was carried out for the local potential (21). Here the equation for discrete spectrum is[28,36]

$$\frac{\Gamma(\ell+1-\nu)}{(2\ell+1)!\, z} M_{\nu,\ell+1/2}(z)\, W_{\nu,\ell+1/2}(z) = q^{-1}, \qquad (23)$$

where $\zeta = 1$, $\nu = \lambda^{-1}$, $z = 2\lambda r_0$ and M,W are the Whittaker functions. At $\zeta \to 0$ eq.(23) takes the form of

$$I_{\ell+1/2}(\lambda r_0)\, K_{\ell+1/2}(\lambda r_0) = q^{-1} \qquad (24)$$

from which it follows that there is precisely one bound state in each partial wave that exists at $q > q_\ell = 2\ell+1$. The parameters $a_\ell^{(s)}$, $r_\ell^{(s)}$ and $a_\ell^{(cs)}$, $r_\ell^{(cs)}$ for this case can be found analytically[17,28,36]. The comparison of the exact (23) and approximate (1) equations for the energy of s-level is given in Fig.8, where it is seen that the uncertainty of energy E calculated according to eq.(1) is not more than 10% untill $\lambda r_0 \lesssim 0.3$ (i.e. the binding energy is less than $0.1\ \hbar^2/2mr_0^2$). Fig.9 illustrates the accuracy of our approximation, eq.(B.1), for the s-scattering length.

A general conclusion that can be drawn from examination of exact solutions is as follows. Eqs.(1),(13) etc. obtained

at $r_0 \ll a_B$ are in fact applicable with a reasonable accuracy up to $r_0/a_B = 0.1 \div 0.2$. Note that to get this accuracy the introduction of terms of order r_s/a_B is essential, see, for example, Fig.8.

10. Rearrangement of the atomic spectrum can always appear in the systems for which the interaction potential splits into two parts with greatly differing radii.

Let us give some more examples: the problem with a short-range potential in magnetic field [37] (a negative ion, H^-, for example); interaction of molecular terms at large distances [38,39]; shifts of Landau levels in homogeneous magnetic field in the presence of center [39]; model-independent account of nuclear interaction for muon sticking probability in $dt\mu \to n + (\mu^- \alpha)$ reaction. One should also mention a collapse of quantum orbit of excited electron in heavy atoms [40] (a characteristic example is a drastic change in the $4f$ - shell wave function in passing from Ba, $Z = 56$, to La, $Z = 57$). And finally, when considering the electron spectrum for a superheavy nucleus [41-43], $Z \sim Z_{cr} > 137$, the energy spectrum undergoes a rearrangement at that value of Z when the bound state dives into the lower continuum (within a limit of very small nuclear radius, $\ln(\hbar/m_e c r_N) \gg 1$ [42]). The list of examples illustrating the Zel'dovich effect can, no doubt, be enlarged.

The author is greatly indebted to Drs.B.M.Karnakov, A.E. Kudryavtsev, V.I.Lisin and V.D.Mur whose assistance considerably promoted to get the results discussed above and to N.S. Libova for the help in preparing the manuscript.

Appendix A

The parameters a_ℓ and r_ℓ in eq.(1) have dimensions $L^{2\ell+1}$ and $L^{1-2\ell}$, respectively (L is the length; note that $[a_\ell] = [r_\ell] = L$ only for s-wave).

At a moment of emergence of the ℓ-level in the short-range potential $V_s(r)$

$$\chi_\ell(r) \equiv r R_\ell(r) = A_\ell r^{-\ell} + \cdots , \quad r \to \infty \qquad (A.1)$$

At $\ell \geq 1$, assuming the normalization condition $\int_0^\infty \chi_\ell^2(r)\,dr = 1$ to be fulfilled, we have [12]

$$r_\ell^{(s)} = -2^{1-2\ell}\left[(2\ell)!/\ell!\,A_\ell\right]^2 \qquad (A.2)$$

It can be easily shown that for the scattering on a hard sphere of radius r_0

$$k^{2\ell+1} \operatorname{ctg} \delta_\ell(k) = -\frac{(2\ell)!\,(2\ell+1)!}{(2^\ell \ell!)^2 r_0^{2\ell+1}} \cdot \frac{\Lambda_{-\nu}(kr_0)}{\Lambda_\nu(kr_0)} ,$$

where $\nu = \ell + 1/2$,

$$\Lambda_\nu(z) = \Gamma(\nu+1)(z/2)^{-\nu} J_\nu(z) \underset{(z\to 0)}{=} 1 - \frac{z^2}{4(\nu+1)} + \cdots$$

Hence there follow formulae for the scattering length and the effective range,

$$\tilde{a}_\ell = \frac{(2^\ell \ell!)^2}{(2\ell)!\,(2\ell+1)!} r_0^{2\ell+1} , \quad \tilde{r}_\ell = -\frac{(2\ell+1)!^2 r_0^{1-2\ell}}{2^{2\ell-1}\ell!^2(2\ell-1)(2\ell+3)} \qquad (A.3)$$

(we denote the value for the hard sphere by tilde). In other cases $a_\ell^{(s)}$ and $z_\ell^{(s)}$ depend on the depth of $V_s(r)$. For some simple potentials they can be obtained analytically. So, for the square well and for delta-potential we have

$$z_\ell^{(s)}/\tilde{z}_\ell = \begin{cases} (2\ell+3)/2(2\ell+1), & \text{square-well} \\ 2/(2\ell+1), & \delta\text{-potential} \end{cases} \qquad (A.4)$$

(here the values of $z_\ell^{(s)}$ refer to the point of emergence of the bound ℓ-state). At $\ell \gg 1$,

$$z_\ell^{(s)} \simeq -C \cdot \ell^\beta \exp\left\{2\ell \ln \frac{\ell}{\bar{\ell}}\right\} \cdot z_0^{1-2\ell}, \qquad (A.5)$$

where $C = 2$, $\beta = 0$ for the square well, $C = 4$, $\beta = -1$ for the δ-potential, $C = 4$, $\beta = 0$ for the hard sphere and $\bar{\ell} = e/2 = 1.36$ for all three potentials.

The asymptotics (A.5) has a general meaning. It can be shown[45] using the 1/n expansion[44] that at $\ell \to \infty$

$$z_\ell^{(s)} \sim -z_0^{1-2\ell} \exp\left\{2\ell \cdot \ln(\ell/\bar{e})\right\} \qquad (A.6)$$

where $\bar{\ell} = \exp(I_0 + 1 - \ln 2)$,

$$I_0 = \ln x_0 + \int_{x_0}^{\infty} \frac{dx}{x}\left\{1 - \left[1 - \frac{x^2 v(x)}{x_0^2 v(x_0)}\right]^{1/2}\right\} \qquad (A.7)$$

$x = z/z_0$, while x_0 is defined from equation $xv' + 2v = 0$. For instance, we find that $x_0 = 1$, $I_0 = 0.7303$ and $\bar{\ell} = 2.82$ for the Yukawa potential, $v(x) = e^{-x}/x$; $I_0 = 0.3652$, $\bar{\ell} = 1.96$ for gaussian, $v(x) = e^{-x^2}$; $x_0 = 1$, $I_0 = 0$ and $\bar{\ell} = 1.36$ for the square well and delta potential.

Using eq.(A.6) one can easily estimate different parameters entering the problem. For instance, at $\ell \gg 1$

$$\alpha_{k\ell} \sim (\zeta z_0)^{\ell-1/2} \exp\{-2\ell \cdot \ln(\ell/\ell_1)\}, \quad k=\ell+1,$$

$$\rho_\ell \sim (\zeta z_0)^{\frac{1-2\ell}{2}} \cdot \exp\left\{\frac{4}{3}\ell \cdot \ln(\ell/\ell_2)\right\} \tag{A.8}$$

where $\alpha_{k\ell}$ and ρ_ℓ – parameters in eqs.(4) and (10), $\ell_1 = \exp(I_0 - 1 - \frac{1}{2}\ln 2) = 0.19\,\bar{\ell}$, $\ell_2 = \exp(\frac{1}{2}I_0 + 1 - \frac{1}{2}\ln 2)$. Hence it is seen that $\alpha_{k\ell}$ and ρ_ℓ also contain, besides powers of z_0/a_B, essential numerical factors strongly depending on ℓ.

Appendix B

Let us briefly discuss Coulomb renormalization of the scattering lengths and effective radii. We denote the Coulomb modified nuclear scattering parameters by (cs) and the parameters belonging to the strong potential V_s at $\zeta = 0$ - by index (s). The Coulomb renormalization is of the greatest importance provided the potential V_s has a level close to zero energy. We shall confine ourselves to this particular case. If $\ell = 0$, then[4]

$$\frac{1}{a_{cs}} = \frac{1}{a_s}(1 + 2b_1\rho) - 2\zeta\left(\ln|\zeta| + c_0 + c_1\rho\right), \tag{B.1}$$

$$z_{cs} = z_s(1 + h_0\rho) \tag{B.2}$$

Here $C_0 = 2C + \ln(2r_c/r_s) = 1.848 - \ln(r_s/r_c)$,

$$r_s = 2\int_0^\infty [1 - \chi_0^2(r)]\,dr,$$

$$r_c = \exp\left\{\ln R + \int_0^\infty \frac{\theta(r-R) - \chi_0^2(r)}{r}\,dr\right\}, \quad R>0 \qquad (B.3)$$

$\chi_0(r)$ is a wave function in potential (14) normlized by condition $\chi_0(\infty) = 1$ and corresponding to the value $g = g_0$ at a moment of appearance of the bound s-state. In eq.(B.1) we denote $\rho = \zeta r_s$ ($|\rho| = r_s/a_B \ll 1$) and the terms $\lesssim \rho^2$ are omitted. The coefficients b_1, c_0, c_1 and h_0 depend only on the shape of strong potential, i.e. they do not change at the scaling $V_s(r) \to \alpha^2 V_s(\alpha r)$.

If one neglects in eq.(B.1) the corrections $\sim \rho$ (formally assuming $b_1 = c_1 = 0$), one arrives at the well-known formula by Schwinger [18] which is asymptotically exact within $\rho \to 0$. Comparing eq.(B.1) with the exact solution of model (21), we see that the range of applicability of this equation is noticeably extended when corrections $\sim \rho$ are taken into account (see, for example, Fig.8).

In the case of arbitrary l "the large logarithm" appears in the Coulomb correction for the coefficient s_l at k^{2l} in the effective range expansion [19]

$$k^{2l+1} \operatorname{ctg} \delta_l(k) = -\frac{1}{a_l} + \frac{1}{2} r_l k^2 + \cdots + s_l k^{2l} + \cdots \qquad (B.4)$$

$$s_l^{(cs)} - s_l^{(s)} = 2\rho \left[\ln(|\rho| r_0) + \text{const}\right] \qquad (B.5)$$

Here $r_0 \sim r_s$ is the radius of nuclear interaction while the constant depends only on the shape of $V_s(r)$. Note that S_l is of dimension L^{-1}; thus, $S_0 = -1/a_0$, $S_1 = \frac{1}{2} r_1$ etc. In particular, at $l \neq 0$ the leading correction is of order ρ, the term in eq.(12) singular at $\rho \to 0$ being written explicitly,[46]

$$1/a_l^{(cs)} - 1/a_l^{(s)} = \\ = \rho \frac{(2l)!^2}{2^{2l-1} l!^2} \int_0^\infty \chi_l^2(r) \frac{dr}{r} + \dots - \frac{2}{l!^2} \rho^{2l+1} \ln|\rho r_0| + \dots, \quad (B.6)$$

(note, however, that at $l \neq 0$ the singular term is but a small correction of order $\rho^{2l+1} \ln \rho$).

For p-wave

$$r_1^{(cs)} - r_1^{(s)} = 4\rho \left(\ln \left| \frac{\rho}{r_1^{(s)}} \right| + \beta_1 \right) + \dots, \quad (B.7)$$

where β_1 is a constant (see Table 2). If $l \neq 1$, then

$$r_l^{(cs)} / r_l^{(s)} = 1 + h_l \rho \left| r_l^{(s)} \right|^{-\frac{1}{2l-1}} + \dots \quad (B.8)$$

i.e. the Coulomb correction to $r_l^{(s)}$ is of order ρ. At $l = 0$ this formula coincides with (B.2). In this case the singularity at $\rho \to 0$ disappears, and expansion (B.8) does not contain logarithms at all. In other cases, $l \geq 1$, the singular term in expansion (B.8) has the structure[46] $r_l^{(cs)} - r_l^{(s)} = \frac{2}{3} \cdot \frac{(l+1)(2l+1)}{(l-1)!\, l!} \rho^{2l-1} \ln|\rho r_0| + \dots$. Estimate (A.6) gives

$$\rho \left| r_l^{(s)} \right|^{-\frac{1}{2l-1}} \sim \rho r_0 \cdot (l/\bar{l})^{-1},$$

i.e. the value of the Coulomb correction decreases with ℓ growing.

Note that eqs.(B.7) and (B.8) were already used to connect the values of effective ranges and energies of near-threshold resonances in charged and neutral channels (for (for $K^{-4}He$-system[27] and for the lightest nuclei ^5He and ^5Li, ^8Li and ^8B, etc.[25]).

The results of numerical calculations[4,20] of the parameters $z_\ell^{(s)}$, C_0 etc. are shown in Table 2. In particular, it contains the data on the Yukawa potential, $v(x) = e^{-x}/x$; the Hulthén potential, $v(x) = (e^x - 1)^{-1}$; the square well, $v(x) = \theta(1-x)$ and the Coulomb well, $v(x) = x^{-1}\theta(1-x)$. The OBEP potential corresponds to $\bar{N}N$-interaction, the first line referring to the state with S=0, while the second - to S=1[4]. All the numericals in Table 2 refer to the depth of potential $V_s(r)$ at a moment of appearance of the first bound state with the angular momentum ℓ .

Footnotes

1) In this connection it suffices to refer to a paper by Zel'dovich[1] devoted to a problem which is rather outside the field of nuclear physics and concerns the electron energy levels in a semiconductor with impurity centres ($\ell = 0$). The condition $r_0 \ll a_B$ is fulfilled there because of the fact that dielectric penetreability of media $\varepsilon \gg 1$. Hereafter we call this phenomenon the Zel'dovich effect (for states with arbitrary ℓ).

2) If $\ell = 0$, then the index j product in eqs.(1) and (2) should be omitted.

3) Eq.(1) can be obtained by analytical continuation of the effective range expansion[49] into the region of discrete spectrum[2,3]. Another way of its obtaining is based on the coupling constant evolution method[5].

4) For example, the annihilation $\bar{p}p \to 2\pi, 3\pi, \cdots$ in the case of proton-antiproton atom.

5) i.e. contours in the ν -plane where $Im\,(a_B/a_{cs}(\nu))$ is kept fixed.

6) According to ref.[21], the shift $\Delta E_{1s} \approx 3$ keV (that should be compared with $E_{2s}^{(0)} - E_{1s}^{(0)} = 0.375\zeta^2 E_c = 9.38$ keV.

7) See ref.[11]. For example, at $\ell = 0$ (ns-states)

$$c_1 = \beta_n + \frac{1}{n^2}\zeta r_{cs} \approx \beta_n ,$$

where $\beta_0 = 3.154$, $\beta_1 = 1.541$, $\beta_2 = 1.018$ and $\beta_n = 3/n$ at $n \gg 1$.

8) For example, in the case of pure Coulomb attraction, $V(r) = -\zeta/r$ at $0 < r < \infty$, we have

$$c_{nl} = \frac{2}{(2l+1)!!}\left(\rho_{nl}/\hbar^3\right)^{1/2},$$

ρ_{nl} is obtained by eq.(5).

Table 1

The basic parameters for some hadronic systems

System	m, MeV	ζ	a_B, fm	E_c, keV
$\bar{p}p$	469.1	1	57.6	25.0
$\Sigma^- p$	526.1	1	51.4	28.0
$\Xi^- p$	548.7	1	49.2	29.2
$K^- p$	323.5	1	83.5	17.8
$K^{-4}He$	436.0	2	31.0	23.2
pp	469.1	-1	57.6	25.0
dd	937.8	-1	28.8	50.0
dt	1124.7	-1	24.0	59.9

Footnote: Here $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass, $a_B = \hbar^2/me^2|\zeta|$ is the Bohr radius, $E_c = me^4/\hbar^2$.

Table 2

$v(x)$	$l=0$						$l=1$		$l=2$
	r_s/r_0	r_c/r_s	c_0	c_1	b_1	h_0	$-r_1^{(s)}\cdot r_0$	β_1	$-r_2^{(s)}\cdot r_0^3$
e^{-x}/x	2.120	0.364	0.837	0.842	1.056	- 0.178	1.374	-	1.429
$(e^x-1)^{-1}$	3.000	0.374	0.865	0.983	1.000	0.138	0.960	-	0.482
$\exp(-x)$	3.541	0.395	0.919	1.215	0.903	0.512	0.834	-	0.326
$\exp(-x^2)$	1.435	0.418	0.975	1.457	0.814	0.936	2.059	-	4.765
$\theta(1-x)$	1.000	0.439	1.024	1.634	0.757	1.189	3.000	0.701	15.0
$x^{-1}\cdot\theta(1-x)$		0.422	0.985	1.500	0.793	1.043	-	-	-
$\delta(1-x)$	1.333	0.455	1.060	1.750	0.750	1.312	2.400	0.856	8.571
OBEP(S=0)	1.43 fm	0.417	0.973	1.35	0.86	-			
OBEP(S=1)	1.23 fm	0.440	1.027	1.65	0.77	-			

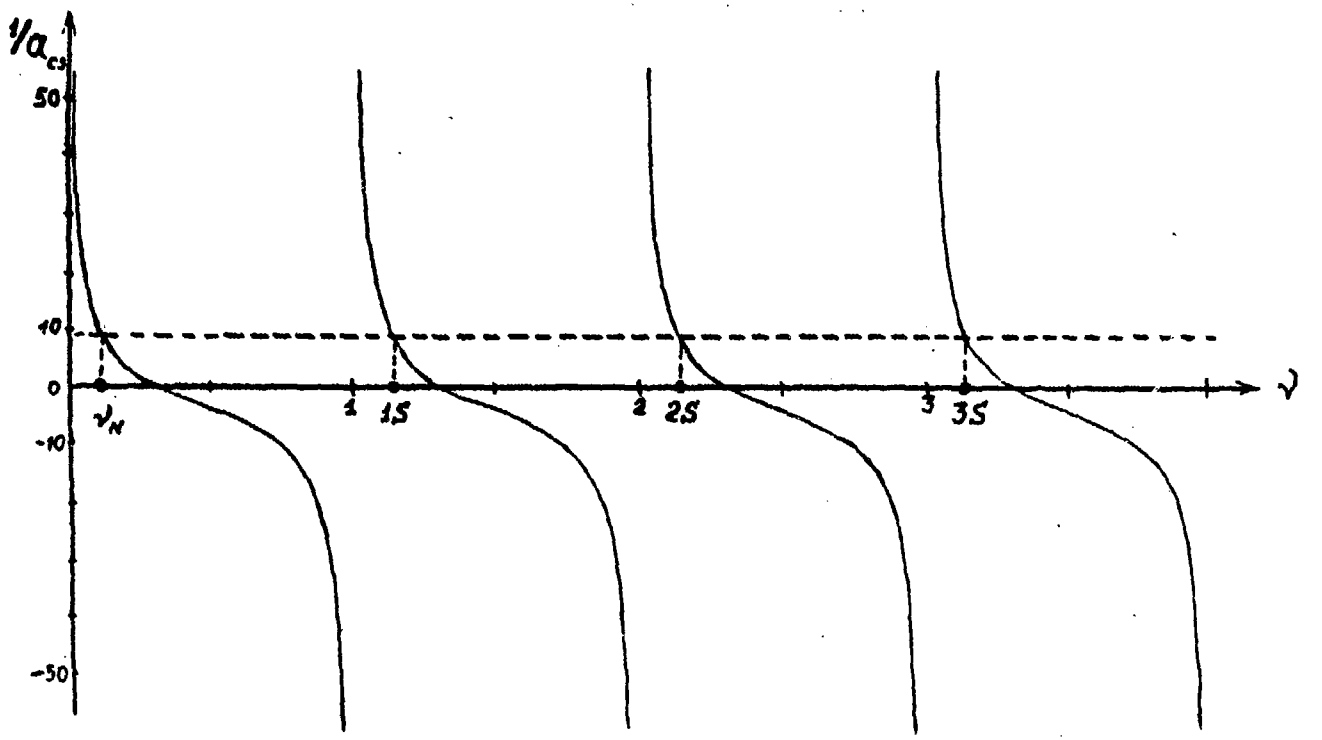

Fig.1. Graphical solution of eq.(1), $l = 0$. The position of atomic ns-levels and of the quasinuclear level ($\nu = \nu_N$) are shown by dots.

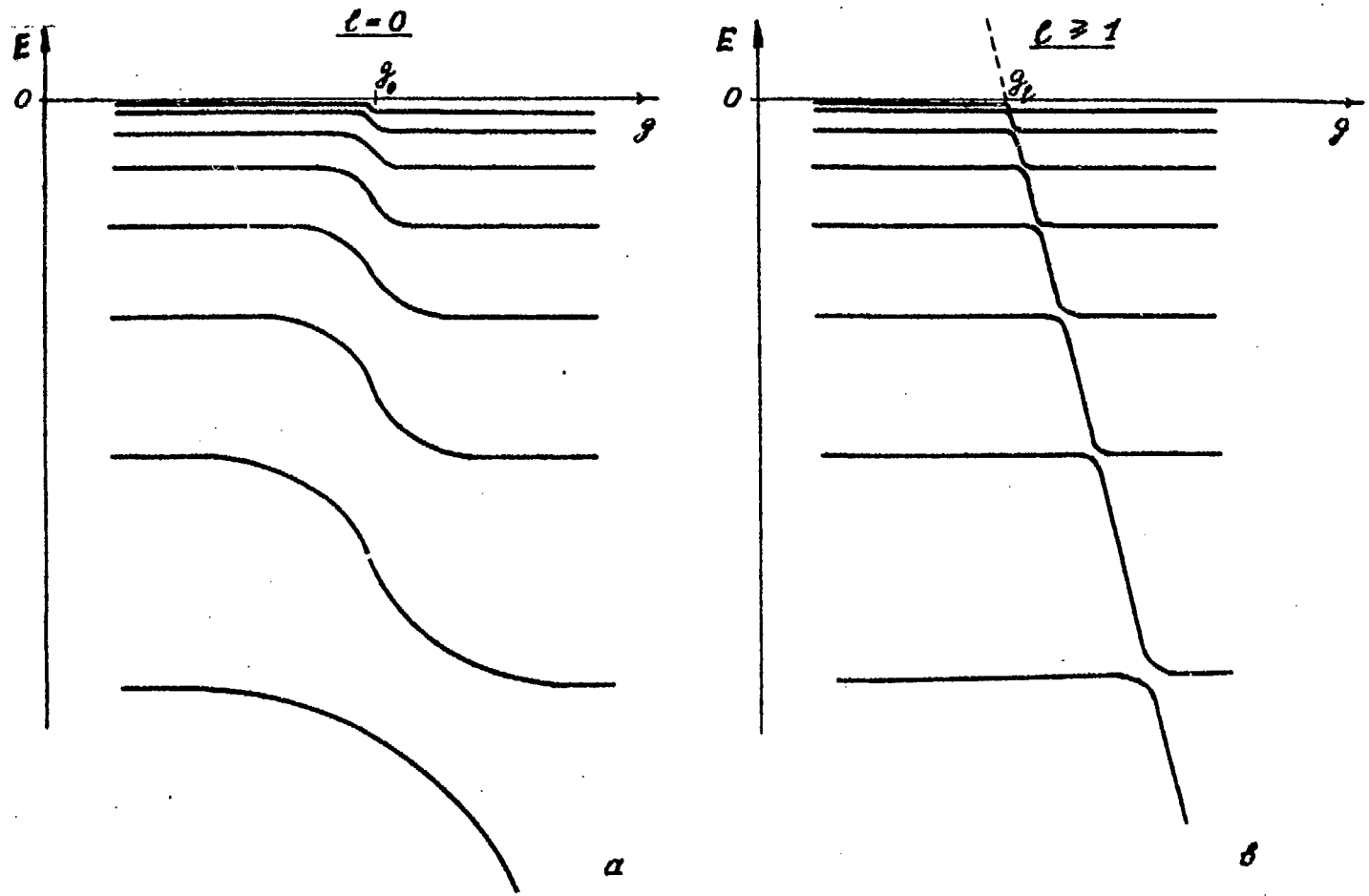

Fig.2. Rearrangement of atomic spectrum with no absorption: a) for s-states; b) for $\ell \neq 0$ states. The dimensionless coupling parameter $q \sim |V_s| r_0^2$, where V_s is the depth of the strong potential and r_0 is its radius.

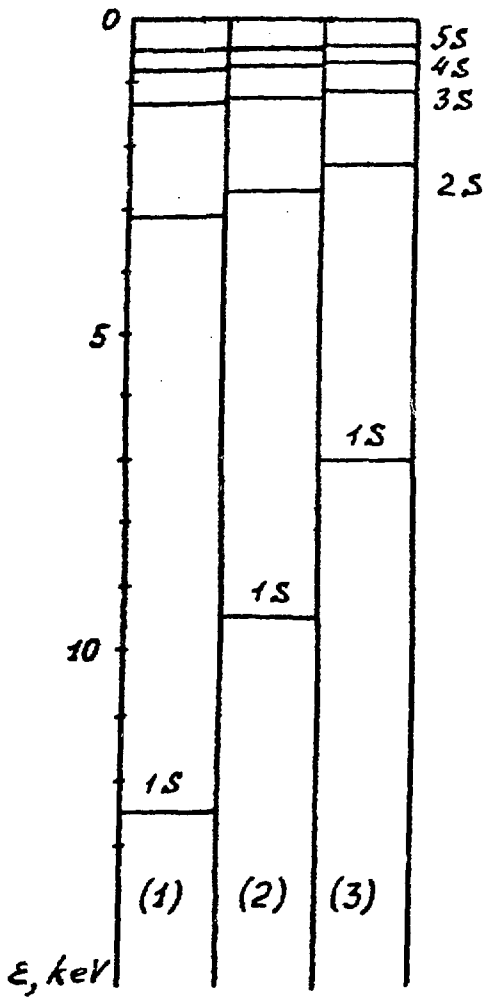

Fig.3. The atomic spectrum of $\bar{p}p$-atom for three values of s-scattering length ($\epsilon = -E$ is the binding energy):
1) $a_s = 0$; 2) $a_s = 6.6$ fm $\simeq 0.12\, a_B$;
3) $a_s = \infty$.

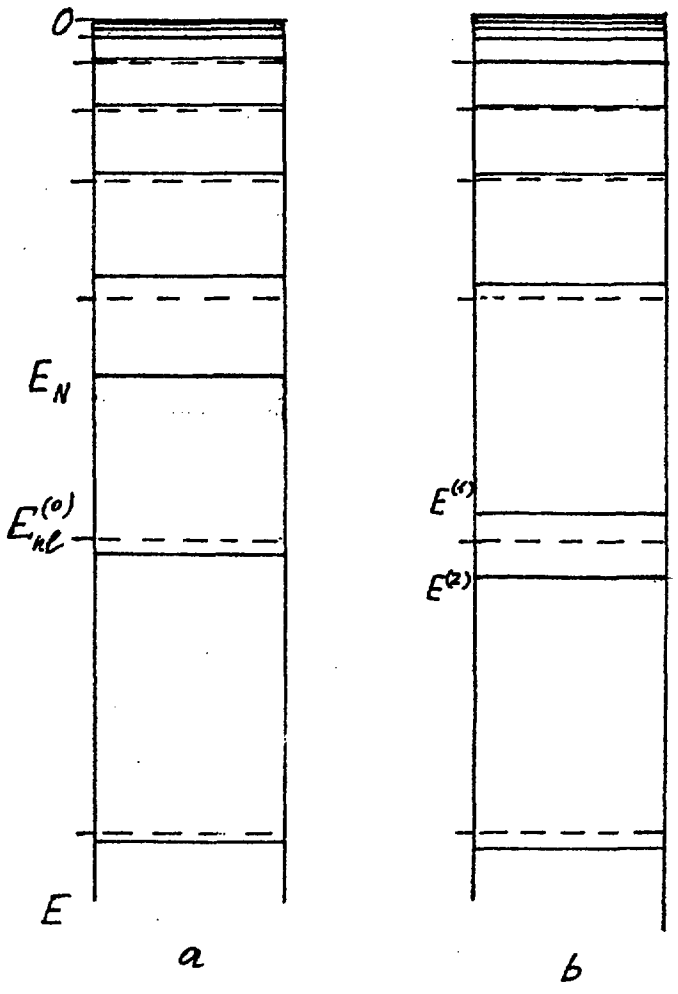

Fig.4. The atomic spectrum with $l \geq 1$:

a) in the case described by eq.(4);

b) in the region of "term quasicrossing". Broken lines show the positions of the nonshifted Coulomb levels, $E_{nl}^{(0)} = -Z^2/2n^2$.

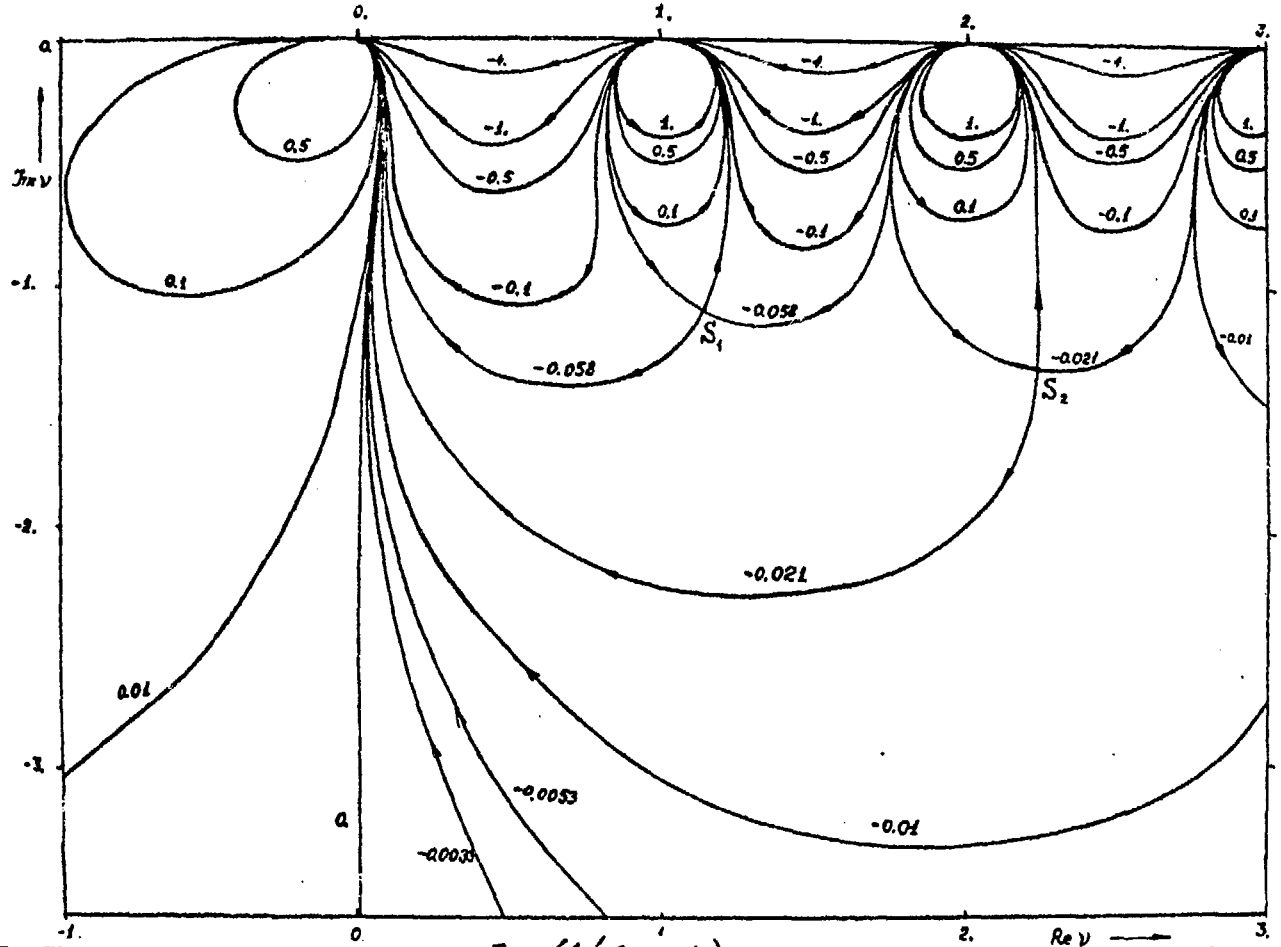

Fig.5. The lines of the level of $Im\,(1/a_{cs}(\nu))$ in the complex ν -plane (l =0, $\zeta = a_\beta = 1$ and $\gamma = \lambda^{-1}$). The values of $Im\,(a_{cs}^{-1}) - 2\pi = 2\pi(\xi - 1)$ are indicated at the curves. The pole motion marked by arrows correspond to the growth of the coupling parameter q .

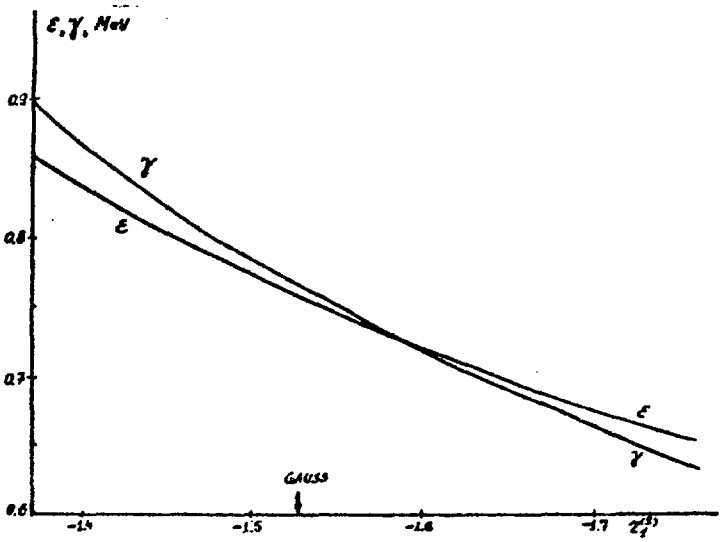

Fig.6. The binding energy $\mathcal{E}$ and the width γ of quasinuclear p-state depending on the effective range $r_1^{(s)}$. The values of $\mathcal{E}$ and γ are given in MeV's, $r_1^{(s)}$ in fm^{-1}. The arrow indicates $r_1^{(s)}$ for the gaussian potential.

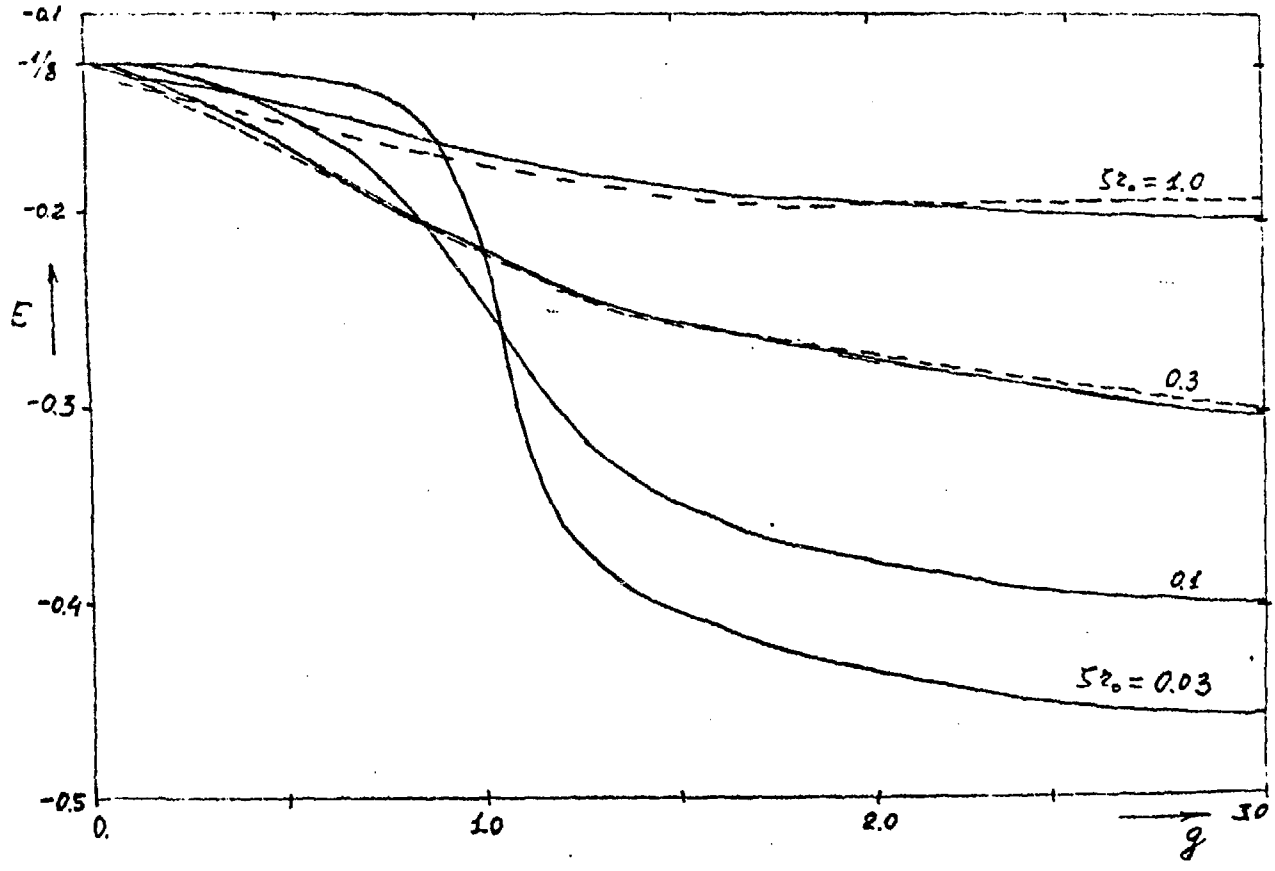

Fig.7. The Coulomb plue Yamaguchi potential. The Exact results obtained with eq.(22) are drawn by the solid lines. The results obtained with the approximate eq.(13) are drawn by the broken lines. Near the curves the ratios $r_3/a_B = 3\mu^{-1}$ are indicated.

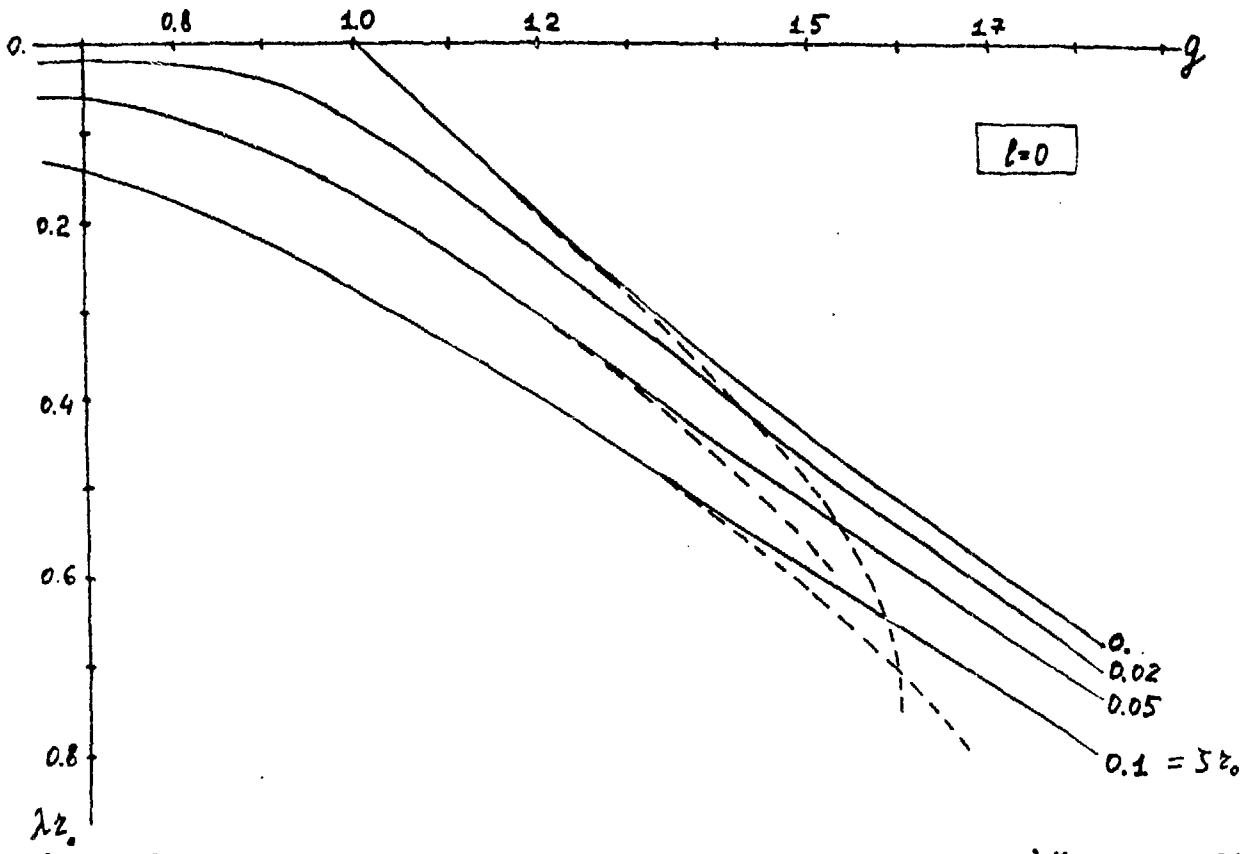

Fig.8. The Coloumb plus delta-potential. The coupling constant dependence of λz_0 . The solid curves are based on the exact eq.(23) and the broken ones on the approximate eq.(1). The values of ζz_0 are indicates at the curves.

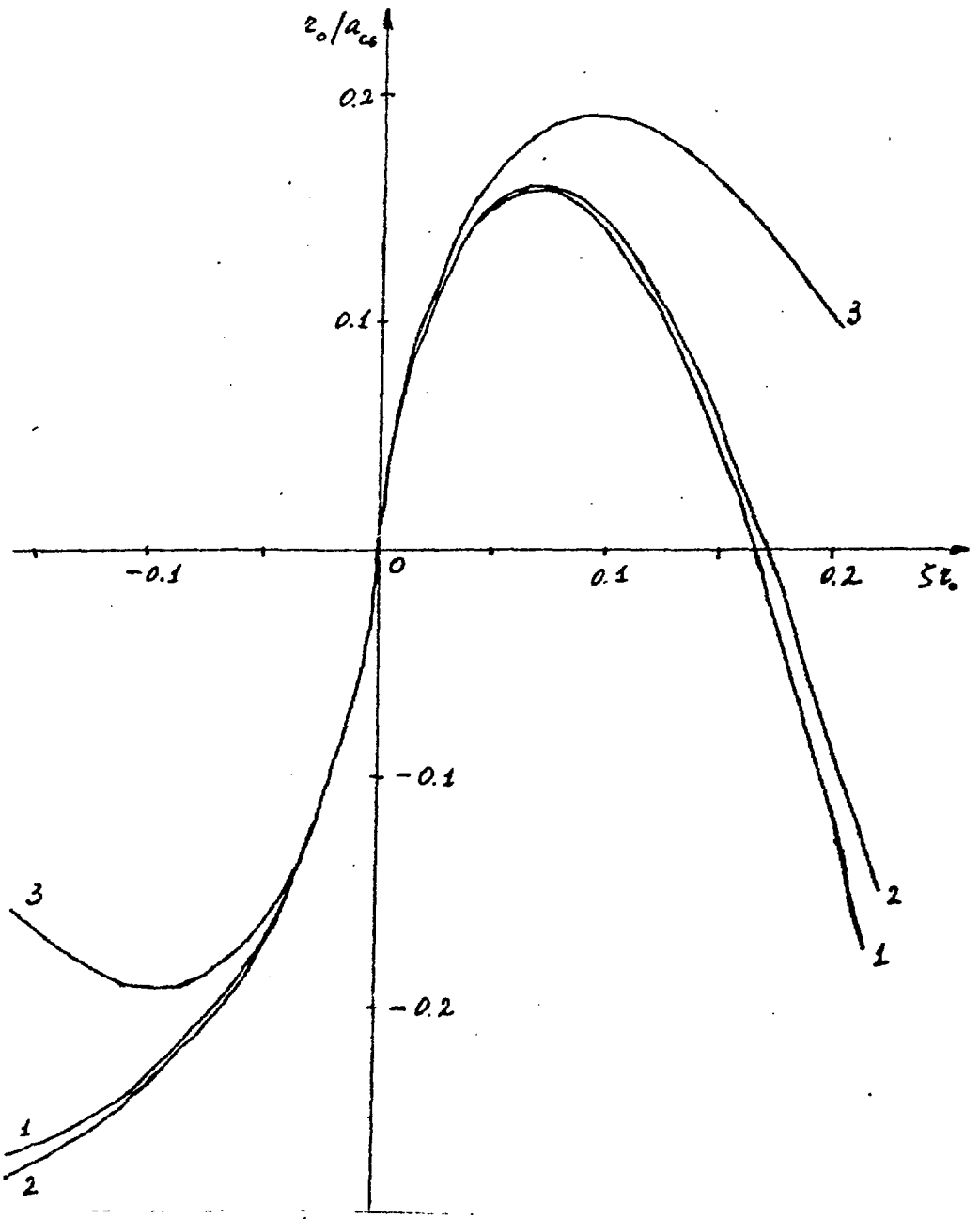

Fig.9. The ratio z_0/a_{cs} for the model described by eq.(21) with with $\ell = 0$, $g = 1$ $(a_s = \infty)$. The curves 1,2 and 3 correspond to exact solution, eq.(B,1) and the Schwinger formula (i.e., $b_1 = c_1 = 0$ in eq.(B.1)), respectively.

References

1. Zel'dovich Ya.B. // Sov.Phys.Solid State, 1959, 1, 1637.
2. Trueman T.L. //Nucl.Phys., 1961, 26, 57.
3. a) Kudryavtsev A.E., Popov V.S. //Pis'ma v ZhETF, 1979, 29,311;
   b) Popov V.S., Kudryavtsev A.E., Mur V.D. // ZhETF,1979,77,1727.
4. Popov V.S., Kudyavtsev A.E., Lisin V.I., Mur V.D. // ZhETF, 1981,80, 1271.
5. Kirzhnits D.A., Pen'kov F.M. // ZhETF, 1982,82,657.
6. Kok L.P. // Phys.Rev.Lett.,1980,45,425.
7. Deser S., Goldberger M.L., Baumann K., Thirring W. // Phys.Rev.,1954' 96,774.
8. Lambert E. // Helv.Phys.Acta, 1969, 42,667.
9. Backenstoss G. // Ann.Rev.Nucl.Sci., 1970,20,467.
10. Kudryavtsev A.E., Markushin V.E., Shapiro I.S. // ZhETF,1978,74, 431.
11. Karnakov B.M., Mur V.D., Popov V.S. // Dokl.Akad.Nauk SSSR,1984, 279,345.
12. Mur V.D., Popov V.S. // Teor.Mat.Fiz., 1976,27,204.
13. Krell M. // Phys.Rev.Lett.,1971,26,584.
14. Koch J.D., Sternheim M.M., Walker J.J. // Phys.Rev.Lett.,1971,26, 1465; Phys.Rev., 1972, C5,381.
15. Karnakov B.M., Kudryavtsev A.E., Mur V.D., Popov V.S. M.,preprint ITEP, 1985, N 158.
16. Kok L.P. // Phys.Rev., 1980, C22, 2204.
17. Kok L.P., de Maag J.W., Brouwer H.H., van Haeringen H. // Phys.Rev. 1982, C26, 2381.
18. Schwinger J. // Phys.Rev., 1947, 72, 724.
19. Popov V.S., Kudryavtsev A.E., Lisin V.I., Mur V.D. // Pis'ma v ZhETF, 1982,36, 207.

20. Mur V.D., Kudryavtsev A.E., Popov V.S. M., Preprint ITEP, 1980, N 180; Yad.Fiz., 1983, 37, 1417.

21. Izycki M., Backenstoss G. et al. // Zeits.Physik, 1980, A297, 1.

22. Ahmad S., Amsler C. et al. // Phys.Lett., 1985, 157B, 33.

23. Corringe T.P., Davies J.D. et al. // Phys.Lett., 1985, 162B, 71.

24. Brückner W. et al. // Phys.Lett., 1985, 158B, 180; Kudryavstev A.E., Markushin V.E. M., Preprint ITEP, 1985, N 179.

25. Kudryavstev A.E., Lisin V.I., Popov V.S. // Pis'ma v ZhETF, 1981, 34, 292.

26. Baird S., Batty C.J. et al. // Nucl.Phys., 1983, A392, 297.

27. Kudryavtsev A.E., Mur V.D., Popov V.S. // Phys.Lett., 1984, B143, 41.

28. Popov V.S., Kudryavstev A.E., Lisin V.I., Mur V.D. // ZhETF, 1985, 88, 715.

29. Zel'dovich Ya.B., Gerstein S.S. // Uspekhi Fiz.Nauk., 1960, 71, 581.

30. Gerstein S.S., Ponomarev L.I. // Phys.Lett., 1977, 72B, 80.

31. Vinitsky S.I., Ponomarev L.I. et al. // ZhETF, 1978, 74, 849; 1980, 79, 698.

32. Bogdanova L.N. et al. // ZhETF, 1981, 81, 829; Yad.Fiz., 1981, 34, 1191.

33. Belyaev V.B., Brener S.E., et al. // J.Phys., 1982, G8, 903.

34. Zachary W.W. // J.Math.Phys., 1971, 12, 1379; 1973, 14, 2013.

35. van Haeringen H., van der Mee C.V.M., van Wageningen R. // J.Math. Phys., 1977, 18, 941.

36. Mur V.D., Popov V.S. // Yad.Fiz., 1985, 42, 1470.

37. Demkov Yu.N., Drukarev G.F. // ZhETF, 1965, 49, 257.

38. Kerselidze T.M., Chibisov M.I. // ZhETF, 1975, 68, 12.

39. Karnakov B.M., Mur V.D. // ZhETF, 1984, 87, 1150.

40. Karaziya R.I. // Uspekhi Fiz.Nauk, 1981, 135, 79.

41. Popov V.S. // Pisma v ZhETF, 1970, 11, 254; Yad.Fiz., 1970, 12, 429.

42. Popov V.S. // ZhETF, 1971, 60, 1228.

43. Zel'dovich Ya.B., Popov V.S. // Uspekhi Fiz.Nauk., 1971, 105, 403.

44. Popov V.S., Mur V.D., Weinberg V.M. // Pis'ma v ZhETF, 1985, 41, 439; Yad.Fiz., 1986, 44, 1103.

45. Mur V.D., Popov V.S. // Pis'ma v ZhETF, 1987, 45, 323.

46. Mur V.D., Popov V.S. // Doklady Akad.Nauk SSSR, 1985, 284, 586.

47. Bethe H.A. // Phys.Rev., 1949, 76, 38.

48. Hamilton J., Overbø I., Tromborg B. // Nucl.Phys., 1973, B60, 443.

49. Burke P.G., Potential Scattering in Atomic Physics. Plenum Press, N.Y. 1977.

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