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MACHIAN EFFECTS IN GENERAL RELATIVITY[1]

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Abstract

As a consequence of Mach's principle, rotating matter should cause local inertial frames or gyroscopes in its vicinity to undergo a small rotation which is not present in the Newtonian picture. H. Thirring and J. Lense were the first to derive similar predictions from the field equations of general relativity. Since these early days of relativity, a lot of exact and approximate solutions to Einstein's equations have been examined under this point of view.

The qualitative features of Machian effects are most easily demonstrated in the cylinder symmetric case, where some exact results are available. For example, space-time is flat inside a uniformly rotating matter shell, and the rotation of this interior with respect to "infinity" (the distant stars) has a clear meaning.

In the more realistic case of what happens near a massive rotating star, one is forced to perform certain approximations. In modern language, Machian effects are described in terms of the twist of timelike Killing vector fields. In the linearized theory, the equations that determine the Machian structure generated by a given matter distribution, resemble to some extent those of classical electrodynamics. This correspondence provides a pedagogical approach how to compute the quantitative extent of inertial frame "dragging".

[1]Contribution to the conference "Ernst Mach and the Development of Science", Prague, 1988

1 Introduction

According to Mach's view of inertial forces, the meaning of "rotating" or "non-rotating" local frames is generated by the relative motion of the bulk of matter in the universe. In order to derive a definite physical effect from this statement, consider a Foucault pendulum fixed at the pole of a planet which is far away from the rest of matter (the "distant stars") and undergoes uniform rotation. In the Newtonian picture, the pendulum plane would remain non-rotating relative to absolute space. Now imagine that all the distant matter surrounding our planet would carry out a collective, cosmic motion to form a "rotating universe". Due to Mach's argumentation, such a motion should be unobservable. In other words, the pendulum plane would be "dragged" by the rotation of the stellar background: It is in fact not fixed with respect to absolute space, but rather to some average over the motion of all the matter in the universe.

The essential point is now that the planet itself is also a part of the universe. Hence, its rotation should cause a dragging of the pendulum plane, with respect to the stars. In other words, the plane follows to some extent the rotation of the planet (Fig. 1).

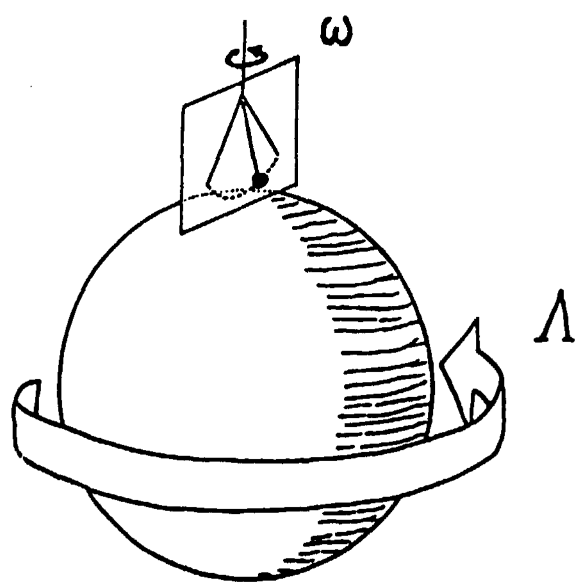

Fig. 1

Let Λ be the angular velocity of the planetary rotation and ω the induced angular velocity of the pendulum plane. Then the ratio ω/Λ is expected to be small but non-vanishing. A similar dragging effect should happen to the frame inside a rotating matter shell. The Foucault pendulum is normally replaced by torque-free gyroscopes carried along by observers. Rotating matter should then cause the angular momentum vector of such gyroscopes (and thus the orientation of local inertial frames) to suffer a certain precession which is not predicted by the Newtonian theory.

In 1918, Thirring and Lense were the first to derive such "Machian effects" as predictions from Einstein's theory of general relativity [1]. For a planet of homogeneous mass density, the numerical value for the Foucault pendulum dragging emerged as

$$\frac{\omega}{\Lambda} = \frac{4M}{5R}, \tag{1.1}$$

R being the radius and $2M \equiv 2GM/c^2$ the Schwarzschild radius. This is about $6 \cdot 10^{-10}$ for the earth. Thirring and Lense also exhibited the Machian modification of planetary orbits in the gravitational field of a rotating central star and the influence of a spherical rotating matter shell onto its interior.

Meanwhile, a lot of physical situations in general relativity have been studied with the same intention. However, since the gravitational field generated by a rotating star is not known to arbitrarily high accuracy, most of these computations involve approximations of weak fields, slow motion or special types of matter. The problem of exhibiting the Machian effects caused by a given matter distribution is intimately related to the generation of the gravitational field, i.e. to the mathematical properties of Einstein's field equations. I just mention the work of Brill and Cohen [2] who pushed forward the accuracy of the results and studied the connections between Machian effects and the formation of horizons. One of their results states that the interior frame of a matter shell is "perfectly dragged" (i.e. it rotates with the same angular velocity as the shell itself) if the radius approaches the Schwarzschild radius. This model has been used to illustrate Mach's principle, the rotating shell representing the distant matter.

Most of the studies of Machian effects (as opposed to "Mach's principle") in general relativity are concerned with isolated rotating objects, the "distant matter" being represented by appropriate boundary conditions at spatial infinity (e.g. asymptotic flatness). Thus, the existence of Machian effects does *not* mean that Mach's principle as a whole is automatically incorporated in general relativity. In some sense, the full "principle" is dealing with the bulk of matter in the universe and not with isolated objects. Nevertheless, Machian effects are not purely local phenomena but emerge from a certain interplay between (local) field generation and boundary conditions.

Here, I would like to show how Machian effects arise in general relativity. First, we will consider the cylinder symmetric case which sheds some light on the more global aspect. As the second point, I will write down the local equations governing Machian effects in some approximation and point out a surprising analogy with electromagnetism.

2 Global Aspects of Machian Effects: a Cylinder Symmetric Example

The domain in which Machian effects are computed most easily and where the most exact results are known is that of stationary, cylinder symmetric space-times. These situations are characterized by the existence of three Killing vector fields associated with the coordinates $(t, z, \varphi) \equiv (x^0, x^2, x^3)$. The metric components and matter variables depend only on the outward directed cylinder coordinate $r \equiv x^1$, thus giving rise to ordinary differential equations ins'ead of partial ones. The general form of the line element is given by

$$ds^2 = -A^2dt^2 + B^2(dr^2 + dz^2) + E^2(d\varphi - \Omega dt)^2, \tag{2.1}$$

A, B, E and Ω being functions of r that are subject to certain physically reasonable boundary conditions near the symmetry axis ($r = 0$) and at spatial infinity ($r \to \infty$), especially

$$\lim_{r \to 0} \Omega = 0. \tag{2.2}$$

This ensures that sufficiently far from the axis, the metric becomes manifestly static, thus providing a region of space-time one may associate with "distant stars", or just a measure for "rotation with respect to infinity".

In order to discuss a concrete example, we consider the case of an infinitely thin matter shell, located at $r = R$. In other words, the energy momentum tensor

$$T^\mu{}_\nu = \rho u^\mu u_\nu + p_r \delta^\mu_1 \delta^1_\nu + p_z \delta^\mu_2 \delta^2_\nu + p_\varphi w^\mu w_\nu \tag{2.3}$$

is concentrated on an infinitely long and thin rotating hollow cylinder. Here,

$$u^\mu = (u^0, 0, 0, u^3), \qquad u^\mu u_\mu = 1 \tag{2.4}$$

is the matter four-velocity, w^μ being defined by

$$w^\mu = (w^0, 0, 0, w^3), \qquad u^\mu w_\mu = 0, \qquad w^\mu w_\mu = 1. \tag{2.5}$$

ρ is the energy density as measured from the rest frame, and p_i are the respective stresses, all these four variables being proportional to $\delta(r - R)$. (2.1) and (2.3) constitute the most general expressions for geometry and matter, u^μ containing only one degree of freedom which may be associated with the angular velocity

$$\Lambda = \frac{d\varphi}{dt}\Big|_{matter\ shell} = \frac{u^3}{u^0} \tag{2.6}$$

of the hollow cylinder.

These ingredients have to be fed into Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{2.7}$$

which reduce the total number of physical parameters to four, for example chosen as R, Λ, ρ, p_z, all other quantities being expressed through these [3]. One important consequence of (2.7) is that there is no stress in the outward direction,

$$p_r = 0. \tag{2.8}$$

The explicit expressions are rather lengthy, and we will not write them all down because we are only interested in Machian effects.

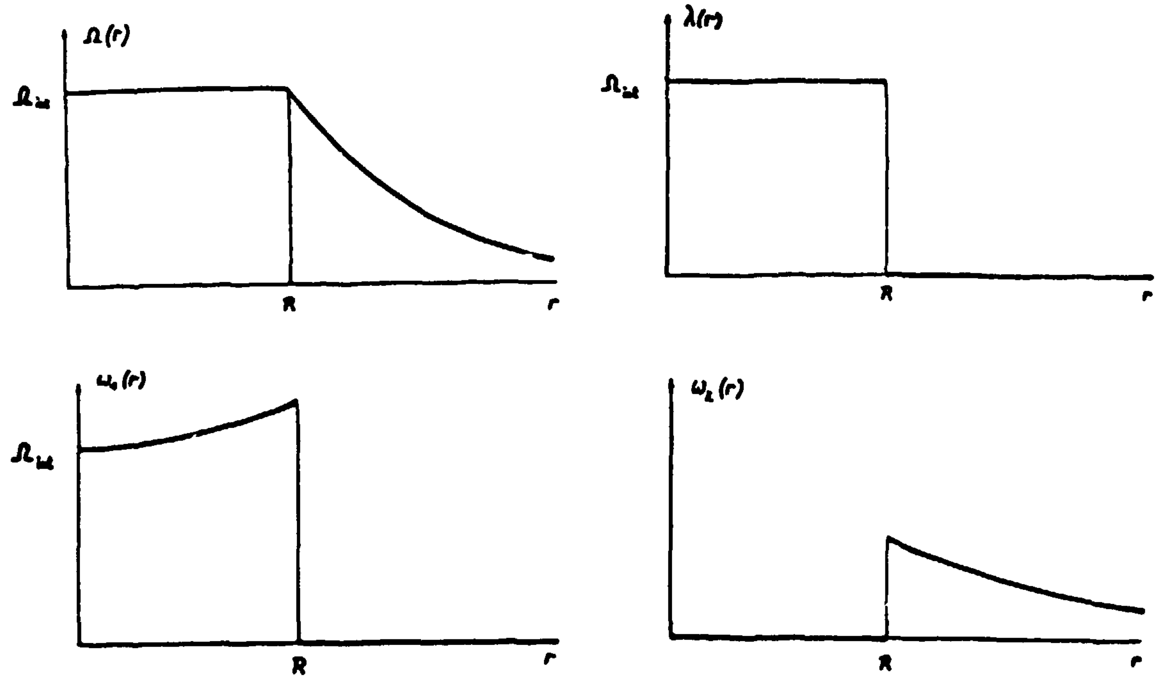

Fig. 2

The metric in the exterior region $r > R$ turns out to be equivalent to the so-called Kasner-metric [4]. Although one obtains a non-trivial expression for $\Omega(r)$, one may choose other coordinates such that the exterior region looks completely *static*. However, when applied globally, such a coordinate transformation would destroy the boundary conditions at the axis. Nevertheless, from the physical point of view, the exterior part of space-time *is* static, and it is not possible to encounter an influence of the rotation by local measurements. This feature occurs only in the cylinder symmetric case, but since we are discussing the global aspects, it is not really a drawback of our model. (Another difference to more realistic situations is the absence of event horizons.) Summarizing we may state that there are *no purely local Machian effects outside the shell.*

The geometry of the interior region $r < R$ may even be more surprising at the first sight: it is *flat*. However, it appears in the form

$$ds^2_{int} = -dt^2 + dr^2 + dz^2 + (d\varphi - \Omega_{int} dt)^2 \tag{2.9}$$

with constant Ω_{int}. It takes a manifest flat form only after the redefinition

$$\varphi_{new} = \varphi - \Omega_{int} t \tag{2.10}$$

which would violate the boundary condition (2.2) if applied globally. The behaviour of $\Omega(r)$ as predicted by general relativity is drawn in Fig. 2. Again, the *absence of purely local Machian effects inside the shell* is clear.

The Machian character of our space-time shows up only when the way these two static regions are glued together is taken into account. The easiest way to illustrate the physical effect of frame dragging is to consider an observer, sitting somewhere in the interior inertial frame, and looking at the "distant stars". The light rays coming from objects at spatial infinity are well defined. The observer in the interior will notice a rotation of the whole celestial pattern with an angular velocity $-\Omega_{int}$. On the other hand, the high symmetry of space-time enables one to state that *the interior flat region is dragged* by the matter shell, and it *undergoes a rotation relative to the distant stars.* The quantity which measures the extent of the effect is the dragging coefficient

$$\chi = \frac{\Omega_{int}}{\Lambda} \tag{2.11}$$

whose absolute value never exceeds 1 if appropriate energy conditions (e.g. $|\rho| > |p_i|$) are imposed. By means of Einstein's equations, χ may be expressed through the four free physical parameters.

Let us write down the values of χ for some physically interesting situations [3]. In the limit of *slow motion* ($\Lambda \approx 0$), one obtains

$$\chi_{slow} = 8\pi \int (T_3^3 - T_0^0)\sqrt{-g}\, dr. \tag{2.12}$$

Here, the dragging is quite nicely expressed by the covariant matter variables. The definite values depend on the kind of matter considered. One immediately sees that the energy density ρ as well as the pressure p_φ enter (2.12). In the limit of *fast rotation* (i.e. the velocity v of the matter shell as measured from the flat inertial frame inside approaches 1), we find

$$\chi_{fast} = \frac{1}{2}\chi_{slow} \tag{2.13}$$

which shows that the relative dragging is stronger for a slowly rotating shell. If the shell velocity v is given, the inequality

$$\chi \leq \frac{1}{1+v} \tag{2.14}$$

provides a general upper bound. (2.14) turns into an equality for a very heavy matter shell. In the limit of vanishing mass, χ goes to zero. In order to give an example for a specific type of matter, we consider a rotating shell of dust (i.e. $p_i = 0$). Here, v is constrained to be smaller than 1/3 (as a consequence of the balance between centripetal and gravitational forces), and

$$\chi_{dust} = -8\pi \int T_0^0 \sqrt{-g}\, dr = 4 \times \text{total energy per unit length in } z\text{-direction} < 3/4. \tag{2.15}$$

One may invent other ways to describe the Machian effects quantitatively. For example, for any r, consider an observer moving on an orbit

$$\varphi(t) = \lambda t \tag{2.16}$$

around the axis. Let him orient a telescope always towards the symmetry axis (in order to fix a kinematical frame). Then the angular momentum of a gyroscope carried along by him will undergo a uniform precession with angular velocity ω, as measured with respect to the kinematical frame. $\omega = 0$ is attained only for a special value of λ: This defines a family of moving observers, characterized by a function $\lambda(r)$ (Fig. 2). On the other hand, if we put $\lambda = 0$ ("stationary observer"), we find a dragging frequency $\omega_1(r)$ (Fig. 2), and for a family of free falling observers ($\lambda = \lambda_{geodesic}$), one gets a function $\omega_2(r)$ (Fig. 2). The last two quantities show that "true" Machian effects and the pure influence of the observer's motion (being already present in globally flat Minkowski space) are closely related. All three functions λ, ω_1 and ω_2 are only meaningful within a global point of view, since we have already stated that purely local Machian effects do not exist in this model (as long as one does not try to describe what happens *at* the points $r = R$, i.e. "in" the shell).

However, if cylindrical symmetry is loosened, such purely local effects emerge. They are described in terms of the so-called Thirring-Lense-frequency.

3 Local Aspects of Machian Effects: Weak Fields and Slow Motion – a Pedagogical Picture

A general stationary space-time is characterized by the existence of a timelike Killing vector field ξ^μ. A convenient way to write down such a line element is

$$ds^2 = -f(dt - g_i dx^i)^2 + \gamma_{ij} dx^i dx^j, \tag{3.1}$$

where

$$\xi^\mu = \delta_0^\mu \tag{3.2}$$

and all functions are time-independent. The Thirring-Lense-frequency (or "twist" of ξ^μ) is the four-vector field defined by [5]

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\xi_\nu \nabla_\rho \xi_\sigma. \tag{3.3}$$

In terms of (3.1), its spatial components are given by

$$\omega^i = -\frac{1}{2}\varepsilon^{ijk}\partial_j g_k. \tag{3.4}$$

The physical meaning of ω is as follows: Consider a freely falling observer, carrying a gyroscope along his world line. The gyroscope's angular momentum $\vec{S}$ will precede relative to the kinematical frame defined by the Killing vector field ("stationary" frame), the angular velocity of this precession just being given by $\vec{\omega}$. This situation is illustrated in Fig. 3. If the gravitational field is weak, $\vec{\omega}$ applies to related phenomena (e.g. Foucault pendulum dragging somewhere on the surface of a planet which is not an inertial frame) as well.

If ω^μ vanishes, the Killing vector field is proportional to a gradient (it is then said to be "hypersurface orthogonal"), and space-time is static.

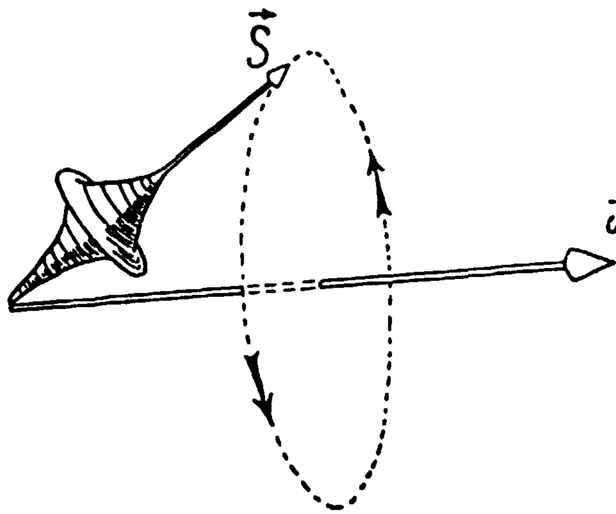

Fig. 3

As one sees from (3.4), $\vec{\omega}$ is connected to the g_{0i} components of the metric. For reasons we will encounter in a minute, $\vec{\omega}$ (or g_{oi}) are said to describe the "magnetic" aspects of the gravitational field.

We will now write down part of the Einstein field equations in a certain approximation. First we restrict the *gravitational field to be weak* and neglect all terms quadratic or of higher order in $g_{\mu\nu} - \eta_{\mu\nu}$. Moreover, the equations governing the dynamics of γ_{ij} are omitted because they do not contribute to the Machian effects we are interested in (this means of course to abandon certain degrees of freedom of the gravitational field). The gravitational field strength is defined by

$$G_i = -\frac{1}{2}\,\partial_i f \tag{3.5}$$

and is responsible for the attractive $1/r^2$ Newtonian force. In other words, we keep only the Newtonian and Machian parts of the metric to the first nontrivial order. To complete the approximation, we restrict the matter generating the gravitational field to be described by a mass density ρ and a matter velocity field $\vec{v}$, whose absolute value is small as compared to 1.

Under these assumptions, the relevant part of Einstein's field equations reduces to [6]

$$\begin{aligned} \mathrm{rot}\,\vec{G} &= 0 \qquad & \mathrm{div}\,\vec{\omega} &= 0 \\ \mathrm{div}\,\vec{G} &= -4\pi\rho \qquad & \mathrm{rot}\,\vec{\omega} &= 8\pi\rho\vec{v}. \end{aligned} \tag{3.6}$$

The $\vec{G}$-equations are well known from Newtonian gravity. For a point-like source ρ, they just reproduce the $1/r^2$ central force field. By comparison with the time-independent Maxwell equations, one sees that the way $\vec{\omega}$ enters (3.6) resembles a magnetic field.

In order to complete our set of equations, we consider a test particle with mass m. Its motion is described by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0. \tag{3.7}$$

Under the assumption that its velocity $\vec{u} = d\vec{x}/dt$ is small as compared to 1, and inserting the approximations already made, (3.7) reduces to the non-relativistic equation of motion

$$m\frac{d\vec{u}}{dt} = m(\vec{G} - 2\vec{u} \times \vec{\omega}), \tag{3.8}$$

which is clearly of the Lorentz force type with $\vec{G}$ and $\vec{\omega}$ corresponding (up to numerical factors) to the electric and the magnetic field.

Equipped with these equations, we are now able

(i) to determine the gravitational field $(\vec{G}, \vec{\omega})$ from a given matter distribution $(\rho, \vec{v})$

(ii) to predict frame dragging by means of $\vec{\omega}$ (cf. Fig. 3) and

(iii) to predict test particle motion.

In order to push the formal analogy to electromagnetism even further, we define the following quantities:

$$\begin{aligned} \mu &= \rho & \vec{E} &= -\vec{G} & \\ \tau &= \tfrac{1}{2}t & \vec{B} &= \vec{\omega} & e &= -4m. \end{aligned} \tag{3.9}$$

Since we have introduced a new time variable, we must also transform the velocities of matter and test particles according to

$$\begin{aligned} \vec{V} = \frac{d\vec{x}}{d\tau}\Big|_{matter} &= 2\frac{d\vec{x}}{dt}\Big|_{matter} = 2\vec{v} \\ \vec{U} = \frac{d\vec{x}}{d\tau}\Big|_{test\ particle} &= 2\vec{u}, \qquad \frac{d\vec{U}}{d\tau} = 4\frac{d\vec{u}}{dt}. \end{aligned} \tag{3.10}$$

Re-expressing the field equations and the test particle equation of motion, we obtain

$$\begin{aligned} \text{rot}\,\vec{E} &= 0 & \text{div}\,\vec{B} &= 0 \\ \text{div}\,\vec{E} &= 4\pi\mu & \text{rot}\,\vec{B} &= 4\pi\mu\vec{V} \equiv 4\pi\vec{j} \end{aligned} \tag{3.11}$$

$$m\frac{d\vec{U}}{d\tau} = e(\vec{E} + \vec{U} \times \vec{B}). \tag{3.12}$$

But these *are* the time-independent Maxwell equations with $(\mu, \vec{j})$ as sources, and the correct Lorentz force law for a test particle with mass m and electric charge e.

Thus, within the approximation considered, we have a complete analogy between Newtonian/Machian aspects of the gravitational field with the electric/magnetic parts

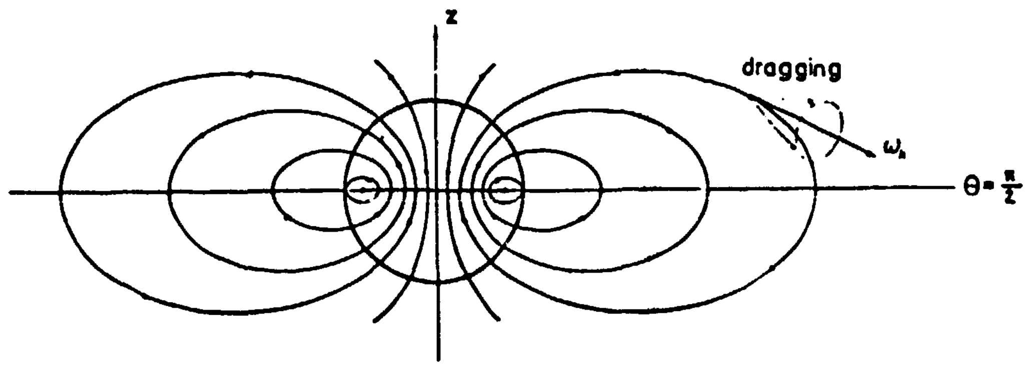

Fig. 4

of time-independent Maxwell's theory. The advantage of this analogy is that Maxwell's equations are accessible to a much larger group of physicists than Einstein's equations, and that physical imagination (which is useful to guess the solution of problems) in electromagnetism is quite familiar to many people. For example, in this picture one may just think of magnetic flow lines and Ampère's law in order to compute the Machian effects near a rotating matter distribution. However, the analogy is a formal one, the factors 2 and 4 in (3.9) arising from the tensorial character of gravity, the minus signs from its attractive nature.

As an example, consider a uniformly rotating spherical star with radius R, homogeneous mass density ρ and angular velocity Λ. The magnetic field generated by the analogous electric current distribution is well known (and has surely often been computed by students), the result for $\vec{\omega}$ being (in spherical coordinates)

$$\begin{aligned} \omega_r &= 8\pi\rho\Lambda(\frac{1}{3}R^2 - \frac{1}{5}r^2)\cos\theta \\ \omega_\theta &= 8\pi\rho\Lambda(\frac{2}{5}r^2 - \frac{1}{3}R^2)\sin\theta \\ \omega_\varphi &= 0 \end{aligned} \tag{3.13}$$

inside the star ($r < R$) and

$$\begin{aligned} \omega_r &= \frac{16\pi}{15}\rho\Lambda\frac{R^5}{r^3}\cos\theta \\ \omega_\theta &= \frac{8\pi}{15}\rho\Lambda\frac{R^5}{r^3}\sin\theta \\ \omega_\varphi &= 0 \end{aligned} \tag{3.14}$$

outside. The flow lines are drawn in Fig. 4.

The dragging coefficient $|\vec{\omega}|/\Lambda$ is now a function of $\vec{x}$. Near the equator, one finds $2M/5R$ (with antiparallel dragging), at the poles $4M/5R$ (cf. equation (1.1)). The best dragging occurs in the center at $r = 0$, where we get

$$\frac{2M}{R} \equiv \frac{\text{Schwarzschild radius}}{\text{radius}}.$$

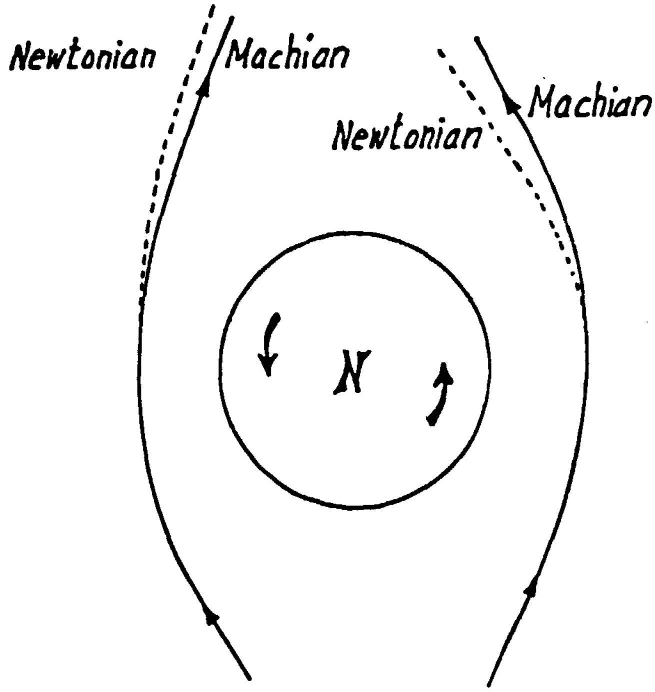

Fig. 5

The equations of motion of a test particle (e.g. a planet) are given by (3.8). From the magnetic character of $\vec{\omega}$ one may immediately estimate the effect of the star's rotation upon a particle passing near by (analogous to the light deflection, Fig. 5). Including the Newtonian field

$$G_r = -\frac{M}{r^2} \tag{3.15}$$

in the exterior region, the equations of motion (3.8), when written down explicitly, become

$$\begin{aligned} \ddot{x} &= \frac{4}{5} M\Lambda \frac{R^2}{r^3}[\dot{y}(1-\frac{3z^2}{r^2}) + 3\dot{z}\frac{yz}{r^2}] - \frac{Mx}{r^3} \\ \ddot{y} &= -\frac{4}{5} M\Lambda \frac{R^2}{r^3}[\dot{x}(1-\frac{3z^2}{r^2}) + 3\dot{z}\frac{xz}{r^2}] - \frac{My}{r^3} \\ \ddot{z} &= \frac{12}{5} M\Lambda \frac{R^2}{r^3}[\dot{y}\frac{xz}{r^2} - \dot{x}\frac{yz}{r^2}] - \frac{Mz}{r^3}. \end{aligned} \tag{3.16}$$

This is exactly the result obtained by Thirring and Lense (in a more complicated derivation).

Pictures like the one presented here might be helpful in attempts to popularize certain fields of theoretical physics which otherwise remain closed for the majority of interested people.

References

[1] H. Thirring, Phys. Z. 19, 33 (1918)
H. Thirring, Phys. Z. 22, 29 (1921)
J. Lense and H. Thirring, Phys. Z. 19, 156 (1918)

[2] D.R. Brill and J.M. Cohen, Phys. Rev. 143, 1011 (1966)
J.M. Cohen and D.R. Brill, Nuovo Cimento 56, 206 (1968)

[3] F. Embacher, J. Math. Phys. 24, 1182 (1983)

[4] M.M. Som, A.F.F. Teixeira and I. Wolk, Gen. Relativ. Gravit. 7, 263 (1967)
E. Frehland, Comm. Math. Phys. 23, 127 (1971)
E. Frehland, Comm. Math. Phys. 26, 307 (1972)

[5] R. Geroch, J. Math. Phys. 12, 918 (1971)

[6] F. Embacher, Found. Phys. 14, 721 (1984)