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N SOLITON SOLUTION FOR A CLASS OF THE SYSTEM OF LS NONLINEAR WAVE INTERACTION *

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ABSTRACT

In this note, the explicit form of the N soliton solutions for a class of the system of LS nonlinear wave interaction have been obtained by using Hirota's method.

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1. INTRODUCTION

Djordjevie and Redekopp [1] found that the evolution equation describing the resonance interaction between the long wave and short wave can be written as

$$i S_t + \lambda S_{xx} = LS \qquad L_t = -\alpha (|S|^2)_x \tag{1.1}$$

In the above equations, S is the envelope of the short wave, while L is the amplitude of long wave and real. λ and α are positive constants. As pointed in [1], the physical significance of equations (1.1) is such that the dispersion of the short wave is balanced by nonlinear interaction of the long wave with the short wave, while the evolution of the long wave is driven by the self interaction of the short wave. These equations also appear in an analysis of internal wave [2], as well as the Rossby wave. The existence and uniqueness of the global solution for the initial value problem and periodic initial value problem of the system (1.1) was proved in [3].

Recently, it is shown that the multi-soliton solutions for some nonlinear evolution equations can be obtained by many methods: the inverse scattering method, Backlund Transformations, and Hirota's method. Much work [4-8] have been done by using the direct method. The basic ideas in this direct method are as follows: Introduce a dependent variable transformation, the transformation should reduce the evolution equation to a so-called bilinear equation, quadratic in the dependent variables. Then introduce a formal perturbation expansion into this bilinear equation. In the case of soliton solutions, the expansion truncates.

In this notes, we have obtained N soliton solution for the system (1.1) by using Hirota's method.

2. N SOLITON SOLUTIONS FOR THE SYSTEM (1.1)

For simplicity, let $s \to \frac{\lambda^{\frac{1}{4}}}{\sqrt{\alpha}} u$, $L \longrightarrow v$, $x \to \sqrt{\lambda} x$, $t \longrightarrow t$. The system (1.1) can be written as

$$\begin{cases} i u_t + u_{xx} - v u = 0 \\ v_t + (|u|^2)_x = 0 \end{cases} \tag{2.1}$$

Suppose $u, v \to 0$, as $|x| \to \infty$.

Let us introduce some notations of "bilinear D operator" [7]:

$$D_x^n D_t^m a(x,t) \cdot b(x,t) = (\partial_x - \partial_{x'})^n (\partial_t - \partial_{t'})^m a(x,t)\, b(x',t') \Big|_{\substack{x'=x \\ t'=t}} \tag{2.2}$$

where $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_t \equiv \frac{\partial}{\partial t}$. From the definition (2.2) of D operators, the following formulas can be obtained easily:

$$D_x^n D_t^m a(x,t) \cdot 1 = \partial_x^n \partial_t^m a(x,t) \tag{2.3}$$

$$D_x^n a(x,t) \cdot b(x,t) = (-1)^n D_x^n b(x,t) \cdot a(x,t) \tag{2.4}$$

$$D_x^n D_t^m e^{(p_1 x + w_1 t)} \cdot e^{(p_2 x + w_2 t)} = (p_1 - p_2)^n (w_1 - w_2)^m \exp[(p_1 + p_2)x + (w_1 + w_2)t] \tag{2.5}$$

and

$$F(D_t, D_x)\, e^{(p_1 x + w_1 t)} \cdot e^{(p_2 x + w_2 t)} = \frac{F(w_1 - w_2, p_1 - p_2)}{F(w_1 + w_2, p_1 + p_2)} F(D_t, D_x)\, e^{(p_1 + p_2)x + (w_1 + w_2)t} \tag{2.6}$$

where $F(D_t, D_x)$ is a polynomial of operator D_t, D_x, and

$F(0,0) = 0$.

In (2.1), let

$$u = g/f, \quad v = -2(\ln f)_{xx} \tag{2.7}$$

Here f is a real valued function, g is a complex valued function. Substituting the transformations (2.7) into the system (2.1) and applying the operator notations (2.2), the system (2,1) can be written as follows:

$$\begin{cases} f^2 [i D_t + D_x^2] g \cdot f = 0 \\ D_t D_x f \cdot f = g g^* \end{cases} \tag{2.8}$$

By using the perturbation method, we expand f, g as a perturbation series of a small parameter ε:

$$\begin{aligned} f &= 1 + \varepsilon^2 f_1 + \varepsilon^4 f_2 + \varepsilon^6 f_3 + \cdots \\ g &= \varepsilon g_1 + \varepsilon^3 g_2 + \varepsilon^5 g_3 + \cdots \end{aligned} \tag{2.9}$$

Let

$$F(D_t, D_x) = D_t D_x, \quad G(D_t, D_x) = i D_t + D_x^2. \tag{2.10}$$

Inserting (2.9) into (2.8), and equating power of ε yields a system of linear partial differential equations of f_k, g_k to be solved.

$$G(D_t, D_x)\, g_k \cdot 1 = -G(D_t, D_x) \sum_{j=1}^{k-1} g_{k-j} \cdot f_j, \quad k=1,2,\cdots$$
$$F(D_t, D_x)[f_k \cdot 1 + 1 \cdot f_k] = \sum_{j=1}^{k} g_{k-j+1} g_j^* - F(D_t, D_x) \sum_{j=1}^{k-1} f_{k-j} \cdot f_j \tag{2.11}$$

i.e.,

$$\begin{cases} G(D_t, D_x)\, g_1 \cdot 1 = 0 & (2.12) \\ F(D_t, D_x)[f_1 \cdot 1 + 1 \cdot f_1] = g_1 \cdot g_1^* & (2.13) \end{cases}$$

$$\begin{cases} G(D_t, D_x)\, g_2 \cdot 1 = -G(D_t, D_x)\, g_1 \cdot f_1 & (2.14) \\ F(D_t, D_x)[f_2 \cdot 1 + 1 \cdot f_2] = g_2 g_1^* + g_1 g_2^* - F(D_t, D_x) f_1 \cdot f_1 & (2.15) \end{cases}$$

$$\begin{cases} G(D_t, D_x)\, g_3 \cdot 1 = -G(D_t, D_x)(g_2 \cdot f_1 + g_1 \cdot f_2) & (2.16) \\ F(D_t, D_x)[f_3 \cdot 1 + 1 \cdot f_3] = g_3 \cdot g_1^* + g_2 g_2^* - F(D_t, D_x)[f_2 \cdot f_1 + f_1 \cdot f_2] & (2.17) \end{cases}$$

Firstly, let us consider a single soliton solution of the system (2.1), N=1. From (2.3) the equation (2.12) can be written as follows:

$$(i\partial_t + \partial_x^2)\, g = 0. \tag{2.18}$$

Now take the solution equation (2.18) in the simple form

$$g_1 = e^{\eta_1}, \quad \eta_1 = p_1 x + w_1 t + \eta_1^{(0)}, \quad w_1 = i p_1^2, \tag{2.19}$$

where p_1 and $\eta_1^{(0)}$ are constants.

Substituting g into (2.13), and using (2.4)(2.6), we have

$$2F(D_t, D_x) f_1 \cdot 1 = e^{\eta_1 + \eta_1^*} = [F(w_1 + w_1^*, p_1 + p_1^*)]^{-1} F(D_t, D_x)\, e^{\eta_1 + \eta_1^*} \cdot 1$$

So

$$f_1 = [2F(w_1 + w_1^*, p_1 + p_1^*)]^{-1} e^{\eta_1 + \eta_1^*} = [2(p_1 + p_1^*)(i p_1^2 - i p_1^{*2})]^{-1} e^{\eta_1 + \eta_1^*}$$
$$\triangleq A_{11}\, e^{\eta_1 + \eta_1^*} \tag{2.20}$$

where w_1^*, p_1^* and η_1^* are the complex conjugations of w_1, p_1 and η_1 respectively. Inserting g_1 and f_1 into (2.14), and using (2.6), it follows

$$G(D_t, D_x)\, g_2 \cdot 1 = -G(D_t, D_x)\, g_1 \cdot f_1 = -G(D_t, D_x)\, e^{\eta_1} \cdot A_{11} e^{\eta_1 + \eta_1^*}$$
$$= -A_{11}\, G(-w_1^*, -p_1^*) / G(2w_1 + w_1^*, 2p_1 + p_1^*)\, G(D_t, D_x)\, e^{2\eta_1 + \eta_1^*} \cdot 1$$

Since $G(-w_1^*, -p_1^*) = 0$, it follows

$$G(D_t, D_x)\, g_2 \cdot 1 = 0.$$

We take $g_2 = 0$. Similarly, substituting g_1 and f_1 into (2.15), we have

$$2F(D_t, D_x) f_2 \cdot 1 = g_2 g_1^* + g_1 g_2^* - F(D_t, D_x) f_1 \cdot f_1 = -F(D_t, D_x) A_{11} e^{\eta_1 + \eta_1^*} \cdot A_{11} e^{\eta_1 + \eta_1^*} = 0.$$

We can also take $f_2=0$ .From the system (2.12) analogously we have

$$f_k = g_k = 0 \quad (k \geq 2) \tag{2.21}$$

Let ε =1, substituting (2.19)(2.20) and (2.21) into (2.9), it follows

$$f = 1 + A_{11} e^{\eta_1+\eta_1^*}, \quad g = e^{\eta_1}, \quad A_{11} = [2(p_1+p_1^*)(ip_1^2-ip_1^{*2})]^{-1}. \tag{2.22}$$

Hence the explicit form of the one-soliton solution of the system (2.1)can be found

$$u = g/f = \frac{1}{2} \operatorname{Sech} \frac{1}{2}[(p_1+p_1^*)x + i(p_1^2-p_1^{*2})t + (\eta_1^{(0)}+\eta_1^{(0)*}) + \varphi_{11}] \cdot$$

$$\cdot \exp \frac{1}{2}[(p_1-p_1^*)x + i(p_1^2+p_1^{*2})t + (\eta_1^{(0)}-\eta_1^{(0)*}) - \varphi_{11}]$$

$$v = -2(\ln f)_x = -\frac{1}{2}(p_1+p_1^*)^2 \operatorname{Sech}^2 \frac{1}{2}[(p_1+p_1^*)x + i(p_1^2-p_1^{*2})t + (\eta_1^{(0)}+\eta_1^{(0)*}) + \varphi_{11}]$$

$$\varphi_{11} = \ln A_{11} = \ln [2(p_1+p_1^*)(ip_1^2-ip_1^{*2})]^{-1}.$$

Now let us consider the case: N=2. Taking a soliton solution of equation (2.13) as follows

$$g_1 = e^{\eta_1} + e^{\eta_2} \tag{2.23}$$

where

$$\eta_k = p_k x + w_k t + \eta_k^{(0)}, \quad w_k = i p_k^2, \quad k=1,2.$$

Inserting (2.23) into (2.14),we have

$$2F(D_t, D_x) f_1 \cdot 1 = (e^{\eta_1}+e^{\eta_2})(e^{\eta_1^*}+e^{\eta_2^*})$$

It follows

$$f_1 = A_{11} e^{\eta_1+\eta_1^*} + A_{12} e^{\eta_1+\eta_2^*} + A_{21} e^{\eta_2+\eta_1^*} + A_{22} e^{\eta_2+\eta_2^*} \tag{2.24}$$

where

$$A_{jk} = [2(p_j+p_k^*)(ip_j^2-ip_k^{*2})]^{-1}, \quad j,k=1,2. \tag{2.25}$$

By using the formula (2.6) in the equation (2.15),and noticing $G(-w_k^*, -p_k^*)=0$ (k=1,2),we have

$$G(D_t, D_x)\, g_2 \cdot 1 = -G(D_t, D_x)\, g_1 \cdot f_1 = -G(D_t, D_x)\,(e^{\eta_1} + e^{\eta_2})$$

$$\cdot\,(A_{11} e^{\eta_1+\eta_1^*} + A_{12} e^{\eta_1+\eta_2^*} + A_{21} e^{\eta_2+\eta_1^*} + A_{22} e^{\eta_2+\eta_2^*})$$

$$= -\left[A_{21} \frac{G(w_1 - w_2 - w_1^*,\ p_1 - p_2 - p_1^*)}{G(w_1 + w_2 + w_1^*,\ p_1 + p_2 + p_1^*)} + A_{11} \frac{G(w_2 - w_1 - w_1^*,\ p_2 - p_1 - p_1^*)}{G(w_2 + w_1 + w_1^*,\ p_2 + p_1 + p_1^*)} \right] \cdot$$

$$G(D_t, D_x)\, e^{\eta_1+\eta_2^*+\eta_1^*} \cdot 1 - \left[A_{22} \frac{G(w_1 - w_2 - w_2^*,\ p_1 - p_2 - p_2^*)}{G(w_1 + w_2 + w_2^*,\ p_1 + p_2 + p_2^*)} + A_{12} \cdot \right.$$

$$\left. \frac{G(w_2 - w_1 - w_2^*,\ p_2 - p_1 + p_2^*)}{G(w_2 + w_1 + w_2^*,\ p_2 + p_1 + p_2^*)} \right] G(D_t, D_x)\, e^{\eta_1+\eta_2+\eta_2^*} \cdot 1$$

$$= 2(p_2 - p_1)(i p_1^2 - i p_1^{*2}) \cdot$$

$$[A_{11} A_{21} G(D_t, D_x)\, e^{\eta_1+\eta_2+\eta_1^*} \cdot 1 + A_{12} A_{22} G(D_t, D_x)\, e^{\eta_1+\eta_2+\eta_2^*} \cdot 1]$$

i.e.,

$$G(D_t, D_x)\, g_2 \cdot 1 = B_{12} A_{11} A_{21} G(D_t, D_x)\, e^{\eta_1+\eta_2+\eta_1^*} \cdot 1 + G(D_t, D_x)\, e^{\eta_1+\eta_2+\eta_2^*} \cdot 1 \qquad (2.26)$$

$$B_{12} \triangleq 2(p_2 - p_1)(i p_2^2 - i p_1^2).$$

From (2.26) it follows

$$g_2 = B_{12} A_{11} A_{21} e^{\eta_1+\eta_2+\eta_1^*} + B_{12} A_{12} A_{22} e^{\eta_1+\eta_2+\eta_2^*} \qquad (2.27)$$

Substituting (2.23)(2.24) and (2.27) into (2.15),it follows

$$2F(D_t, D_x)\, f_2 \cdot 1 = g_2^* g_1 + g_1^* g_2 = -F(D_t, D_x)\, f_1 \cdot f_1$$

$$= B_{12} A_{11} A_{21} (e^{\eta_1+\eta_2+2\eta_1^*} + e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}) + B_{12} A_{12} A_{22} (e^{\eta_1+\eta_2+2\eta_2^*} +$$

$$e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}) + B_{12}^* A_{11}^* A_{21}^* (e^{2\eta_1+\eta_1^*+\eta_2^*} + e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*})$$

$$+ B_{12}^* A_{12}^* A_{22}^* (e^{2\eta_2+\eta_1^*+\eta_2^*} + e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*})$$

$$- F(D_t, D_x) \left[\sum_{k=1}^{2} \sum_{j=1}^{2} A_{kj} e^{\eta_k+\eta_j^*} \cdot \sum_{k=1}^{2} \sum_{j=1}^{2} A_{kj} e^{\eta_k+\eta_j^*} \right]$$

(2.28)

where B_{12}^* and A_{kj}^* are complex conjugations of B_{12} and A_{kj} respectively, and $A_{kj}^* = A_{jk}$ (j,k=1,2).It is noticed in the equation (2.28)

$$B_{12}^* A_{11}^* A_{21}^* e^{2\eta_1+\eta_1^*+\eta_2^*} - F(D_t, D_x)[A_{11} e^{\eta_1+\eta_1^*} \cdot A_{12} e^{\eta_1+\eta_2^*} + A_{12} e^{\eta_1+\eta_2^*} \cdot A_{11} e^{\eta_1+\eta_1^*}]$$

$$= 2(p_2^* - p_1^*)(i p_1^{*2} - i p_2^{*2}) A_{11} A_{12} e^{2\eta_1+\eta_1^*+\eta_2^*} - 2 A_{11} A_{12} F(w_2^* - w_1^*,\ p_2 - p_1^*)\, e^{2\eta_1+\eta_1^*+\eta_2^*} = 0$$

Similarly, the coefficients of the terms $e^{2\eta_1+\eta_1^*+\eta_2^*}$, $e^{\eta_1+\eta_2+2\eta_1^*}$ and $e^{\eta_1+\eta_2+2\eta_2^*}$ all are equal to zero in the equation (2.28). So from equation (2.28) we can obtain

$$2F(D_t,D_x)f_2\cdot 1=(B_{12}A_{11}A_{21}+B_{12}A_{12}A_{22}+B_{12}^*A_{11}A_{12}+B_{12}^*A_{21}A_{22})\cdot$$
$$\cdot e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}-2F(D_t,D_x)(A_{11}e^{\eta_1+\eta_1^*}A_{22}e^{\eta_2+\eta_2^*}+A_{12}e^{\eta_1+\eta_2^*}A_{21}e^{\eta_2+\eta_1^*})$$
$$=2B_{12}B_{12}^*A_{11}A_{12}A_{21}A_{22}(p_1+p_2+p_1^*+p_2^*)\cdot$$

i.e.,

$$2F(D_t,D_x)f_2\cdot 1=2B_{12}B_{12}^*A_{12}A_{11}A_{22}F(D_t,D_x)e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*} \tag{2.29}$$

Hence we have

$$f_2=B_{12}^*B_{12}A_{11}A_{12}A_{21}A_{22}\,e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*} \tag{2.30}$$

Inserting f_1, f_2, g_1 and g_2 into (2.16) and (2.17), and by the direct computations, it follows

$$G(D_t,D_x)g_3\cdot 1=-G(D_t,D_x)(g_2\cdot f_1+g_1\cdot f_2)=0$$

$$2F(D_t,D_x)f_3\cdot 1=g_3g_1^*+g_2g_2^*+g_1g_3^*-F(D_t,D_x)(f_2\cdot f_1+f_1\cdot f_2)=0$$

We take $g_3=f_3=0$. From (2.11) it follows

$$f_k=g_k=0,\quad k\geq 3 \tag{2.31}$$

Let $\varepsilon=1$. The solution of the system (2.8) can be found

$$\begin{aligned} f&=1+\sum_{k=1}^{2}\sum_{j=1}^{2}A_{kj}e^{\eta_k+\eta_j^*}+B_{12}B_{12}^*A_{11}A_{12}A_{21}A_{22}e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}\\ g&=e^{\eta_1}+e^{\eta_2}+B_{12}A_{11}A_{21}e^{\eta_1+\eta_2+\eta_1^*}+B_{12}A_{12}A_{22}e^{\eta_1+\eta_2+\eta_2^*}\end{aligned} \tag{2.32}$$

where

$$\eta_k=p_kx+ip_k^2t+\eta_k^{(0)},\ k=1,2.$$

$$B_{12}=2(p_2-p_1)(ip_2^2-ip_1^2)\qquad A_{kj}=[2(p_k+p_j^*)(ip_k^2-ip_j^{*2})]^{-1}$$

and

$$w_{j+N}=w_j^*,\quad p_{j+N}=p_j^*,\quad \eta_{j+N}=\eta_j^*,\quad j=1,2,\ N=2$$

$$e^{\varphi_{kj}}=\begin{cases}2(p_k+p_j)(ip_k^2-ip_j^2), & k=1,2,\ j=3,4\\ 2(p_k-p_j)(w_k-w_j), & k=1,2,\ j=1,2\ \text{or}\ k=3,4,\ j=3,4.\end{cases}$$

The expression (2.32) can be written as

$$\begin{aligned} f&=1+\sum_{k=1}^{2}\sum_{j=3}^{4}\exp(\eta_k+\eta_j+\varphi_{kj})+\exp\Big(\sum_{k=1}^{4}\eta_k+\sum_{1\leq k<j}^{4}\varphi_{kj}\Big)\\ g&=\sum_{k=1}^{2}e^{\eta_k}+e^{\eta_1+\eta_2+\eta_3+\varphi_{12}+\varphi_{13}+\varphi_{23}}+e^{\eta_1+\eta_2+\eta_4+\varphi_{12}+\varphi_{14}+\varphi_{24}}\end{aligned} \tag{2.33}$$

From (2.7) and (2.33),the 2-soliton solution of the system (2.1) can be obtained. Similarly,we have the explicit form of N-soliton solution of the system (2.1) as follows [8]:

$$u = g/f, \quad v = -2(\ln f)_{xx}$$

$$f = \sum_{\mu=0,1} D_1(\mu) \exp\Big(\sum_{k=1}^{2N} \mu_k \eta_k + \sum_{1\le k<j}^{2N} \varphi_{kj} \mu_k \mu_j\Big)$$

$$g = \sum_{\mu=0,1} D_2(\mu) \exp\Big(\sum_{k=1}^{2N} \mu_k \eta_k + \sum_{1\le k<j}^{2N} \varphi_{kj} \mu_k \mu_j\Big)$$

where

$$\eta_k = p_k x + w_k t + \eta_k^{(0)}, \quad w_k = i p_k^2$$

$$p_{k+N} = p_k^*, \quad \eta_{k+N} = \eta_k^*, \quad k=1,2,\cdots,N.$$

$$e^{\varphi_{kj}} = \begin{cases} [2(p_k+p_j)(i p_i^2 - i p_j^2)]^{-1}, & k=1,2,\cdots N, \; j=N+1, N+2,\cdots, 2N. \\ 2(p_k-p_j)(w_k-w_j), & k=1,2,\cdots,N, \; j=1,2,\cdots,N. \\ & \text{or } k=N+1, N+2,\cdots 2N, \; j=N+1,\cdots, 2N \end{cases}$$

$$D_1(\mu) = \begin{cases} 1 & \sum_{k=1}^{N} \mu_k = \sum_{k=1}^{N} \mu_{k+N} \\ 0 & \text{others} \end{cases}$$

$$D_2(\mu) = \begin{cases} 1 & 1 + \sum_{k=1}^{N} \mu_{k+N} = \sum_{k=1}^{N} \mu_k \\ 0 & \text{others} \end{cases}$$

and

$\sum_{\mu=0,1}$ denotes summations for all possible combinations $\mu_1=0,1;\ \mu_2=0,1;\ \cdots,\ \mu_N=0,1.$

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