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PINCHING CONDITIONS FOR YANG-MILLS INSTABILITY
OF HYPERSURFACES

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PINCHING CONDITIONS FOR YANG-MILLS INSTABILITY OF HYPERSURFACES *

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ABSTRACT

A compact Riemannian manifold M is said to be Yang-Mills instable, if for every choice of compact Lie group G and every principle G-bundle P over M, none of the nonflat Yang-Mills connection in P is weakly stable. This paper gives curvature pinching condition for the Yang-Mills instability of hypersurfaces in space form.

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1 Introduction

Let M be a compact Riemannian manifold and P a principal G-bundle over M, where G is a compact Lie group. On the space of connections in P, we consider the Yang-Mills functional $\mathcal{J} : \mathcal{C} \rightarrow R$ defind by

$$J(w) = \frac{1}{2}\int_M \|\Omega\|^2 \quad , \quad w \in \mathcal{C} , \tag{1}$$

where Ω is the curvature of the connection and the norm $\|\Omega\|$ is defined by the Riemannian metric of M and a fixed Ad_G -invariant inner product on the Lie algebra $\mathfrak{g}$ of G.

A Yang-Mills connection is a critical point of J and its curvature is a Yang-Mills field. A Yang-Mills connection w is called a weakly stable if for any family of connections w^t, $|t|<$ with $w^c = w$, the second variation of the functional at w is non-negative, i.e.

$$\frac{d^2}{dt^2} J(w^t)\Big|_{t=c} \geqslant 0 \ . \tag{2}$$

M is said to be Yang-Mills instable, if for every choice of compact Lie group G and every principal G-bundle P over M, none of the nonflat Yang-Mills connection in P is weakly stable. A typical example of Yang-Mills instable manifold is the Euclidean sphere S^n with $n \geqslant 5$ ([1]). For the case that M is a hypersurface or submanifold in the Euclidean space E^m or S^m , some conditions for the Yang-Mills instability of M can be found in [2] and [3]. In this paper , we give an intrinsic condition for the Yang-Mills instability of M where M is a hypersurface in a space form $S^{n+1}(c)$ of constant curvature c with $c \geqslant 0$.

2 Preliminaries

Let M^n be an isometrically immersed compact submanifold in a space form $S^{n+p}(c)$ $(c \geq 0)$. As known, $S^{n+p}(c)$ can be isometrically immersed into the Euclidean space E^{n+p+1} in a standard manner. We choose a local field of orthonormal base $\{ e_i, e_\alpha, e_0 \}$ of E^{n+p+1} such that, restricted to M^n, $\{e_i\}$ span the tangent space of M, $\{e_\alpha\}$ span the normal space of M^n in $S^{n+p}(c)$ and e_c is the unit positionvector. Throughout this paper we agree on the following ranges of indices unless otherwised stated :

$$1 \leq i,j,k,\ldots, \leq n, \quad n+1 \leq \alpha,\beta,\gamma,\ldots, \leq n+p, \quad 1 \leq a,b,c,\ldots, \leq \dim G$$

Now let w be a Yang-Mills connection in a principal G-bundle P over M^n with compact Lie group G. Let

$$w^t = w + A^t, \tag{3}$$

where A^t is a $\mathcal{G}$- valued 1-form on M^n and let

$$B = \frac{dA^t}{dt}\Big|_{t=0} . \tag{4}$$

Evidently, B is a $\mathcal{G}$ - valued 1-form on M^n.

It is known ([1]) that the second variation of the Yang-Mills functional is

$$\frac{d^2}{dt^2} J(w^t)\Big|_{t=0} = \int_{M^n} \langle \delta^w d^w B + \mathcal{R}^w(B) , B \rangle , \tag{5}$$

where d^w is the gauge covariant differential operator, δ^w is the adjoint operator of d^w, and $\mathcal{R}^w(B)$ is an operator defined by

$$\mathcal{R}^w(B)(X) = \sum_i [\Omega(e_i, X), B(e_i)] , \quad \forall X \in T_p(M). \tag{6}$$

If $\delta^w B = 0$, (6) can be rewritten ([1]) as

$$\frac{d^2}{dt^2} J(w^t)\Big|_{t=0} = \int_M \langle -\nabla^w_* \nabla^w B + B\, Ric^M + 2\mathcal{R}^w(B), B \rangle , \tag{7}$$

where $\nabla^w_* \nabla^w$ is the trace Laplacian operator.

Set

$$B(e_i) = b_i = \sum b_i^a X_a , \quad \Omega(e_i, e_j) = \sum f_{ij}^a X_a , \quad [X_a, X_b] = \sum C_{ab}^c X_c , \tag{8}$$

where $\{ X_a \}$ is an orthonormal base of $\mathcal{G}$, i.e. $\langle X_a, X_b \rangle = \delta_{ab}$, and C_{ab}^c are the structure constants of the Lie group G .

Let V be a fixed unit vector in E^{n+p+1}. V_M denotes the tangent projection to M^n of V . Locally, V_M can be expressed as

$$V_M = \sum \langle e_i , V \rangle e_i = \sum v_i e_i \ , \ v_i = \langle V , e_i \rangle \ . \tag{9}$$

For each V , we can define a $\mathfrak{g}$ -valued 1-form B_V as follows.

$$B_V = \sum v_i f_{ij}^a w_j X_a \ , \tag{10}$$

where $\{w_i\}$ is the dual base of $\{e_i\}$.

It is easy to see that $\delta^w B_V = 0$. So formula (7) applies this case. For each V , denote the correspond second variation of w by $Q_W(V)$ and $Q_W(V)$ can be considered as a quadratic form on E^{n+p+1} . We have ([3])

$$\text{trace } Q_W = \int_M q_W = \int_M \{ \sum h_{kj}^\alpha h_{mj}^\alpha f_{ik}^a f_{im}^a + c \sum f_{ik}^a f_{ik}^a + \sum R_l^k f_{ik}^a f_{li}^a + 2R^k{}_{ilj} f_{jk}^a f_{li}^a \} \ , \tag{11}$$

where $\sum h_{ij}^\alpha w_i \otimes w_j \otimes e_\alpha$ is the second fundamental form of M^n in $S^{n+p}(c)$ R_j^i and $R^k{}_{lij}$ are Ricci curvature tensor and Riemannian curvature tensor of M^n .

3 Main results

Now we consider the case where M^n is a hypersurface i.e., p = 1 . Due to the Gauss equation of M^n in $S^{n+1}(c)$, (11) becomes

$$\text{trace } Q_W = \int_M \{ (4-n) c \sum f_{ik}^a f_{ik}^a + \sum h_{kk} h_{ij} f_{lj}^a f_{il}^a + 2 \sum h_{kj} h_{mj} f_{ik}^a f_{im}^a - 2 \sum h_{il} h_{jk} f_{lj}^a f_{ki}^a \} . \tag{12}$$

Choose $\{e_i\}$ such that $h_{ij} = \lambda_i \delta_{ij}$, where $\lambda_1 , \ldots , \lambda_n$ are principal curvatures of M^n . From (12) , we have , setting $H = \sum \lambda_i$,

$$\begin{aligned} \text{trace } Q_W &= \int_M \{ (4-n) c \sum f_{ik}^a f_{ik}^a - H \sum \lambda_i f_{ik}^a f_{ik}^a + 2 \sum \lambda_i^2 f_{ik}^a f_{ik}^a + 2 \sum \lambda_i \lambda_k f_{ik}^a f_{ik}^a \} \\ &= \int_M \sum \{ (4-n) c + 2\lambda_i^2 + 2\lambda_i \lambda_k - \lambda_i H \} f_{ik}^a f_{ik}^a \ . \end{aligned} \tag{13}$$

It is obvious that if trace Q is negative then w is weakly stable if and only if w is flat. So we have the following

<u>Theorem 1</u>. If M^n is a compact hypersurface in $S^{n+1}(c)$ $(c \geq 0)$ such that its principal curvatures $\lambda_1 , \ldots , \lambda_n$ satisfy

$$(4-n)c + \lambda_i (2\lambda_i + 2\lambda_j - H) < 0, \text{ for any } i \neq j, \tag{14}$$

at every point of M^n, then M^n is Yang-Mills instable.

Now we give a pinching condition on the curvatures for a compact hypersurface in $S^{n+1}(c)$ $(c\geqslant 0)$ to be Yang-Mills instable.

Theorem 2. Let $S^{n+1}(c)$ be an (n+1)-dimensional simply connected space form with constant sectional curvature $c\geqslant 0$. Suppose that M^n $(n\geqslant 5)$ is a compact hypersurface in $S^{n+1}(c)$ of which the sectional curvatures $Riem^M$ satisfy the following pinching condition :

$$c + 3a^2/[(n-4)c + (n-1)a] \leqslant Riem^M < c + a \tag{15}$$

for some constant $a>0$ .Then M^n is Yang-Mills instable.

From the Gauss equations $R_{ijij} = c + \lambda_i\lambda_j$, (15) is equivalent to

$$a^2/[(n-4)c + (n-1)a] \leqslant \lambda_i\lambda_j < a \text{ , for any } i\neq j \text{ .} \tag{16}$$

Since M^n is convex , without loss of generality, we may assume

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_n \text{ .} \tag{17}$$

Setting

$$A_k = H - 2\lambda_k \text{ ,} \tag{18}$$

we have $A_n \leqslant A_{n-1} \leqslant \dots \leqslant A_1$ and (14) is equivalent to

$$2\lambda_i^2 - \lambda_i A_j - (n-4)c < 0 \text{ , } i\neq j \text{ .} \tag{19}$$

Due to (17) it is equivalent to

$$\lambda_i < \frac{1}{4}[\, A_j + \sqrt{A_j^2 + 8(n-4)c}\,] \text{ , } i\neq j \text{ .} \tag{20}$$

Lemma 1. If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_n$ and $\lambda_n < \frac{1}{4}[A_n + \sqrt{A_n^2 + 8(n-4)c}]$, then (14) holds .

Proof Since , for any k, we have

$$\begin{aligned} A_k^2 - A_n^2 &= (H - 2\lambda_k)^2 - (H - 2\lambda_n)^2 \\ &= H^2 - 4\lambda_k H + 4\lambda_k^2 - H^2 + 4\lambda_n H - 4\lambda_n^2 \\ &= 4(\lambda_n - \lambda_k)(H - \lambda_n - \lambda_k) > 0 \text{ ,} \end{aligned}$$

hence for any i,k,

$$\lambda_i \leqslant \lambda_n < \frac{1}{4}[\, A_n + \sqrt{A_n^2 + 8(n-4)c}\,] < \frac{1}{4}[\, A_k + \sqrt{A_k^2 + 8(n-4)c}\,] \text{ .}$$

The Lemma is proved.

Lemma 2. If

$$\lambda_n < \frac{1}{8}[\ H + \sqrt{H^2 + 16(n-4)c}\ ], \qquad (21)$$

then

$$\lambda_n < \frac{1}{4}[\ A_n + \sqrt{A_n^2 + 8(n-4)c}\ ]. \qquad (22)$$

Proof From (21), we have

$$4\lambda_n^2 - \lambda_n H - (n-4)c < 0 .$$

It follows that

$$(\ 6\lambda_n - H\ )^2 < A_n^2 + 8(n-4)c .$$

Consequently,

$$4\lambda_n < (\ H - 2\lambda_n) + \sqrt{A_n^2 + 8(n-4)c} .$$

So

$$\lambda_n < \frac{1}{4}[\ A_n + \sqrt{A_n^2 + 8(n-4)c} \quad .$$

Now let

$$b^2 \leqslant \lambda_i \lambda_j < B^2,\ i \neq j , \qquad (23)$$

where b,B are positive constants.

Lemma 3. If $\lambda_1 \geqslant b$ and b,B in (23) satisfy

$$B^2 - Lb + (n-1)b^2 = 0 , \qquad (24)$$

where

$$L = \frac{1}{6}[\ 7(n-1)b + \sqrt{(n-1)^2 b^2 + 12(n-4)c}\ ] , \qquad (25)$$

then (21) holds.

Proof Since $\lambda_1 \lambda_n < B^2$, from (24) we have

$$\lambda_n < \frac{B^2}{\lambda_1} \leqslant \frac{B^2}{b} = L - (n-1)b . \qquad (26)$$

From (25) , we have

$$3L^2 - 7(n-1)bL + 4(n-1)^2 b^2 - (n-4)c = 0 \quad . \qquad (27)$$

Hence

$$L - (n-1)b = \frac{1}{8}[\ L + \sqrt{L^2 + 16(n-4)c}\ ] . \qquad (28)$$

Thus

$$\lambda_n < \frac{1}{8}[\ L + \sqrt{L^2 + 16(n-4)c}\ ] . \qquad (29)$$

If $L \leqslant H$, then (29) implies (21) . Suppose $L > H$. Set $L - H = K$

We have

$$\lambda_n = H - \sum_{k\neq n} \lambda_k < H - (n-1)b = L - (n-1)b - K$$
$$= \frac{1}{8}[\, L + \sqrt{L^2 + 16(n-4)c} - 8K \,]$$
$$= \frac{1}{8}[\, H + \sqrt{L^2 + 16(n-4)c} - 7K \,] \ . \qquad (30)$$

On the other hand, it is easy to see

$$(\sqrt{H^2 + 16(n-4)c} + 7K)^2 > L^2 + 16(n-4)c \ . \qquad (31)$$

Hence

$$\sqrt{L^2 + 16(n-4)c} - 7K < \sqrt{H^2 + 16(n-4)c} \ . \qquad (32)$$

From (30) and (32) , we still have

$$\lambda_n < \frac{1}{8}[\, H + \sqrt{H^2 + 16(n-4)c} \,] \ .$$

The Lemma is proved.

Lemma 4. If $n \geqslant 5$ and (24) holds, then (21) is true.

Proof It suffices to prove the Lemma in the case of $\lambda_1 < b$. Since $n \geqslant 5$, there exist λ_1 and λ_2 ($\leqslant \lambda_3$) such that $\lambda_1\lambda_2 \geqslant b^2$. Construct

$$\lambda_1' = \lambda_2' = \frac{1}{2}(\lambda_1 + \lambda_2) \ . \qquad (33)$$

Then

$$(\lambda_1')^2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 \geqslant \lambda_1\lambda_2 \geqslant b^2 ,$$

so that $\lambda_1' \geqslant b$. Obviously we still have $\lambda_1' = \lambda_2' \leqslant \lambda_3$ and $\lambda_1' + \lambda_2' + \sum_{k=3}^{n} \lambda_k = H$ unchanged. Thus , applying Lemma 3 to the case where $0 < \lambda_1' = \lambda_2' \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n$, we can prove this lemma.

The proof of Theorem 2. (15) is equivalent to

$$3a^2/[(n-4)c + (n-1)a] < \lambda_i\lambda_j < a \ . \qquad (34)$$

Set

$$b^2 = 3a^2/[(n-4)c + (n-1)a] \text{ and } B^2 = a \ . \qquad (35)$$

we have

$$3a^2 = (n-4)cb^2 + (n-1)b^2 a \ . \qquad (36)$$

Thus

$$a = \frac{b}{6}[(n-1)b + \sqrt{(n-1)^2 b^2 + 12(n-4)c}]$$
$$= \frac{b}{6}[7(n-1)b + \sqrt{(n-1)^2 b^2 + 12(n-4)c}] - (n-1)b^2 \ . \qquad (37)$$

It follows from (35) and (37) that

$$B^2 - bL + (n-1)b^2 = 0 , \tag{38}$$

where

$$L = \frac{1}{6}[7(n-1)b + \sqrt{(n-1)b^2 + 12(n-4)c}\ ] . \tag{39}$$

Now applying Lemma 4 we complete the proof.

Corollary. If $n \geq 5$ and M^n is a compact n-dimensional hypersurface in the Euclidean space E^{n+1} satisfying the condition

$$3a/(n-1) < \mathrm{Riem}^M < a ,$$

for some $a > 0$, then M^n is Yang-Mills instable .

Remark. The constant a can be replaced by a positive function $a \in C^0(M)$ such that the pinching condition holds at every point $x \in M$.

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REFERENCES

[1] Bourguignon,J.P. and Lawson,H.B. Stability and isolation phenomena for Yang-Mills fields, Commun. Math. Phys. 79, 189-230(1981)

[2] Kobayashi,S. Ohnita,Y. and Takeuchi,M. On instability of Yang-Mills connections, Math. Z. 193,165-189(1986).

[3] Shen Chun-Li, Weakly stability of Yang-Mills fields over the submanifold of the sphere. Arch.Math. 39, 78-84(1982) .

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