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FROM SUFFICIENTLY PINCHED SIMPLY CONNECTED RIEMANNIAN MANIFOLDS

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NONEXISTENCE OF STABLE HARMONIC MAPS FROM SUFFICIENTLY PINCHED SIMPLY CONNECTED RIEMANNIAN MANIFOLDS *

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ABSTRACT

It is proved that for $n \geq 3$ there exists a constant $\delta(n)$ with $\frac{1}{4} < \delta(n) < 1$ such that if M is a simply connected Riemannian manifold of dimension n with $\delta(n)$-pinched curvatures then for every Riemannian manifold N every stable harmonic map $\phi: M \to N$ is constant. Together with Howard's result, it shows that a simply connected sufficiently pinched Riemannian manifold is weakly E-unstable.

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§1 Introduction

A harmonic map is a critical point of the energy functional. A harmonic map is said to be stable if for any deformation vector field, its second variation is always non-negative. For simplicity, we give

Definition 1. Let M be a compact Riemannian manifold. M is said to be weakly E-unstable, if the following two conditions are fulfilled :

(A) For any compact Riemannian manifold N there are no nonconstant stable harmonic maps from N to M,

(B) For any Riemannian manifold N there are no nonconstant stable harmonic maps from M to N.

Several classes of weakly E-unstable manifolds have been founded in recent years. A typical case is the Euclidean sphere S^n with $n \geqslant 3$. It is due to a combination of Xin's result for (B) [8] and Leung's result for (A) [4] . Taking S^n as model manifold, one might expect the weakly E-unstability for sufficiently pinched Riemannian manifolds, i.e., for compact Riemannian manifolds whose sectional curvatures are between the interval $[\delta K , K]$ with constants $K \geqslant 0$ and $1 \geqslant \delta > 0$. But Urakawa [7] shows that the identity maps of any non-simply connected manifold with positive constant curvatures are stable. So the condition of simply connectness is necessary. In 1985, Howard proved the following

Theorem 1 (Howard). Let $n \geqslant 3$. There is a number $\delta_1(n)$ with δ such that if M is simply connected with $\delta_1(n)$-pinched curvatures then for every compact Riemannian manifold N every stable harmonic map $\phi : N \to M$ is constant.

It means for such a manifold the condition (A) is satisfied. This is a theorem determined only by the intrinsic geometry of the manifold.

In this paper, we establish the following

Theorem 2. Let $n \geq 3$. There is a number $\delta_2(n)$ with $\frac{1}{4} \leq \delta_2(n) < 1$ such that if M is a simply connected Riemannian manifold of dimension n with $\delta_2(n)$-pinched curvatures then for any Riemannian manifold N every stable harmonic map $\phi : M \to N$ is constant.

Combining the above two theorems, letting $\delta(n) = \max(\delta_1(n), \delta_2(n))$ we obtain

Theorem 3. Let $n \geq 3$. There is a number $\delta(n)$ with $\frac{1}{4} \leq \delta(n) < 1$ such that if M is simply connected Riemannian manifold of dimension n with $\delta(n)$-pinched curvatures then M is weakly E-unstable.

The proof of Theorem 2 goes in a way similar to that of Howard. That is, we make an integral average for the second variation formula over a continuous family of deformations and show the result is negative.

2 Second variation formula

Let M and N be Riemannian manifolds with dimension n and m respectively. M is compact without boundary, and ∇, ∇' represent the Riemannian connections of M and N respectively. Suppose that $\phi : M \to N$ is a harmonic map, $\phi_* : TM \to TN$ is the induced map, where TM and TN are the tangent bundle of M and N respectively. We also can consider ϕ_* as a $\phi^{-1}TN$ valued 1-form $d\phi$, i.e., $d\phi(X) = \phi_* X$, for $X \in TM$. The induced bundle $\phi^{-1}TN \to M$ possesses the induced Riemannian connection as follows

$$\tilde{\nabla}_X S = \nabla'_{\phi_* X} S, \tag{2.1}$$

where $X \in TM$, $S \in \Gamma(\phi^{-1}TN)$.

Choose local fields of orthonormal frames $\{e_i\}$ and $\{e'_\alpha\}$ in M and N, respectively, and let $\{w_i\}$ and $\{w'_\alpha\}$ be the fields of dual forms. we shall make the following convention on the ranges of indices: $1 \leq i, j, k, \ldots, \leq n$, $1 \leq \alpha, \beta, \cdots, \leq m$, and use the summation convention.

Let $W \in (\phi^{-1}TN)$ be a deformation vector field, ϕ_t the one parametric family of maps generated by W, $\phi_0 = \phi$. It is well known that the second variation of the energy functional $E(\phi_t)$ is given by

$$\frac{d^2}{dt^2}E(\phi_t)\Big|_{t=0} = I(W,W) = -\int_M \langle \tilde{\nabla}\cdot\tilde{\nabla} W + R(\phi_* e_i, W)\phi_* e_i,\ W\rangle_N \,\Omega_M\,, \tag{2.2}$$

where $\tilde{\nabla}\cdot\tilde{\nabla}$ is the trace Laplacian with respect to $\tilde{\nabla}$, R is the curvature operator of N, and Ω_M is the volume element of M [6].

If we take $\phi_* V$, where $V \in TM$, as the deformation vector, then, using Weitzenböck formula, we can rewrite (2.2) as [5]

$$I(\phi_* V, \phi_* V) = \int_M \langle d\phi(\nabla_{e_i}\nabla_{e_i} V) - 2\tilde{\nabla}_{e_i}(d\phi(\nabla_{e_i} V)) - \phi_*(\mathrm{Ric}^M(V)),\ \phi_* V\rangle_N \,\Omega_M\,, \tag{2.3}$$

where Ric^M is the Ricci curvature operator of M, $\mathrm{Ric}^M(e_i) = R_{ij}e_j$.

Under the map ϕ suppose the pull back of w'_α is $\phi^*(w'_\alpha) = a_{\alpha i} w_i$. Then the energy density of ϕ is $e(\phi) = \frac{1}{2}\sum a^2_{\alpha i}$, the energy of ϕ is $E(\phi) = \frac{1}{2}\int_M \sum a^2_{\alpha i}\,\Omega_M$, and the tension field of ϕ is $\tau = \sum_{\alpha, i} a_{\alpha i i} e'_\alpha$, where $a_{\alpha i j}$ is the covariant derivative of $a_{\alpha i}$. For harmonic map ϕ , $\tau = 0$.

Let $V = V_i e_i$, we compute the quantities in (2.3) as follows.

$\nabla_{e_i} V = V_{ji} e_j$, $\nabla_{e_i}\nabla_{e_i} V = V_{jii} e_j = (\Delta V_j) e_j$, where V_{ji} , V_{jii} are covariant derivatives and Δ is the Laplacian of M .

$$d\phi(\nabla_{e_i}\nabla_{e_i} V) = a_{\alpha j}(\Delta V_j)\, e'_\alpha$$

$$\tilde{\nabla}_{e_i}(d\phi(\nabla_{e_i} V)) = \tilde{\nabla}_{e_i}(a_{\alpha j} V_{ji} e'_\alpha) = (V_{jii} a_{\alpha j} + V_{ji} a_{\alpha j i})\, e'_\alpha$$

$$= (a_{\alpha j}(\Delta V_j) + a_{\alpha j i} V_{ji})\, e'_\alpha\,,$$

$$\phi_*(\mathrm{Ric}^M(V)) = a_{\alpha j} R_{ij} V_i\, e'_\alpha\,.$$

Thus (2.3) becomes

$$I(\phi_* V, \phi_* V) = -\int_M (a_{\alpha j} a_{\alpha l}(\Delta V_l) V_j + 2a_{\alpha j i} a_{\alpha l} V_{ji} V_l + a_{\alpha j} a_{\alpha l} R_{ij} V_i V_l)\,. \tag{2.4}$$

§ 3 Some estimates

In this section, we shall list some useful estimates obtained by Howard. Details can be found in [3]. Suppose V is the gradient of a smooth function f on M. For V we define a smooth field of linear endomorphisms of the tangent spaces to M by

$$\mathcal{Q}^V(X) = \nabla_X V, \quad \text{for } X \in TM_x, \ x \in M. \tag{3.1}$$

For the gradient vector field of f a straightforward calculation shows

$$\langle \mathcal{Q}^V X , Y \rangle_M = D^2 f(X,Y), \tag{3.2}$$

where $D^2 f$ is the Hessian of f. So $\mathcal{Q}^V$ is self-adjoint and has real eigenvalues $\lambda_1, \ \ldots \ , \lambda_n$.

From now on we always suppose M is a compact simply connected with curvatures between δ and 1, where $1 > \delta \geq \frac{1}{4}$. Due to Klingenberg's result, the injective radius of such a manifold is greater than π.

If $x \in M$, let $\rho_x(y)$ deenote the geodesic distance at y from x. Define a function $f: R \to R$ by

$$f(t) = \begin{cases} -\cos t & |t| \leq \pi , \\ 1 & |t| \geq \pi . \end{cases} \tag{3.3}$$

Then f and f' are continuous. For any $x \in M$, let V^x be the vector field defined by

$$V^x = \nabla (f \circ \rho_x) = f'(\rho_x) \nabla \rho_x . \tag{3.4}$$

Then V^x is continuous and smooth off the locus defined by $\rho_x = \pi$, and $V^x = 0$ on the set defined by $\rho_x > \pi$.

Noting (3.2) and using the Hessian comparison theorem of Greene-Wu[2], Howard gives the following estimates for the eigenvalues of $\mathcal{Q}^{V^x}$, $\{\lambda_i\}$:

At any point y at a geodesic distance from $x = \rho$, with $\rho < \pi$, we have

$$\cos \rho < \lambda_i < \sqrt{\delta} \sin \rho \cot \sqrt{\delta} \rho , \qquad 1 \leq i \leq n. \tag{3.5}$$

Set

$$\begin{aligned} \tilde{g}_1(t,\delta) &= \text{middle value of } \{ \cos t, \ 0, \sqrt{\delta} \sin t \cot \sqrt{\delta} t \} , \\ g_1(t,\delta) &= (\tilde{g}_1(t,\delta))^2 , \quad 0 \leq t \leq \pi , \\ g_2(t,\delta) &= \max\{ \cos^2 t, \ \delta \sin^2 t \cot^2 \sqrt{\delta} t \}, \quad 0 \leq t \leq \pi, \\ g_1(t,\delta) &= g_2(t,\delta) = 0, \quad t > \pi . \end{aligned} \tag{3.6}$$

Then we have

$$g_1(\rho, \delta) \leq \lambda_i^2 \leq g_2(\rho, \delta) . \tag{3.7}$$

The other comparison theorem needed is due to Bishop and Grittenden[1] . If $y \in M$ then let UM_y be the unit sphere in TM_y. Then, letting $\rho = \rho_y$, and $u \in UM_y$, we view (ρ, u) as polar coordinates on M near y in the obvious way. Let Ω_{UM_y} be the volume density on UM_y . Then on the open set M\cut(y), where cut(y) is the cut locus of y,

$$\sin^{n-1}\rho \, d\rho \, \Omega_{UM_y}(u) \leq \Omega_M \leq \left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho \, \Omega_{UM_y}(u), \tag{3.8}$$

where the lower bound only holds up to π [1].

The following lemma can be found in [3].

<u>Lemma 1</u> . Let $Q : R^n \to R$ be a quadratic form, then

$$\int_{S^{n-1}} Q(u)\Omega_{S^{n-1}}(u) = \frac{1}{n} \operatorname{vol}(S^{n-1}) \operatorname{trace}(Q) \ . \tag{3.9}$$

§ 4 Proof of the Theorem 2

Without loss of generality, we can assume the curvatures of M are between $[\delta, 1]$. Let x be a point of M and V^x the deformation vector field defined by (3.4). Write the second variation determined by V^x as $I_x(\phi_* V^x, \phi_* V^x)$, and let $V^x(y) = V^x_i(y)e_i$. Then from (2.4) we have

$$I_x(\phi_* V^x, \phi_* V^x) = -\int_{M(y)} \{ a_{\alpha j}(y)a_{\alpha l}(y)(\Delta V^x_l(y))V^x_j(y) + 2a_{\alpha j,i}(y)a_{\alpha l}(y)V^x_{j,i}(y)V^x_l(y) + a_{\alpha j}(y)a_{\alpha l}(y)R_{ij}(y)V^x_i(y)V^x_l(y)\} \Omega_M(y). \tag{4.1}$$

In the following, we compute the integration

$$\int_{M(x)} I_x(\phi_* V^x, \phi_* V^x)\Omega_M(x)$$

and show there exists a certain constant $\delta_2(n)$ such that for $1 \geq \delta > \delta_2(n)$, the integration is negative. Hence, at least for a certain point x, $I_x(\phi_* V^x, \phi_* V^x)$ is negative.

First of all, we need to transform (4.1) into another suitable form . For simplicity, we omit the variable y, then no confusion is caused. We have

$$2a_{\alpha j,i}a_{\alpha l}V^x_{j,i}V^x_l = 2(a_{\alpha j}a_{\alpha l}V^x_{j,i}V^x_l)_{,i} - 2a_{\alpha j}a_{\alpha l,i}V^x_{j,i}V^x_l - 2a_{\alpha j}a_{\alpha l}V^x_{j,ii}V^x_l - 2a_{\alpha j}a_{\alpha l}V^x_{j,i}V^x_{l,i}. \tag{4.2}$$

Thus by divergence theorem (4.1) becomes

$$I_x(\phi_* V^x,\ \phi_* V^x) = \int_{M(y)} \{ a_{\alpha j} a_{\alpha l} V^x_{ji} V^x_{li} + 2 a_{\alpha j} a_{\alpha l i} V^x_{ji} V^x_{l} + 2 a_{\alpha j} a_{\alpha l} V^x_{ji} V^x_{li} - a_{\alpha j} a_{\alpha l} R_{ij} V^x_{i} V^x_{l} \} \Omega_M(y) \quad . \tag{4.3}$$

Noting $a_{\alpha ij} = a_{\alpha ji}$ and $V^x_{ji} = V^x_{ij}$, from

$$a_{\alpha j} a_{\alpha l i} V^x_{ji} V^x_{l} = (a_{\alpha j} a_{\alpha i} V^x_{ji} V^x_{l})_{,l} - a_{\alpha j l} a_{\alpha i} V^x_{ji} V^x_{l} - a_{\alpha j} a_{\alpha i} V^x_{jil} V^x_{l} - a_{\alpha j} a_{\alpha i} V^x_{ji} V^x_{ll} \ ,$$

we have

$$2\int_{M(y)} a_{\alpha j} a_{\alpha l i} V^x_{ji} V^x_{l} \Omega_M(y) = -\int_{M(y)} (a_{\alpha j} a_{\alpha i} V^x_{jil} V^x_{l} + a_{\alpha j} a_{\alpha i} V^x_{ji} V^x_{ll}) \Omega_M(y) \quad . \tag{4.4}$$

Using Ricci identities and noting $R_{ij} = -R_{ikjk}$, we have

$$\begin{aligned} a_{\alpha j} a_{\alpha l} V^x_{jil} V^x_{l} &= a_{\alpha j} a_{\alpha l} (V^x_{iij} - V^x_{k} R_{kiji}) V^x_{l} \\ &= a_{\alpha j} a_{\alpha l} V^x_{iij} V^x_{l} + a_{\alpha j} a_{\alpha l} R_{kj} V^x_{k} V^x_{l} \ , \end{aligned} \tag{4.5}$$

Noting $a_{\alpha jj} = 0$ due to the harmonicity of ϕ , we have

$$\begin{aligned} a_{\alpha j} a_{\alpha l} V^x_{iij} V^x_{l} &= (a_{\alpha j} a_{\alpha l} V^x_{ii} V^x_{l})_{,j} - a_{\alpha jj} a_{\alpha l} V^x_{ii} V^x_{l} - a_{\alpha j} a_{\alpha lj} V^x_{ii} V^x_{l} - a_{\alpha j} a_{\alpha l} V^x_{ii} V^x_{lj} \\ &= (a_{\alpha j} a_{\alpha l} V^x_{ii} V^x_{l})_{,j} - a_{\alpha j} a_{\alpha lj} V^x_{ii} V^x_{l} - a_{\alpha j} a_{\alpha l} V^x_{ii} V^x_{lj} \ . \end{aligned} \tag{4.6}$$

From (4.4) – (4.6) , it follows that

$$I_x(\phi_* V^x,\ \phi_* V^x) = \int_{M(y)} \{ -a_{\alpha j} a_{\alpha lj} V^x_{ii} V^x_{l} - 2 a_{\alpha j} a_{\alpha l} V^x_{ii} V^x_{lj} - a_{\alpha j} a_{\alpha i} V^x_{jil} V^x_{l} + 2 a_{\alpha j} a_{\alpha l} V^x_{ji} V^x_{li} \} \Omega_M(y) \quad . \tag{4.7}$$

Now since

$$a_{\alpha j} a_{\alpha lj} V^x_{ii} V^x_{l} = e(\phi)_{,l} V^x_{ii} V^x_{l} = (e(\phi) V^x_{ii} V^x_{l})_{,l} - e(\phi) V^x_{iil} V^x_{l} - e(\phi) V^x_{ii} V^x_{ll} \ ,$$

(4.7) becomes

$$I_x(\phi_* V^x,\ \phi_* V^x) = \int_{M(y)} \{ e(\phi) V^x_{iil} V^x_{l} + e(\phi) V^x_{ii} V^x_{ll} - 2 a_{\alpha j} a_{\alpha l} V^x_{ii} V^x_{lj} - a_{\alpha j} a_{\alpha i} V^x_{jil} V^x_{l} + 2 a_{\alpha j} a_{\alpha l} V^x_{ji} V^x_{li} \} \Omega_M(y) \tag{4.8}$$

Since V^x is the gradient vector of the function $f \circ \rho_x$, at the point y with $\rho_x(y) = \rho$ and $\rho < \pi$, we have $V^x(y) = \sin\rho \frac{\partial}{\partial\rho}$ and since $\nabla_{\frac{\partial}{\partial\rho}} \frac{\partial}{\partial\rho} = 0$,

$$\nabla_{V^x} V^x(y) = \cos\rho \ V^x(y) \quad . \tag{4.9}$$

So at the point y, with respect to the frame $\{e_i\}$, we have from (4.9)

$$V^x_i V^x_{ji} = \cos\rho \ V^x_j \ . \tag{4.10}$$

Differentiating (4.10), we get

$$V^x_{il} V^x_{ji} + V^x_i V^x_{jil} = -V^x_l V^x_j + \cos\rho \ V^x_{jl} \quad . \tag{4.11}$$

It follows

$$V^x_{jil} V^x_l = -V^x_l V^x_j + \cos\rho V^x_{jl} - V^x_{il} V^x_{ji} \ . \tag{4.12}$$

Now, using Ricci identities, from (4.12) we have

$$V_{j\ell i}V^{x}_{i} = -V^{x}_{\ell}V^{\lambda}_{j} + \cos\rho V^{\lambda}_{j\ell} - V^{\lambda}_{i\ell}V^{x}_{j i} - V^{x}_{k}V^{x}_{i}R_{kj\ell i} \quad . \tag{4.13}$$

Summing on the indices j and ℓ, we get

$$V^{x}_{\ell\ell i}V^{\lambda}_{i} = -\sin^2\rho + \cos\rho V^{\lambda}_{\ell\ell} - V^{x}_{i\ell}V^{\lambda}_{i\ell} - R_{ik}V^{x}_{i}V^{\lambda}_{k} \quad . \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.8), (4.8) becomes

$$\begin{aligned} I_x(\phi_* V^x, \phi_* V^x) = \int_{M(y)} \{ & -e(\phi)\sin^2\rho + e(\phi)\cos\rho V^{x}_{\ell\ell} - e(\phi)V^{x}_{i\ell}V^{x}_{\ell i} \\ & -e(\phi)R_{ik}V^{x}_{i}V^{x}_{k} + e(\phi)V^{x}_{ii}V^{x}_{\ell\ell} - 2a_{\alpha j}a_{\alpha\ell}V^{x}_{ii}V^{x}_{\ell j} + a_{\alpha j}a_{\alpha i}V^{x}_{i}V^{x}_{j} \\ & -\cos\rho\, a_{\alpha j}a_{\alpha i}V^{\lambda}_{ij} + 3a_{\alpha j}a_{\alpha i}V^{x}_{j\ell}V^{x}_{\ell i} + a_{\alpha j}a_{\alpha\ell}V^{x}_{k}V^{x}_{\ell}R_{kji\ell} \}\Omega_M(y) \end{aligned} \tag{4.15}$$

At the point y, we take the unit eigenvectors of $\mathfrak{U}^{V^x}$ as $\{e_i\}$, and we have

$$V^{x}_{ij} = \lambda_i \delta_{ij} \quad , \qquad i, j, = 1, 2, \ldots, n, \tag{4.16}$$

where λ_i is the eigenvalue of $\mathfrak{U}^{V^x}$.

Now we can estimate each term of the integration (4.15) just in the same way as in [3]. Using (3.5), (3.7), (3.8) and (4.16) we get

$$\int_{M(x)}\int_{M(y)}(-e(\phi)\sin^2\rho)\Omega_M(y)\Omega_M(x) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi}(-\sin^{n+1}\rho\, d\rho)\Omega_M(y) \quad , \tag{4.17}$$

$$\begin{aligned}\int_{M(x)}\int_{M(y)} e(\phi)\cos\rho V^{x}_{ii}\Omega_M(y)\Omega_M(x) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\{ & \int_0^{\frac{\pi}{2}} n\sqrt{\varepsilon}\sin\rho\cos\rho\cot\sqrt{\varepsilon}\rho(\frac{\sin\sqrt{\varepsilon}\rho}{\sqrt{\varepsilon}})^{n-1}d\rho \\ & +\int_{\frac{\pi}{2}}^{\pi} n\cos^2\rho(\frac{\sin\sqrt{\varepsilon}\rho}{\sqrt{\varepsilon}})^{n-1}d\rho\}\Omega_M(y) \quad ,\end{aligned} \tag{4.18}$$

$$\int_{M(x)}\int_{M(y)}(-e(\phi)V^{x}_{i\ell}V^{x}_{i\ell}\Omega_M(y)\Omega_M(x) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi}(-ng_1(\rho,\delta)\sin^{n-1}\rho d\rho\,\Omega_M(y) \tag{4.19}$$

$$\int_{M(x)}\int_{M(y)} e(\phi)V^{x}_{ii}V^{x}_{\ell\ell}\Omega_M(y)\Omega_M(x) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi} n^2 g_2(\rho,\delta)(\frac{\sin\sqrt{\varepsilon}\rho}{\sqrt{\varepsilon}})^{n-1}d\rho\,\Omega_M(y), \tag{4.20}$$

$$\begin{aligned}\int_{M(x)}\int_{M(y)}(-\cos\rho\, a_{\alpha i}a_{\alpha j}V^{\lambda}_{ij})\Omega_M(y)\Omega_M(x) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\{ & \int_0^{\frac{\pi}{2}} -2\cos^2\rho\sin^{n-1}\rho\, d\rho \\ & +\int_{\frac{\pi}{2}}^{\frac{\pi}{2\sqrt{\varepsilon}}} -2\sqrt{\varepsilon}\sin\rho\cos\rho\cot\sqrt{\varepsilon}\rho(\frac{\sin\sqrt{\varepsilon}\rho}{\sqrt{\varepsilon}})^{n-1}d\rho \\ & +\int_{\frac{\pi}{2\sqrt{\varepsilon}}}^{\pi} -2\sqrt{\varepsilon}\sin^{n}\rho\cos\rho\cot\sqrt{\varepsilon}\rho\, d\rho\}\Omega_M(y),\end{aligned} \tag{4.21}$$

and

$$\int_{M(x)}\int_{M(y)} a_{\alpha i}a_{\alpha j}V^{x}_{\ell i}V^{x}_{\ell j}\Omega_M(y)\Omega_M(x) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi} 2g_2(\rho,\delta)(\frac{\sin\sqrt{\varepsilon}\rho}{\sqrt{\varepsilon}})^{n-1}d\rho\,\Omega_M(y). \tag{4.22}$$

Using Lemma 1 and (3.8), we have

$$\int_{M(y)}\int_{M(x)} a_{\alpha i}a_{\alpha j}V_i^x V_j^x \,\Omega_M(x)\Omega_M(y) \leq \int_{M(y)}\left(\int a_{\alpha i}a_{\alpha j}V_i^x V_j^x \left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho\,\Omega_{U_{M_y}}(u)\right)\Omega_M(y)$$
$$= \int_{M(y)}\left(\int_0^{\pi} \frac{1}{n}\mathrm{vol}(S^{n-1})\sum_{\alpha,i}(a_{\alpha i}^2)\cdot \sin^2\rho \left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho\right)\Omega_M(y)$$
$$= \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi} \frac{2}{n}\sin^2\rho \left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho\,\Omega_M(y) \;. \tag{4.23}$$

From the assumption on the curvatures of M , it follows

$$-\int_{M(y)}\int_{M(x)} e(\phi)R_{ik}V_i^x V_k^x \,\Omega_M(x)\Omega_M(y) \leq -\int_{M(y)}\int_{M(x)} e(\phi)\cdot(n-1)\delta\sin^2\rho\,\Omega_M(x)\,\Omega_M(y)$$
$$< -\mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi}(n-1)\delta\sin^{n+1}\rho\, d\rho\,\Omega_M(y), \tag{4.24}$$

and using Lemma 1 and the assumption on curvatures, we have

$$\int_{M(y)}\int_{M(x)} a_{\alpha i}a_{\alpha j}V_k^x V_\ell^x R_{kji\ell}\,\Omega_M(x)\Omega_M(y)$$
$$\leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\int_0^{\pi}\frac{2(n-1)}{n}\sin^2\rho\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho\,\Omega_M(y)\,. \tag{4.25}$$

It is also easy to get the following estimate

$$-2\int_{M(y)}\int_{M(x)} a_{\alpha j}a_{\alpha i}V_i^x V_j^x\,\Omega_M(x)\,\Omega_M(y) \leq \mathrm{vol}(S^{n-1})\int_{M(y)} e(\phi)\Big\{\int_0^{\frac{\pi}{2}} -4n\cos^2\rho\sin^{n-1}\rho\, d\rho$$
$$+\int_{\frac{\pi}{2}}^{\frac{\pi}{2\sqrt{\delta}}} -4n\delta\sin^{n+1}\rho\,\cot^2\sqrt{\delta}\rho\, d\rho + \int_{\frac{\pi}{2\sqrt{\delta}}}^{\pi} -4n\sqrt{\delta}\sin^n\rho\cos\rho\cot\sqrt{\delta}\rho\, d\rho\Big\}\Omega_M(y)\,. \tag{4.26}$$

By means of the above estimates (4.17)-(4.26) and Fubini theorem, we get

$$\int_{M(x)} I_x(\phi_* V^x,\ \phi_* V^x)\,\Omega_M(x) \leq \mathrm{vol}(S^{n-1})E(\phi)F(n,\delta)\ , \tag{4.27}$$

where $F(n,\delta)$ is a continuous function of δ given by

$$F(n,\delta) = \int_0^{\pi}\sin^{n+1}\rho\, d\rho + \int_0^{\frac{\pi}{2}} n\sqrt{\delta}\sin\rho\cos\rho\cot\sqrt{\delta}\rho\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho$$
$$+\int_{\frac{\pi}{2}}^{\pi} n\cos^2\rho\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho - \int_0^{\pi} ng_1(\rho,\delta)\sin^{n-1}\rho\, d\rho$$
$$+\int_0^{\pi} ng_2(\rho,\delta)\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho + \int_0^{\pi}\frac{2}{n}\sin^2\rho\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho$$
$$-\int_0^{\pi}(n-1)\delta\sin^{n+1}\rho\, d\rho - \int_0^{\frac{\pi}{2}} 2\cos^2\rho\sin^{n-1}\rho\, d\rho$$
$$-\int_{\frac{\pi}{2}}^{\frac{\pi}{2\sqrt{\delta}}} 2\sin\rho\cos\rho\cot\sqrt{\delta}\rho\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho - \int_{\frac{\pi}{2\sqrt{\delta}}}^{\pi} 2\sqrt{\delta}\sin^n\rho\cos\rho\cot\sqrt{\delta}\rho\, d\rho$$
$$+\int_0^{\pi} 6g_2(\rho,\delta)\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho + \int_0^{\pi}\frac{2(n-1)}{n}\sin^2\rho\left(\frac{\sin\sqrt{\delta}\rho}{\sqrt{\delta}}\right)^{n-1} d\rho$$
$$-\int_0^{\frac{\pi}{2}} 4n\cos^2\rho\sin^{n-1}\rho\, d\rho - \int_{\frac{\pi}{2}}^{\frac{\pi}{2\sqrt{\delta}}} 4n\delta\sin^{n+1}\rho\cot^2\sqrt{\delta}\rho\, d\rho$$
$$-\int_{\frac{\pi}{2\sqrt{\delta}}}^{\pi} 4n\sqrt{\delta}\sin^n\rho\cos\rho\cot\sqrt{\delta}\rho\, d\rho \quad . \tag{4.28}$$

If $\delta = 1$, then $g_1(\rho,\delta) = g_2(\rho,\delta) = \cos^2\rho$ and consequently

$$F(n,1) = -2(n-2)\int_0^{\pi}\cos^2\rho\sin^{n-1}\rho\, d\rho \quad , \tag{4.29}$$

which is negative when $n \geq 3$.

Set

$$\delta_2(n) = \inf\{\delta : \delta > \tfrac{1}{4} \text{ and } F(n,\delta) \leqslant 0\}. \qquad (4.30)$$

First note that $F(n,1) < 0$ and so $\frac{1}{4} \leq \delta_2(n) < 1$. Then (4.27) shows that for any nonconstant harmonic map $\phi: M \to N$ there is a certain vector field V^* such that the second variation for V^* is negative. If V^* were smooth we might conclude that ϕ is stable iff ϕ is constant. So the only problem is that the vector field V^* is not smooth . But this difficulty can be overcome just in the same way as in [3] . So the proof is complete .

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