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LOCAL AND GLOBAL BIFURCATION AND ITS APPLICATIONS IN A PREDATOR-PREY SYSTEM WITH SEVERAL PARAMETERS *

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ABSTRACT

A predator-prey system, depending on several parameters, is investigated for bifurcation of equilibria, Hopf bifurcation, global bifurcation occurring saddle connection, and global existence and non-existence of limit cycles, and changes of the topological structure of trajectory as parameters are varied.

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§1. INTRODUCTION

In [1,2] G.W.Harrison analyzed the mathematical model for a predator-prey system with several parameters:

$$\dot{X} = a(X)-f(X)g(Y)$$
$$\dot{Y} = f(X)g(Y)-c(Y) \qquad (1)$$

where $X(t)$, $Y(t)$ are the prey and predator densities, respectively, and

$$a(X) = \rho X \min\left[1, \frac{k-X}{k-\alpha}\right]; \quad \rho>0,\ k>0,\ \alpha\geq 0.$$

$$f(X) = \frac{wX}{X+\varphi}; \quad w>0,\ \varphi>0.$$

$$g(Y) = \frac{Y}{1+\beta Y}; \quad \beta\geq 0. \qquad (2)$$

$$c(Y) = rY+\delta Y^2; \quad r>0,\ \delta\geq 0.$$

The function $a(X)$ indicates the growth rate of the prey in the absence of the predator. $c(Y)$ indicates the growth (or decreased) rate of the predator, $\delta>0$ can be used to model predator intraspecific competition that is not direct competition for food, such as some type of territorility. The product $f(X)g(Y)$ gives the rate at which prey is consumed. $f(X)g(Y)/Y$ was termed the functional response by Solomon [3]. If $\beta=0$, $g(Y)$ reduces to the traditional form $g(Y)=Y$, and indicates that the prey consumed is proportional to the number of predators, but there is evidence that there is mutual interference among predators searching for food, resulting in decreased consumption per predator as predator density increases. The product $f(X)g(Y)$ indicates the numerical response of the predator population. The detailed explication of (2) is given in [1].

The system (1) is investigated by many authors [1, 2, 4]. Because nine parameters are involved in (1), many papers have only analyzed the existence and stability of equilibria, but did show the qualitative behaviour of solutions to system (1) as the parameters varied. In the present paper, this analysis is continued and deepened. We are mainly interested in establishing results on bifurcation of equilibria, Hopf bifurcation, and global bifurcation occurring saddle connection, and global existence and non-existence of limit cycles, and changes of the topological structure of trajectory to system (1) as parametes are varied.

All the results obtained in this paper are established in the domain $\bar{D}=\{(X,Y)\mid X\geq 0, Y\geq 0\}$, and $D=\{(X,Y)\mid X>0, Y>0\}$.

Let us sketch the contents of the paper. By rescalling, we first reduce the number of parameters from nine to seven. In §2 we describe in detail the location of the equilibria as function of the parameters, and bifurcation of equilibria, Hopf bifurcation, global bifurcation occurring saddle connection, and give the conditions for the nonexistence of limit cycle. Also, for some different parameter sets, the corresponding equilibria are shown to be globally attractive. Some specific cases ($\beta=0, \delta\neq 0$; or $\beta\neq 0, \delta=0$) will be pointed out in §3. The results and methods of the proof are similar to that of §2. But, at more simple conditions of the parameters we can obtain more results: for a wide range of parameters the existence and nonexistence of limit cycle are proved. In §4, we discuss the biological explanation for the mathematical results.

The calculations of the proofs for the results are lengthy but straightforward, because more parameters are involved. So, we give only the main results and expressions.

§2. QUALITATIVE ANALYSIS TO SYSTEM (I)(ALL THE PARAMETERS ARE INVOLVED)

We first reduce the number of parameters, by rescaling $\bar{t}=\rho t$, $\bar{Y}=\beta Y$, $\bar{X}=X$. By further setting $w=\frac{w}{\rho\beta}$, $\delta_1=\frac{\delta}{\rho\beta^2}$, $\sigma_1=\frac{\sigma\rho\beta}{\delta}$, $r=\frac{r\beta}{\delta}$. We transform system(I) into($\bar{X},\bar{Y},\bar{t}$ are written by X,Y,t)

$$\begin{aligned}\dot{X} &= X\left(1-\frac{w_1 Y}{(X+\varphi)(1+Y)}\right) \\ \dot{Y} &= \delta_1 Y\left(\frac{\sigma_1 w_1 X}{(X+\varphi)(1+Y)} - r_1 - Y\right)\end{aligned} \qquad X\in[0,\alpha] \qquad (3)_a$$

$$\begin{aligned}\dot{X} &= X\left(\frac{k-X}{k-\alpha} - \frac{w_1 Y}{(X+\varphi)(1+Y)}\right) \\ \dot{Y} &= \delta_1 Y\left(\frac{\sigma_1 w_1 X}{(X+\varphi)(1+Y)} - r_1 - Y\right)\end{aligned} \qquad X\in(\alpha,\infty) \qquad (3)_b$$

The isocline $\dot{X}=0$ consists of strictly increasing curve h_{11} ($Y=\frac{X+\varphi}{w_1-(X+\varphi)}$, $0\le X\le\alpha$) and strictly decreasing curve h_{12} ($Y=\frac{(k-X)(X+\varphi)}{w(k-\alpha)-(k-X)(X+\varphi)}$, $X>\alpha$); the isocline $\dot{Y}=0$ is strictly increasing curve h_2 ($Y=P(X)$). Here the parameters satisfy the inequalities:

$$k-\alpha>0,\quad \sigma_1 w_1-r_1>0,\quad w_1-(\alpha+\varphi)>0$$

The equilibria of system (3) are (0,0), (k,0) and the intersection points of the curves h_{11}, h_{12} with the curve h_2 (see Fig I).

<u>Lemma I</u>. If $k\le\frac{r_1\varphi}{\sigma_1 w_1-r_1}$, the system (3) has only the equilibria (0,0)(is a saddle point) and (k,0)(is a sink), and (k,0) is globally attractive in the domain D. If $k>\frac{r_1\varphi}{\sigma_1 w_1-r_1}$, the equilibria (0,0) and (k,0) are the saddle points.

<u>Proof</u>. For $k<\frac{r_1\varphi}{\sigma_1 w_1-r_1}$, it is easy to show that (0,0) is a saddle, (k,0) is a sink from the eigenvalues of the linearized system (3) at the equilibrium (k,0). Now, let $k=\frac{r_1\varphi}{\sigma_1 w_1-r_1}$, the eigenvalues are $\lambda_1=\frac{-k}{k-\alpha}<0$, $\lambda_2=0$. We can prove that (k,0) is a stable node. In fact, rescaling $t=(X+\varphi)(1+Y)(K-\alpha)t'$, $\bar{X}=\bar{X}-k$, $\bar{Y}=Y$, the system $(3)_b$ becomes ($\bar{X}\to X$, $\bar{Y}\to Y$)

$$\begin{aligned}\dot{X} &= -k(k+\varphi)X-w_1(k-\alpha)kY+(w_1(k-\alpha)+k(k+\varphi))XY-(2k+\varphi)X^2-(1+k^2+k\varphi)X^2Y-X^3-X^3Y \\ \dot{Y} &= \delta(k-\alpha)(\sigma_1 w_1-r_1)XY-\delta_1(k-\alpha)(k+\varphi)(1+r_1)Y^2-\delta_1(k-\alpha)(1+r_1)XY^2-\delta_1(k-\alpha)(k+\varphi)Y^3 \\ &\quad -\delta_1(k-\alpha)XY^3\end{aligned} \qquad (4)$$

We study the stability of the trivial solution of (4) by the centre manifold theorem (see [5] or [6]. Therefore, we can prove that the system (4) has a centre manifold $X=h(Y)=\frac{-w_1(k-\alpha)Y}{k+\varphi}+o(Y^2)$. The flow on the centre manifold is governed by the equation

$$\dot{Y}=-\left[\frac{\delta_1 w_1(k-\alpha)^2(\sigma_1 w_1-r_1)}{k+\varphi}+\delta_1(k-\alpha)(k+\varphi)(1+r_1)\right]Y^2+o(Y^3). \qquad (5)$$

So we get that the equilibrium (k,0) of system (3) is a stable node.

Now we consider the compact rectangle $D_1=[0,a]\times[0,b]$ put $b=\max\left[\frac{\alpha+\varphi}{w_1-(\alpha+\varphi)}, \sigma_1 w_1-r_1\right]$, choose $a>k$, then get $\dot{Y}|_{Y=b}<0$, $\dot{X}|_{X=a}<0$ ($\forall Y>0$), $\dot{X}|_{X=0}<0$ ($\forall Y>0$), and the Y=0 is the integral line. So D_1 is positively invariant. The positive trajectory C starting at $t=t_0$ goes into D_1 for all $t>t_0$. The equilibria are only saddle point (0,0) and stable node (k,0) in D_1. By the Poincaré theorem (see [6]) the closed orbits around (0,0) and (k,0) cannot exist, and also the closed orbits around (k,0) cannot exist because the line Y=0 is the integral line, hence $(0,0)\notin\Omega(C)$, $(k,0)\notin\Omega(C)$, moreover, $\Omega(C)=[(k,0)]$.

This Lemma I means that for $k\le\frac{r_1\varphi}{\sigma_1 w_1-r_1}$ the predator population will ultimately die out, so a long-term interrelation between predator and prey is impossible. Therefore, from now on, the parameters in(3) are assumed to satisfy $k>\frac{\varphi r_1}{\sigma_1 w_1-r_1}$ first. It is easy to get the following Lemma 2 from the monotone of the curves h_{11}, h_{12} and h_2.

<u>Lemma 2</u>. (1°) If $\alpha\le\frac{r_1\varphi}{\sigma_1 w_1-r_1}$ then there exists only the intersection point E(a focus-node) of the curve h_{11} with h_2 ($X\in[0,\alpha]$).(the E and (k.0) lie on the curve h_{12}).

(2°) If $\alpha>\frac{r_1\varphi}{\sigma_1 w_1-r_1}$ and $\frac{\alpha+\varphi}{w_1-(\alpha+\varphi)}\le P(\alpha)$, then there exists only the intersection E of the curve h_{11} with h_{12}, or exist two intersection points E_1 and E_2

(3°) If $\alpha>\frac{r_1\varphi}{\sigma_1 w_1-r_1}$ and $\frac{\alpha+\varphi}{w_1-(\alpha+\varphi)}>P(\alpha)$, then there exist two intersection points E_i ($i=1,2,3$), and E_1 and E_2 lie on the region $0<X\le\alpha$, E_3 lies on the region $X>\alpha$. See Fig I.

<u>Remark</u>. It is simple for the analysis of stability of the equilibria in Lemma 2 (1), (2), then it is omitted here.

In this paper we investigate the existence of Hopf and global bifurcation and other trajectory behaviours under the conditions:

$$k > \alpha > \frac{r_1\varphi}{\sigma_1 w_1 - r_1}, \quad \frac{\alpha+\varphi}{w_1-(\alpha+\varphi)} > P(\alpha). \qquad (6)$$

It is assumed that intersection points of curve h_{11} with h_2 satisfy the equations

$$\Phi(r_1, \sigma_1, \varphi, w_1, Y) = Y^3 + Y^2(r_1+1) + Y(r_1+\sigma_1\varphi-\sigma_1 w_1) + \sigma_1\varphi = 0 . \qquad (7)$$

and

$$X = \frac{Y(r_1+Y)}{\sigma_1} = f_1(Y). \qquad (8)$$

Any solution $X, Y \in R_+$ of (7) and (8) corresponds to one nontrivial equilibrium of (3) and vice versa. Thus, the set

$$M = \left[(r_1,\sigma_1,w_1,\varphi,\alpha,X,Y) \,\middle|\, r_1+\sigma_1\varphi-\sigma_1 w_1<0, 0<X\le\alpha, 0<Y<\frac{\alpha+\varphi}{w-(\alpha+\varphi)}, (6)\text{and}(7)\text{and}(8) \text{ are satisfied}\right\} \qquad (9)$$

describes in a one-to-one way the nontrivial equilibria in their dependence on parameters. The number of positive real solutions of (7) is determined (by the sign of the discriminant Δ of $\Phi=0$,

$$\Delta = \frac{1}{108}\{27\sigma_1^2\varphi^2+\sigma_1\varphi(4(r_1+1)^3-18(r_1+1)(r_1+\sigma_1\varphi-\sigma_1 w))+4(r_1+\sigma_1\varphi-\sigma_1 w)^3-(r_1+1)^2(r_1+\sigma_1\varphi-\sigma_1 w)^2$$
$$= \frac{1}{108}\Delta(\sigma_1\varphi) \qquad (10)$$

The equation (7) has three simple real roots (positive real root Y_1, $Y_2 \in M$, negative real root $Y \notin M$) if $\Delta<0$ and $r_1+\sigma_1\varphi-\sigma_1 w_1<0$, one negative real root $Y\notin M$ and two complex roots $Y \notin M$ if $\Delta>0$ and $r_1+\sigma_1\varphi-\sigma_1 w_1<0$, and at most two of real roots are different and only positive real root $Y_+ \in M$ if $\Delta=0$ and $r_1+\sigma_1\varphi-\sigma_1 w_1<0$.

The intersection point is unique if the curve h_{12} intersects with h_2.

<u>Lemma 3.</u> For $k>\alpha>\max\left[\frac{\varphi r_1}{\sigma_1 w_1-r_1}, \frac{k-\varphi}{2}\right]$, $\frac{\alpha+\varphi}{w_1-(\alpha+\varphi)} > P(\alpha)$ and $r_1+\sigma_1\varphi-\sigma_1 w_1<0$, then

(1°). If $\Delta<0$, the system (3) has three equilibria in the domain D: $E_1(X_1,Y_1)\in M$ is a focus-node, $E_2(X_2,Y_2)\in M$ is a saddle, $E_3(X_3,Y_3)\in h_{12}\cap h_2$ is a sink($\frac{k-\varphi}{2}<X_3<k$ and the occurrence of Hopf bifurcation is impossible).

(2°). If $\Delta=0$, the system (3) has two equilibria in D: $E_0(X_0,Y_0)\in M_0=\{(r_1,\sigma_1, w_1,\varphi,X,Y)\in M \mid \Delta=0, \Phi'_Y=0\}$ is a saddle-node with a stable node region and a saddle region if $\delta_1<\frac{1+Y_0}{Y_0}$; and with a unstable node region and with a saddle region if $\delta_1 > \frac{1+Y_0}{Y_0}$; $E_3(X_3,Y_3)\in h_{12}\cap h_2$ is a sink.

(3°). If $\Delta>0$, the system (3) has only one equilibrium $E_3(X_3,Y_3)\in h_{12}\cap h_2$ in D which is a sink. The bifurcation set $\Delta=0$ is a saddle-node bifurcation for $X\in[0,\alpha]$

<u>Proof</u>. (1°). We linearize system $(3)_b$ at the equilibrium E_3, then get the characteristic equation

$$\lambda^2-\lambda P_3+Q_3=0$$

where the coefficients P_3, Q_3 are given by

$$P_3 = \frac{X_3(k-2X_3-\varphi)}{(X_3+\varphi)(k-\alpha)} - \frac{\delta_1 Y_3(2Y_3+r_1+1)}{1+Y_3} < 0, \quad (X_3 > \frac{k-\varphi}{2})$$

$$Q_3 = \frac{\delta_1 X_3 Y_3(\varphi+2X_3-k)(2Y_3+r_1+1)}{(X_3+\varphi)(k-\alpha)(1+Y_3)} + \frac{\delta_1\sigma_1 w_1^2\varphi X_3 Y_3}{(1+r_3)^3(X_3+\varphi)^2} > 0.$$

The fixed point E_3 is a sink obviously, and the occurrence of Hopf bifurcation forms the change of the stability at equilibrium E_3 is impossible because $P_3<0$ for any parameters.

The equilibria E_1, E_2, E_3 lie on the isocline h_2, by the Poincaré theorem E_2 is a saddle, E_1 is a focus-node.

We linearize system $(3)_a$ at E_1 and E_2, the coefficients P_1, Q_1 ; P_2, Q_2 of the characteristic equations are

$$P_i = \frac{X_i}{X_i+\varphi} - \frac{\delta_1 Y_i(2Y_i+r_1+1)}{1+Y_i},$$
$$Q_i = \frac{\delta_1(r+Y_i)}{(X_i+\varphi)(1+Y_i)}\left[\sigma_1\varphi - Y_i(2Y_i+r+1)\right]. \qquad i=1,2. \qquad (11)$$

(2°). We linearize system $(3)_a$ at E_0, get $Q(Y_0)=0$ and the eigenvalues $\lambda_1=0$ and $\lambda_2<0$, then the codimension one bifurcation of equilibria occur. So $E_0\in M_0=(r_1,\sigma_1,\varphi,w_1,X,Y)\in M, \Delta=0, \Phi'_Y(Y_0)=0, Q(Y_0)=0]$. Now we have to investigate the behaviour of the flow in the neighbourhood of E_0. Applying the transformation: $t=(X+\varphi)(1+Y)t'$, $X=X_0+\bar{X}$, $Y=Y_0+\bar{Y}$, $V=\delta_1 Y_0(X_0+\varphi)(2Y_0+r_1+1)\bar{X} - \frac{\varphi X_0}{+Y}\bar{Y}$, $U=\bar{Y}$, one transforms the system $(3)_a$ into the following from:

$$\dot{V} = UV\left[\frac{X_0(Y_0-1)-\varphi}{Y_0} + \frac{2(1+Y_0)X_0}{\delta_1 Y_0^2} - \frac{w_1 X_0(\sigma_1 w_1-(r_1+Y_0)(1+2Y_0)-Y_0(1+Y_0))}{Y\ (X\ +\ )(2Y\ +r+1)(1+Y\ )}\right.$$
$$+U^2\left[\frac{\delta_1(X_0+\varphi)(2Y_0+r_1+1)(X_0(Y_0-1)-\varphi)X_0}{Y_0^2} + \frac{(X_0+\varphi)(2Y_0+r_1+1)(1+Y_0)X_0^2}{\delta_1 Y_0^2}\right.$$
$$\left.-\frac{w_1 X_0^2}{Y_0^2(1+Y_0)}(\sigma_1 w_1-(r_1+Y_0)(1+2Y_0)-Y_0(1+Y_0)) + \frac{w_1 X_0}{1+Y_0}\delta_1(X_0+\varphi)(1+3Y_0+r_1)\right]$$
$$+V^2\frac{1+Y_0}{\delta_1 Y_0(X_0+\varphi)(2Y_0+r_1+1)} + \cdots\cdots\cdots = UV\xi_{11}+U^2\xi_{02}+V^2\xi_{20}+\cdots\cdots$$

$$\dot{U} = V\frac{Y_0(1+Y_0)}{X_0+\varphi} + U(2Y_0+r_1+1)(1+Y_0-\delta_1 Y_0) - UV\frac{(\sigma_1 w_1-(r_1+Y_0)(1+2Y_0)-Y_0(1+Y_0))}{Y_0(X_0+\varphi)(2Y_0+r_1+1)}$$
$$-U^2\left(\frac{(\sigma_1 w_1-(r_1+Y_0)(1+2Y_0)-Y_0(1+Y_0))X_0}{Y_0^2} + \delta_1(X_0+\varphi)(1+3Y_0+r_1)\right) + \cdots\cdots$$
$$= \eta_{10}V+\eta_{01}U-\eta_{11}UV-\eta_{02}U^2+\cdots\cdots \qquad (12)$$

By centre manifold theory, after a lengthy but straightforward calculation, we get that the system (I3) has a centre manifold

$$U=h(V)=\frac{-Y_0(I+Y_0)}{(X_0+\varphi)(2Y_0+r_1+I)(I+Y_0-\delta_1 Y_0)}+o(V^2)$$

The flow on the centre manifold is governed by the equation

$$\dot V=\xi_{11}h(V)V+\xi_{02}h^2(V)+\xi_{20}V^2+o(V^3)=\Delta_M V^2+o(V^3)$$

Therefore, we come to a conclusion of part(2)(E_3 is a sink obviously).

We easily obtain the proof of the part(3).

For later use (to determine the existence of Hopf bifurcation and homoclinic orbit) we give the following Lemma.

<u>Lemma4.</u> For $\delta_1>\delta_{12}$ the equilibrium $E_1\in M$ is a stable node. For $0<\delta_1<\delta_{11}$ the E_1 is an unstable node. For $\delta_{11}<\delta_1<\delta_{12}$ the E_1 is a focus or centre.

<u>Proof.</u> Applying the transformation: $t=(X+\varphi)(I+Y)t'$, $X=X_1+\bar X$, $Y=Y_1+\bar Y$, the system $(3)_a$ becames as (the new variables $\bar X,\bar Y$ here are written as X,Y)

$$\dot X=X_1(I+Y_1)X-\frac{w_1X_1}{I+Y_1}Y+\frac{X_1(Y_1-I)-\varphi}{Y_1}XY+(I+Y_1)X^2+X^2Y$$

$$\begin{aligned}\dot Y=&\frac{\sigma_1 w_1\varphi}{X_1+\varphi}\delta_1Y_1X-\delta_1Y_1(X_1+\varphi)(2Y_1+r_1+I)Y+(\sigma_1 w_1-(r_1+Y_1)(I+2Y_1)-Y_1(I+Y_1))\delta_1XY\\&-\delta_1(X_1+\varphi)(I+3Y_1+r_1)Y^2-\delta_1(I+3Y_1+r_1)XY^2-\delta_1(X_1+\varphi)Y^3-\delta_1XY^3\end{aligned}\tag{I3}$$

The eigenvalues λ of the corresponding equilibrium (0,0) of the system (I3) are

$$\lambda_{1,2}=\frac{I}{2}(P_1\pm\sqrt{P_1^2-4Q_1})\tag{I4}$$

where

$$P_1=X_1(I+Y_1)-\delta_1Y_1(X_1+\varphi)(2Y_1+r_1+1)$$
$$Q_1=\delta_1 w_1X_1(\sigma_1\varphi-Y_1^2(2Y_1+r_1+1))>0$$

Let

$$\begin{aligned}f(\delta_1)=P_1^2-4Q_1=&\delta_1^2Y_1^2(X_1+\varphi)^2(2Y_1+r_1+I)^2+\delta_1(2X_1Y_1(I+Y_1)(X_1+\varphi)(2Y_1+r_1+I)\\&-4\sigma_1w\varphi X_1)+X_1^2(I+Y_1)^2=0\end{aligned}\tag{I5}$$

Solve the equation (I5), we can show that there exist two values δ_{11} and δ_{12} such that $f(\delta_{11})=f(\delta_{12})=0$. Therefore, for $0<\delta_1<\delta_{11}$ and $\delta_1>\delta_{12}$, $f(\delta_1)>0$, the equilibrium E_1 is a node; for $\delta_{11}<\delta_1<\delta_{12}$, $f(\delta_1)<0$, the E_1 is a centre or focus. As is well known for $\delta_1>\delta_0$ E_1 is a sink; for $\delta_1<\delta_0$ E_1 is a source. So we obtain the conclusion of Lemma4.

<u>Theorem 1</u>. (1°). In case $\alpha\geq\frac{k-\varphi}{2}$, $\delta_1>\frac{w_1}{4\varphi}$ the system (3) has no closed orbit in D.

(2°). In case $\alpha\geq\frac{k-\varphi}{2}$ the system (3) has no closed orbit around the sink E_3.

<u>Proof</u>. Take Dulac's function $B(X,Y)=(XY)^{-1}$. From $(3)_a$ we get by a short calculation

$$\operatorname{div}\Big|_{(3)_a}=(X^{-1})\left[\frac{w_1X-\delta_1(X+\varphi)^2(1+Y)}{(1+Y)(X+\varphi)^2}-\frac{\delta_1\sigma_1w_1X}{(X+\varphi)(I+Y)^2}\right]\tag{I6}$$

Let

$$w_1X-\delta_1(X+\varphi)(1+Y)<w_1X-\delta_1(X+\varphi)^2=f(X)$$

It follows that for $w_1-4\delta_1\varphi<0$, $f(X)<0$. Therefore, $\operatorname{div}\big|_{(3)_a}<0$.

Rescaling $t=(k-\alpha)(I+Y)(X+\varphi)t'$, we transform system $(3)_b$ into

$$\begin{aligned}\dot X&=X(k-\alpha)(I+Y)(X+\varphi)-w_1(k-\alpha)XY\\\dot Y&=\delta_1\sigma_1w_1(k-\alpha)XY-\delta_1(k-\alpha)Y(r_1+Y)(1+Y)(X+\varphi)\end{aligned}\tag{I7}$$

Take Dulac's function $B(X,Y)=(XY)^{-1}$. From (I7) we obtain by a short calculation

$$\operatorname{div}\Big|_{(3)_b}=Y^{-1}\left[(I+Y)(k-2X-\varphi)-\delta_1X^{-1}(X+\varphi)(k-\alpha)(2Y+r_1+1)\right]<0.\tag{I8}$$

$$(\forall\delta_1>0,\ X\in[\tfrac{k-\varphi}{2},\infty))$$

From (16) and (I8) we obtain that for $\delta_1>\frac{w_1}{4\varphi}$ the system (3) has no closed orbit in D. From (I8) we get that the system (3) has no closed orbit around E_3.

<u>Theorem 2.</u> If the parameters satisfy: $r_1-\sigma_1\varphi>0$, $w_1-\varphi>0$, $k>\alpha\geq\max\left[\frac{r_1\varphi}{\sigma_1w_1-r_1},\frac{k-\varphi}{2}\right]$, $\frac{\alpha+\varphi}{w_1-(\alpha+\varphi)}>P(\alpha)$ and $\Delta<0$. By further assuming that the system (3) has no closed orbit around the node E_1. Then

(1°). There exists a Hopf bifurcation value δ_0:

$$\delta_0=\frac{X_1(I+Y_1)}{Y_1(X_1+\varphi)(2Y_1+r_1+I)}\quad\text{(respect to the contral parameter }\delta_1)$$

such that for $\delta_1<\delta_0$ sufficiently close to δ_0, there exists a small oscillatory attracting(subcritical) Hopf limit cycle around E_1. Moreover, $\delta_{11}<\delta_0<\delta_{12}$, $\delta_0\leq\frac{w_1}{4\varphi}$. Here E_1 is a sink for $\delta_1<\delta_0$, and E_1 is a source for $\delta_1>\delta_0$.

(2°). There exists a global saddle connection bifurcation value δ^* such that for $\delta_1=\delta^*$ saddle loop $L_0\subset L_1^s(E_2)\cap L_4^u(E_2)$ occurs.

(3°). There exists a value

$$\delta^0=\frac{X_2(I+Y_2)}{Y_2(X_2+\varphi)(2Y_2+r_1+I)}$$

and for $\delta_1<\delta^0$, the saddle loop L_0 is a unstable; and for $\delta_1>\delta^0$, the L_0 is a stable. Further, if $r_1-\sigma_1\ >0$, $w_1-\ >0$, then $\delta^0<\delta_0$.

(4°). $\delta^0<\delta^*<\delta_0$.

(5°). From (1°-4°) we conclude that for $\delta_1>\delta_1$ the topological structure of the trajectory is illustrated in Fig.1; for $\delta_1<\delta_0$ sufficiently closed to δ_0, there exists a small oscillatory attracting Hopf limit cycle; as δ_1 decreases to δ^* the unstable manifold L_4 and stable manifold L_1 form a loop L_0 $L_1^s(E_2)$ $L_4^u(E_2)$ and the Hopf limit cylce disappears (is "swallowed" by L_0); as δ_1 exceeds δ^* the loop L_0 breaks up. The topological structures of the trajectory are illustrated in Figs.1-4 as δ_1 is varied.

Proof.(1°). From (11) we set $P_1=0$, we readily find that δ_0 and $P'_{1\delta_1}(\delta_0)<0$. For (13) we calculate α_3 (the formula is derived by Andornov, see [6]).

$$\alpha_3=\pi/4\varphi\{\varphi\delta_0(1+Y_1)^2(r_1+Y_1)(X_1(Y_1-1)-\varphi)^2/Y_1^2-\sigma_1 w_1\varphi\delta_0^2X_1(1+Y_1)(1+r_1+3Y_1)(X_1(Y_1-1)-\varphi)$$
$$-w_1X_1^2(\sigma_1w_1-r_1-3Y_1^2-2Y_1(r_1+1))^2-w_1X_1^2(1+Y_1)(\sigma_1w_1-r_1-3Y_1^2-2Y_1(r_1+1))$$
$$-2\sigma_1w_1\varphi\delta_0^3X_1Y_1(1+Y_1)(X_1+\varphi)(1+r_1+3Y_1)^2+2w_1X_1^2(1+Y_1)^2$$
$$+\sigma_1w_1\varphi\delta_0^2X_1(X_1+\varphi)(1+r_1+3Y_1)(\sigma_1w_1-r_1-3Y_1^2-2Y(r_1+1))$$
$$+2\delta_0X_1^2(1+Y_1)^2(X_1+\varphi)(1+r_1+3Y_1)(\sigma_1w_1-r_1-3Y_1^2-2Y_1(r_1+1))$$
$$+w_1\varphi\delta_0(r_1+Y_1)(1+Y_1)(X_1(Y_1-1)-\varphi)+2X_1^2(1+Y_1)^2(X_1(Y_1-1)-\varphi)/Y_1$$
$$+3\sigma_1w_1\varphi\delta_0^2X_1^2Y_1(1+Y_1)^2-3\sigma_1^2w_1^2\varphi^2\delta_0^3X_1Y_1-2X_1^3(1+Y_1)^2+2w_1\varphi\delta_0(r_1+Y_1)(1+Y_1)$$
$$-2\sigma_1w_1\varphi\delta_0^2X_1^2(1+Y_1)(1+r_1+3Y_1)+2\delta_0X_1^3(1+Y_1)^2(1+r_1+3Y_1)\}$$

Now, we have to investigate the sign of α_3 or equivalently. If $\alpha_3\neq 0$, then the following two cases can occur. Case 1. $\alpha_3<0$. In this case for $\delta_1<\delta_0$ sufficiently close to δ_0, there exists an attractive (subcritical) Hopf limit cycle. Case 2. $\alpha_3>$ In this case for $\delta_1>\delta_0$ sufficiently close to δ_0 there exists a repelling (supercritical) Hopf limit cycle.

To calculate analytically the zero set of α_3 seems rather difficult, since α_3 is a function of the independent parameters σ_1, w_1, φ, r_1, δ_0, while $(X_1,Y_1)\in M$. We can, however, prove that the only possible case is $\alpha_3<0$ from the conclusion of the following (2°) and (3°).

(2°). Let us first establish some properties of the unstable manifold $L_4^u(E_2)$ and stable manifold $L_1^s(E_2)$. The changes of the vector fields are indicated (Fig 1) in D. The variable is X_2 and Y_2 instead of X_1 and Y_1 in (13), then we obtain (13)'. The equation, characterizing directions of the manifold for the saddle point E_2, has the form from (13)'

$$S^2-S\frac{[X_2(1+Y_2)+\delta_1Y_2(X_2+\varphi)(2Y_2+r_1+1)](1+Y_2)}{w_1X_2}+\frac{\varphi\sigma_1\delta_1Y_2(1+Y_2)}{X_2(X_2+\varphi)}=0 \qquad (19)$$

The two roots S_1 and S_2 of (19) are positive; it follows that the segments of the manifold near E_2 lie in subregions 1 — 6 (see Fig 1). Taking into consideration the direction of the field of the system (3), the configuration of the manifold $L_{i(i=1,2,3,4)}$ near E_2 can be represented as follows:

Choose $Y_m=\frac{(\alpha+\varphi)}{w_1-(\alpha+\varphi)}$ and $X_m=k$, then get $\dot{X}|_{Y=Y_m}<0$, $\dot{Y}|_{Y=Y_m}<0$ and $\dot{X}|_{X=X_m}<0$. We consider the compact region $D_2=[(X,Y)|X=0,\ Y=0,\ X=X_m,\ Y=Y_m]$, $X=0$ and $Y=0$ are the integral lines. The trajectory goes into D_2 eventually as $t\to+\infty$. The manifold L_2 goes to E_3 as $t\to+\infty$. Because E_3 is a sink and has no closed orbit around E_3 (see Theorem 1(2°)). The manifold L_3 goes to infinite or crosses

the subregion 6, and goes to infinite as $t\to-\infty$. The manifolds L_2 and L_3 merge into a loop which is impossible, because there is no closed orbit arount E_3.

The behaviour of the manifolds L_1 and L_4 is changing as the parameter δ_1 is varied. For $0<\delta_1\leq\delta_{11}$ E_1 is an unstable node and assume that there is no closed orbit around E_1, in this case L_1 goes to E_1 as $t\to-\infty$; L_4 crosses the isoclines h_{11} and h_2, crosses the subregion 6 again, goes to E_3 finally as $t\to+\infty$, and L_4 lies outside L_1 (see Fig 4). For $\delta_1\geq\delta_{12}$, E_1 is a stable node, and assume that there is no closed orbit around E_1. In this case L_4 goes to E_1 as $t\to+\infty$. and L_1 crosses the isoclines h_2, h_{11}, does to infinite finally as $t\to-\infty$, and L_1 lies outside L_4 (see Fig 1). By the continuity of the solution with respect to the parameter δ_1, we conclude that there exists a value δ^* such that the manifolds L_1 and L_4 merge into a loop $L_0\subset L_1^s(E_2)\cap L_4^u(E_2)$. (see Fig 3). Moreover,

(3°). Let us consider the "saddle quantity"

$$V(X_2,Y_2)=X_2(1+Y_2)-\delta_1Y_2(X_2+\varphi)(2Y_2+r_1+1)$$

Set $V(X_2,Y_2)=0$, get $\delta^*=\frac{X_2(1+Y_2)}{Y_2(X+\varphi)(2Y_2+r_1+1)}$. For $\delta_1<\delta^*$ the saddle loop L_0 is unstable, and for $\delta_1>\delta^*$ the saddle loop L_0 is stable. We can prove $\delta_0>\delta^\circ$. In fact, $\delta_0\in N_1=[(r_1,\sigma_1,w_1,\varphi,X,Y)|(X_1,Y_1)\in M,\ P_1=0,\ Q_1>0]$, $\delta^\circ\in N_2=[(r_1,\sigma_1,w_1,\varphi,X,Y)|(X_2,Y_2)\in M,\ P_2\neq 0,\ Q_2<0]$, so $\delta_0=\frac{Y_1^2+Y_1(1+\bar{\varphi}+\sigma_1w_1)+\bar{\varphi}}{\sigma_1w_1Y_1(2Y_1+r_1+1)}$, $\delta^\circ=\frac{Y_2^2+Y_2(1+\bar{\varphi}+\sigma_1w_1)+\bar{\varphi}}{\sigma_1w_1Y_2(2Y_2+r_1+1)}$.

where $\bar{\varphi}=r_1-\sigma_1\varphi>0$.

Let us try to establish the relation between δ° and δ_0. We study the function

$$F(Y)=\frac{Y^2+Y(1+\bar{\varphi}+\sigma_1w_1)+\bar{\varphi}}{Y(2Y+r_1+1)} \qquad (20)$$

we obtain

$$F'(Y)=\frac{[-(r_1+1)-2\sigma_1(w_1-\bar{\varphi})]Y^2-4\bar{\varphi}Y-(r_1+1)\bar{\varphi}}{Y^2(2Y+r_1+1)^2} \qquad (21)$$

Therefore, for $w_1-\bar{\varphi}>0$ and $\bar{\varphi}>0$, $F'(Y)<0(\forall Y>0)$. Because $Y_1<Y_2$, it turns out that $F(Y_2)<F(Y_1)$. Thus we have $\delta^0<\delta_0$.

(4°). If $V>0$, then L_0 is unstable, we thus have $\delta^*<\delta_0$. At $\delta_1=\delta^*$, the point (X_2,Y_2) is stable, and there is no closed orbit in the saddle loop L_0. The case is impossible. If $V<0$, then L_0 is stable, the following arrangement is possible: $\delta^*>\delta_0$ or $\delta^0<\delta^*<\delta_0$. If $\delta^*>\delta_0$, at $\delta_1=\delta^*$, the point (X_2,Y_2) is an unstable focus, and L_0 is stable; there is no closed orbit in the saddle loop L_0. This case also is impossible. Therefore, we $\delta^0<\delta^*<\delta_0$.

(5°). Finally, we obtain (5) by (1)-(4).

This completes the proof of Theorem 2.

§3. QUALITATIVE ANALYSIS TO SYSTEM (1) (some specific cases).

Case (I). $\beta \neq 0$, $\delta = 0$.

If $\beta \neq 0$, $\delta = 0$, the system(I) become into

$$\dot{X} = \rho X \min\left\{ I, \frac{k-X}{k-\alpha} \right\} - \frac{wXY}{(I+\beta Y)(X+\varphi)},$$

$$\rho > 0,\ k > 0,\ \alpha \geq 0. \qquad (22)$$

$$\dot{Y} = \frac{\sigma wXY}{(I+\beta Y)(X+\varphi)} - r Y,$$

Here the parameters satisfy the inequalities:

$$k-\alpha > 0,\ w_1 - r_1 > 0,\ w_1 - (\alpha+\varphi) > 0. \quad \left(r_1 = \frac{r}{\beta},\ w_1 = \sigma w \right) \qquad (23)$$

The equilibria of system (22) are (0,0), (k,0) and the intersection points of the curves h_{11} with the curve h_2.

For system (22), by using anologous methods to §2, we can get the following results which are similar to that of §2.

(I) If $k \leq \frac{r_1\varphi}{w_1 - r_1}$, the system (22) has only the equilibria (0,0)(is a saddle-point) and (k,0) is globally attractive in D.

The proof of (I) is similar to Lemma I. The result (I) means that for $\beta \neq 0$, $\delta = 0$ if $k \leq \frac{r_1\varphi}{w_1 - r_1}$, then the predatory population will ultimately die out.

(II). If $k > \frac{r_1\varphi}{w_1 - r_1}$, and (23) is satisfied, then

(1°). If $\alpha \leq \frac{r_1\varphi}{w_1 - r_1}$, there exists only the intersection point E of the curve h_{11} with h_2 ($X \in [0, \alpha]$). E is a focus-node.

(2°). If $\alpha > \frac{r_1\varphi}{w_1 - r_1}$, $\frac{\alpha+\varphi}{w_1 - (\alpha+\varphi)} < P(\alpha)$, there exists only the intersection E of h_{11} with h_{12}.

(3°). If $\alpha > \frac{r_1\varphi}{w_1 - r_1}$, $\frac{\alpha+\varphi}{w_1 - (\alpha+\varphi)} = P(\alpha)$, there exist two intersection points E_1 (saddle point) and E_2 (focus-node).

(4°). If $\alpha > \frac{r_1\varphi}{w_1 - r_1}$, $\frac{\alpha+\varphi}{w_1 - (\alpha+\varphi)} > P(\alpha)$, there exist at most three intersection points E_i (i=I,2,3), and E_1 and E_2 lie on the region $0 < X \leq \alpha$, E_3 lies on the region $X > \alpha$.

(III). The intersection points of curve h_{11} with h_2 satisfy the equations

$$\Phi_1(w_1, r_1, \varphi, X) = X^2 + (r_1 + \varphi - w_1)X + r_1\varphi = 0 \qquad (24)$$

$$Y = \frac{X(w_1 - r_1) - r_1\varphi}{r_1(X+\varphi)} = f_2(X) \qquad (25)$$

The intersection points $(X_i, Y_i) \in M_1 = \left[(w_1, r_1, \varphi, X, Y) \,\middle|\, k > \alpha \geq \max\left(\frac{k-\varphi}{2}, \frac{\varphi r_1}{w_1 - r_1}\right),\ r_1 + \varphi - w_1 < 0,\ 0 < X \leq \alpha,\ \Phi_1(w_1, r_1, \varphi, X_i) = 0,\ Y_i = f_2(X_i) \right]$.

The set $\Delta_1 = (r_1 + \varphi - w_1)^2 - 4r_1\varphi = 0$ is a saddle-node bifurcation(codimension one) for $X \in [0, \alpha]$.

(1°). If $\Delta_1 > 0$, this system (22) has three equilibria in D: $E_1(X_1, Y_1) \in M$ (is a focus-node), $(X_2, Y_2) \in M$ (is a saddle point), $(X_3, Y_3) \in h_{12} \cap h_2$ (is a sink, and $\frac{k-\varphi}{2} < X_3 < k$, moreover, the occurrence of Hopf bifurcation is impossible at (X_3, Y_3).

(2°). If $\Delta_1 = 0$, the system (22) has two equilibria in D: (X_0, Y_0) is a saddle-node($X_0 = \frac{w_1 - r_1 - \varphi}{2}$, $Y_0 = f_2(X_0)$, $\varphi_0 = \frac{\sigma_1 r_1(\varphi - X_0 Y_0)}{(I+Y_0)(X_0+\varphi)} = 0$), $(X_3, Y_3) \in h_{12} \cap h_2$ is a sink.

(3°). If $\Delta_1 < 0$, then the system (22) has only one equilibrium $E_3 \in h_{12} \cap h_2$ in D which is a sink.

(IV). We have

(1°). If $\alpha \geq \frac{k-\varphi}{2}$ and $\sigma_1 > \frac{w_1}{4r_1\varphi}$, then the system (22) has no closed orbit in D.

(2°). If $\alpha \geq \frac{k-\varphi}{2}$, then the system (22) has no closed orbit around E_3. In fact, setting Dulac's function $B(X,Y) = ((X+\varphi)XY)^{-1}$. From $(22)_a$ we get by a short calculation

$$\operatorname{div}\Big|_{(22)_a} = \left[(X+\varphi)X\right]^{-1} \frac{w_1 X - \sigma_1 r_1 (X+\varphi)^2}{X+\varphi} < 0, \left(\forall \sigma_1 > \frac{w_1}{4r_1\varphi},\ 0 < X \leq \alpha\right).$$

By setting Dulac's function $B(X,Y) = (XY)^{-1}$, from $(22)_b$ we get

$$\operatorname{div}\Big|_{(22)_b} = B(X,Y)\left[X(I+Y)(k-2X-\varphi) - \sigma_1 r_1 Y(k-\alpha)(X+\varphi)\right] < 0, \left(\forall \alpha \geq \frac{k-\varphi}{2}, X > \alpha \geq \frac{k-\varphi}{2}\right).$$

(V). This is similar to the conclusions of Theorem 2, i.e. there is a Hopf bifurcation value $\sigma_0 = \frac{w_1 r_1 Y_1}{(X_1+\varphi)^2}$ and a global saddle connection bifurcation value $\sigma_1 = \sigma^{**} < \sigma_0$, moreover, $\sigma^* < \sigma^{**} < \sigma_0$, where $\sigma^* = \frac{w_1 r_1 Y_2}{(X_2+\varphi)^2}$. For $\sigma_1 < \sigma_0$ sufficiently close to σ_0 there exists a small oscillatory attracting(subcritical) Hopf limit saddle around E_1. As σ_1 decreases to σ^{**}, the limit cycle disappears and the saddle loop occurs. As σ_1 decreases continuously, the saddle loop disappears. The topologic structures of the trajectory are similar to Figs. I-4 as the parameter σ_1 is variable

Case (II). $\beta = 0$, $\delta \neq 0$.

By rescaling : $\sigma_1 = \frac{\sigma}{\delta}$, $r_1 = \frac{r}{\delta}$, the system (I) becomes into

$$\dot{X} = X\left[\rho - \frac{wY}{X+\varphi}\right] \quad (\forall X \in [0, \alpha]).$$

$$\dot{Y} = \delta Y\left[\frac{\sigma_1 w X}{X+\varphi} - r_1 - Y\right] \qquad (26)_a$$

$$\dot{X} = X\left[\frac{\rho(k-X)}{k-\alpha} - \frac{wY}{X+\varphi}\right] \quad (\forall X \in (\alpha, \infty))$$

$$\dot{Y} = \delta Y\left[\frac{\sigma_1 w X}{X+\varphi} - r_1 - Y\right] \qquad (26)_b$$

For the system (26), by using analogous methods to §2, we can get the following results which are similar to that of §2.

(I). If $k \leq \frac{r_1\varphi}{\sigma_1 w_1 - r_1}$, the system (26) has only the equilibria(0,0)(is a saddle point) and (k,0)(is a sink), moreover, (k,0) is globally attractive in D.

(II). If $k > \frac{r_1\varphi}{\sigma_1 w_1 - r_1}$, then the system (26) has three equilibria $(X_i, Y_i) \in M_3 = \left[(\rho, \varphi, w, \sigma_1, r_1, X, Y) \,\middle|\, k > \frac{r_1\varphi}{\sigma_1 w_1 - r_1},\ \frac{w}{\rho(\alpha+\varphi)} > \frac{\sigma_1 w \alpha}{\alpha+\varphi} - r_1,\ w(\sigma_1 w - r_1) - 2\varphi\rho > 0,\ 0 < X \leq \alpha,\ \Phi_3 = \rho X^2 + (2\varphi\rho - w(\sigma_1 w - r_1))X + \rho\varphi^2 + r_1\varphi w = 0 \right]$, (i=I,2), and $(X_3, Y_3) \in h_{12} \cap h_2$ in D.

The set $\Delta_2=(\sigma_1 w-r_1)^2-4\rho\sigma_1\varphi=0$ is a saddle-node bifurcation (codimension one) for $X\in[0,\alpha]$.

(1°). If $\Delta_2>0$, the system (26) has three equilibria in D: focus-node $E_1(X_1,Y_1)\in M_3$, saddle point $E_2(X_2,Y_2)\in M_3$, sink $E_3(X_3,Y_3)\in h_{12}\cap h_2$.

(2°). If $\Delta_2=0$, $E_0(X_0,Y_0)\in M_3$ (saddle-node), $E_3\in h_{12}\cap h_2$ (sink).

(3°). If $\Delta_2<0$, $E_3\in h_{12}\cap h_2$ (sink).

(III). If $k>\frac{r_1\varphi}{\sigma_1 w-r_1}$

(1°). For $\alpha\geq\frac{k-\varphi}{2}$ and $\delta>\frac{w}{4\varphi}$, then the system (26) has no closed orbit in D.

(2°). For $\alpha\geq\frac{k-\varphi}{2}$, then the system (26) has no closed orbit around E_3.

(IV). If $\Delta_2>0$, for the system (26) there exists a Hopf bifurcation value $\delta_0=\frac{wX_1}{(X_1+\varphi)^2}$ and a global saddle connection bifurcation value $\delta^*<\delta_0$, moreover, $\delta^0<\delta^*<\delta_0$, where $\delta^0=\frac{wX_2}{(X_2+\varphi)^2}$. For $\delta<\delta_0$ sufficiently close to δ_0 there exists a small oscillatory attracting Hopf limit cycle around E_1. As the parameter δ is varied, the limit cycle occurs and disappears, and the saddle loop occurs and disappears—the changes can be seen in Figs. 1–4 also.

Case (III). $\beta=0$, $\delta=0$.

For $\beta=\delta=0$, the system (I) can be written as $(I)_0$. If $k\leq\frac{r\varphi}{\sigma w-r}$, then $(I)_0$ has only a saddle point (0,0) and a global attracting equilibria (k,0) in D. If $k>\frac{r\varphi}{\sigma w-r}$, the $(I)_0$ has an equilibrium (X_0,Y_0) except the saddle point (0,0) and (k,0) in D, moreover, for $0<\frac{r\varphi}{\sigma w-r}\leq\alpha$, the equilibrium (X_0,Y_0) is a source, and for $\frac{r\varphi}{\sigma w-r}>\alpha$, (X_0,Y_0) is a sink.

For the system $(I)_0$ we get the following global results.

Theorem 3. (1°). For $0<\frac{r\varphi}{\sigma w-r}\leq\alpha$, the system $(I)_0$ has at least one limit (k,0) cycle in D.

(2°). For $\frac{r\varphi}{\sigma w-r}>\alpha$, the system $(I)_0$ has no closed orbit in D. Moreover, is a global attractive in D.

Proof. (1°). The line X=k is a line without contact, in fact, $\dot{X}\big|_{X=k}=\frac{-wkY}{k+\varphi}<0\ (\forall Y>0)$, again, we consider the line $h=\sigma X+Y-A$, and

$$\dot{h}\big|_{h=0}=\sigma X(\rho+r)-rA.\quad(\forall X\in[0,\alpha]) \tag{27}$$

$$\dot{h}\big|_{h=0}=\frac{\sigma\rho X(k-X)}{k-\alpha}+r\sigma X-rA.\quad(\forall X\in(\alpha,\infty)) \tag{28}$$

Now taking

$$A>\max\left[\max_{0<X\leq\alpha}\left(\frac{\sigma}{r}(\rho+r)X\right),\ \max_{\alpha<X\leq k}\left(\frac{\sigma\rho X(k-X)}{r(k-\alpha)}+\sigma X\right),\ \frac{\rho(k+\varphi)}{2w}\right]$$

We well know that (27) and (28) are always negative. So the line h=0 is a line without contact for $X\in[0,k]$. The outer boundary of the annular region of Poincaré Bendixson is formed by the lines h=0, X=k and X=Y=0 in D. The equilibrium (X_0,Y_0) is a source. Therefore, we know that the system $(I)_0$ has at least one limit cycle in D.

(2°). For $\alpha<\frac{r\varphi}{\sigma w-r}<k$, the equilibria lie in always the region $X>\alpha$. By rescaling $t=(X+\varphi)(k-\alpha)t'$, the system $(I)_0$ becomes into

$$\begin{aligned}\dot{X}&=\rho X(k-X)(X+\varphi)-w(k-\alpha)XY\\ \dot{Y}&=\sigma w(k-\alpha)XY-r(k-\alpha)(X+\varphi)Y\end{aligned} \tag{29}$$

Taking Dulac's function $B(X,Y)=(XY)^{-1}$, then we get

$$\operatorname{div}\big|_{(29)}=\frac{\rho}{Y}(k-2X-\varphi)<0.\quad\left(\forall X\in\left(\frac{k-\varphi}{2},\infty\right)\right).$$

If this is a closed orbit, then the closed orbit crosses the line $X=\frac{k-\varphi}{2}$ for two time. But this is $\dot{Y}<0$ at the life region of vertical line. So existence of closed orbit in D is impossible, Therefore we get that (X_0,Y_0) is unique w-limit set in D.

The detailed proofs of the systems (22) and (25) are given in [7, 8].

We will study the codimension two bifurcation in a separate paper.

4. BIOLOGICAL EXPLANATION.

The biological implications for these two different classes of equilibrium (the equilibrium on the Y-axis and in D) and the limit cycle and the saddle loop are quite different, which indicate different results of the interaction of a predator-prey system. (We call the equilibrium on the Y-axis as the extinct equilibrium, call the equilibrium in D as the compromise equilibrium).

(I). If the trajectories tend to extinct equilibrium (k,0) as $t\to+\infty$, then this means that the predator population will ultimately tend to extinction, and prey population with different initiative condition will ultimately get to the balance's density k.

(II). If the trajectories tend to two stable equilibria (X_1,Y_1) and (X_3,Y_3) as $t\to+\infty$, then this means that the predator-prey interactions will ultimately tend to the balance's behaviour. Moreover, the trajectories with different initiative condition will ultimately tend to different equilibrium, and the predator population coexist with prey population at different equilibrium where production of prey equals consumption of prey and hence the system (1) regulation are the more likely to make two of the equilibria stable.

(III). If there will be three equilibria, then it means that for some prey density X constant predator consumption exceeds production, but for some large prey density X production exceeds consumption because production has increased, but predator intraspecific competition keeps consumption in check.

(IV). If the stable limit cycle around the equilibrium (X_4, Y_4) arises, then this indicates that the predator coexists with prey at another balance's behaviour

(V). If the Hopf bifurcation and global saddle connection bifurcation arise as the contral parameter is varied, then these biological interpretations for the procedure are similar to the case stated above, but the variant procedures of the biological phenomenons are very complex and interesting.

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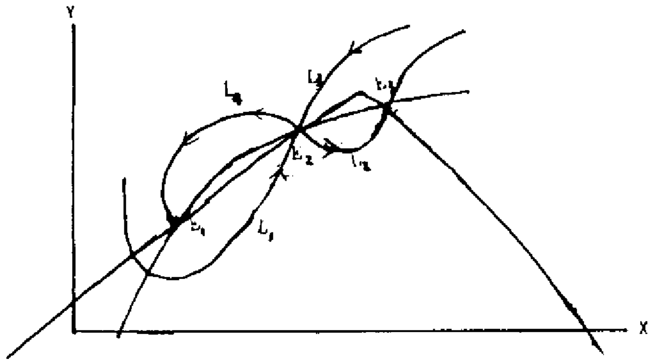

Fig.1 $\delta_1 > \delta_0$.

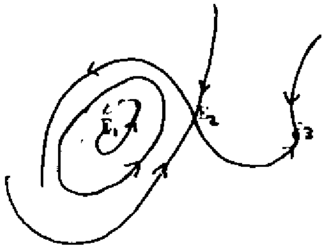

Fig.2 $\delta^* < \delta_1 < \delta_0$ ($|\delta_1 - \delta_0|$ small).

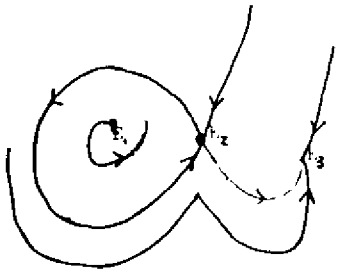

Fig.3 $\delta_1 = \delta^*$.

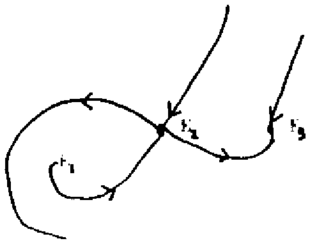

Fig.4 $0 < \delta_1 < \delta^*$.

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