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MODULAR TRANSFORMATIONS OF CONFORMAL BLOCKS IN WZW MODELS ON RIEMANN SURFACES OF HIGHER GENUS *

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ABSTRACT

We derive the modular transformations for conformal blocks in Wess-Zumino-Witten models on Riemann surfaces of higher genus. The basic ingredient consists of using the Chern-Simons theory developed by Witten. We find that the modular transformations generated by Dehn twists are linear combinations of Wilson line operators, which can be expressed in terms of braiding matrices. It can also be shown that modular transformation matrices for $g > 0$ Riemann surfaces depend only on those for $g \leq 3$.

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1. Introduction

There has been steady progress in understanding 2D conformal field theory (CFT)[1,2] recently. Verlinde's work on fusion rules is the first indication that one may classify and construct CFT by starting from some basic data and relations among them[3]. Subsequently, Moore and Seiberg obtained polynomial equations for data such as duality matrices which enable them to prove the conjecture of Verlinde[3] as a by-product. They showed that classifying rational conformal theories might be equivalent to classifying all solutions to those polynomial equations.

Rather surprisingly, Witten found recently that one can attack the WZW models[5] by studying three dimensional Chern-Simons theories[6]. The idea is that in the canonical quantization of Chern-Simons (CS) theory of a certain group, the Hilbert space is isomorphic to the space of conformal blocks in the corresponding WZW model. In a subsequent paper, Witten found a kind of basis of conformal blocks on any Riemann surface[7]. In this paper we work out modular transformations acting on this basis. We found that the modular transformations under basic Dehn twists around homotopy cycles on genus g Riemann surfaces are linear combinations of the holonomy matrices around the corresponding cycles. The nontrivial holonomy matrices in Witten's basis are the ones for b_j ($j = 1, \cdots, g$) cycles which we found to be expressible in terms of braiding matrices. The fact that modular transformations can be expressed so is not surprising, Moore and Seiberg already suggested it in their fundamental works[4], using different methods. However, we found our method simpler and less data involved.

The method we use is similar to the one used by Dijkgraaf and Verlinde in proving Verlinde's conjecture[3]. We find that in WZW models one can extend the operators defined by Verlinde for genus one to higher genus, and in fact the operators can be realized by path integrals in the Chern-Simons theory. The result is a generalization of what was obtained by Verlinde in the case of genus one, namely the holonomy matrices around b cycles can be diagonalized to the form of holonomy matrices arround a cycles. The fact that our method is the generalization of the one used in [3] convinces us that our result is valid for any rational conformal field theory, not only for the WZW models we consider here for convenience.

Let $S(b_1)$, say, be the modular transformation corresponding to the Dehn twist along cycle b_1. One of our main results is eq.(3.10), for the genus 2 case. This result directly generalizes to higher genus, with help of eqs.(3.1), (3.11) and (3.12). The modular transformations generated by Dehn twists along a cycles, in some sense, are trivial. Since the modular groups are generated by basic Dehn twists, our result thus provides the basic blocks for constructing representations of modular groups.

Moore and Seiberg conjectured in [9] that all rational conformal field theories can be constructed from 2+1 Chern-Simons theories. If this could be finally proven true, then our consideration here is essentially complete.

In section 2 we define the Verlinde operators for WZW models and describe the method we will use for higher genus in the case of genus one. This proves Verlinde's conjectures again. We then present our main result in section 3 and discuss some application in sectoin 4.

2. Realization of Verlinde operators in the Chern-Simons theory

As shown by Witten, three dimensional Chern-Simons theory is closely related to the Wess-Zumino-Witten model in two dimensions. Here we shall consider the WZW model of a given compact, simply connected and simply laced group G. Let A be a connection on a three manifold of a bundle with structure group G. The Lagrangian

$$\mathcal{L} = \frac{k}{4\pi}\int_M \mathrm{Tr}(A\wedge dA + \frac{2}{3}A^3) \tag{2.1}$$

is a general covariant one. The Chern-Simons form (2.1) is not well-defined for a group when its third homotopy group is not zero, unless the coupling constant k is quantized. Here k is an integer, corresponding to the level of Kac-Moody algebra in the WZW model.

To quantize the theory canonically, one starts from a three manifold of the form

$\Sigma\times R$, where Σ is a compact Riemann surface. The Hilbert space depends only on the two dimensional surface. In fact, one can only treat the Hilbert space as a (projective) flat vector bundle over the moduli space of Riemann surfaces Σ as follows. The action (2.1) pertains to a constrained system so one must work in the reduced phase space. The reduced phase space is the set of all flat connections modulo gauge transformations. If the Riemann surface has inherited a conformal structure, then the phase space is naturally endowed with a conformal structure and the usual symplectic form restricted on this phase space is Kähler. By this Kähler polarization one can quantize the theory consistently. This procedure seems to be the unique way to canonically quantize the theory for a general group with our requirements[(1)]. But note that our theory will not depend on a specific conformal structure. We then consider all possible conformal structures, and at a point of the moduli space the Hilbert space coincides with the fiber of a flat holomorphic vector bundle. This holomorphic vector bundle is just that of the conformal blocks of the WZW model on Riemann surfaces Σ.

The moduli space is the quotient of Techmüler space by the modular group. A flat vector bundle over it can be completely determined by its twist properties under the modular group, namely the modular transformations. Knowing all modular transformations for a given genus, one can in principle construct the flat bundle over the moduli space, hence calculate conformal blocks as holomorphic sections of the bundle.

Now we recall what has been defined for a vector in the Hilbert space on a Riemann surface for the CS theory. The genus zero case is trivial since the Hilbert space is one dimensional. The first nontrivial case is of genus one. Now the Hilbert space is isomorphic to the space of characters. Let us consider the character of the primary field corresponding to the integrable representation R_i. In a three manifold, given a loop C, we define the

(1) The theory of other kind of groups can be quantized by proper constructions, see [9].

Wilson line as the holonomy around C

$$W_i(C) = \mathrm{Tr}Pexp\oint_C A. \tag{2.2}$$

Consider the solid torus with boundary Σ, as illustrated in fig.1, the vector the Hilbert space corresponding to the character χ_i is given by a path integral with insertion of Wilson line $W_i(C)$[6]. Now C is parallel to the b cycle on the torus.

As for a Riemann surface of higher genus, one considers the handlebody with boundary Σ (fig.3). Instead of Wilson lines, we use graphs constructed from the basic baryon graph (fig.2 shows the case of genus two). Given three representations R_i, R_j and R_l, we associate the graph with two indices ϵ and $\bar{\epsilon}$, each one corresponds to a chiral vertex of type (ijl) and $(i^*j^*l^*)$. Here we use i^* to denote the conjugate of i, we reserve $\bar{i}$ for the representation in the right moving sector coupled to i. The vectors in the Hilbert space are given by path integrals with insertion of these graphs as in fig.3. Note that the number of choices of each vertex in the graph is given by the fusion rule N_{ijl}, so we find that the number of the independent graphs is just the number of conformal blocks. Note that in the path integrals, one must use a specific framing, here we adopt the prescription given in [7]. Also the nomalizations of the baryon graphs are the same as in [7].

We define the generalization of Verlinde's operators here. Consider any loop γ on Σ. Given a representation q, we construct the operator $T_q(\gamma)$ as follows. To set the problem of framing, we first consider the operators associated with the canonical basis of the homology group. The framing of a_i or b_i(2) is given by requiring that the outward vectors are tangent to the surface (thus untwisted). Then any other loop is framed by smoothly connecting those generators by which the loop is presented. Now suppose there is a graph inside the handlebody which gives the vector $|\Psi\rangle$ in the Hilbert space. We send continuously another loop γ on Σ into the handlebody, associating this loop with the

(2) Without further noticing, by a_i cycles we mean the homotopy equivalent class of δ_i cycles in the notation used in the third of ref.[4], to be more specific for later considerations.

Wilson line $W_q(\gamma)$. The new state after the action of the operator $T_q(\gamma)$ is given by the path integral with the presence of the original graph and the additional Wilson line. It is not too difficult to show that this definition is the realization of the holonomy operators in canonical quantization. The abelian case has been considered in [8].

We show that this is the generalization of Verlinde's operators. It is sufficient to calculate the matrices of $T_q(a)$ and $T_q(b)$ in the genus one case (3). We calculate by the method developed in [6] the partition functions corresponding to figs.4 and 5, finding

$$\begin{aligned} \langle m|T_q(a)|n\rangle &= \frac{S_{q,n}}{S_{0,n}}\delta_{mn} \\ \langle m|T_q(b)|n\rangle &= N_{qn}{}^{m}. \end{aligned} \tag{2.3}$$

These formulas are just the ones obtained in [3].

In fact, given an arbitrary graph Γ, we can also defined an operator $T(\Gamma)$ associated to it. The use of this general kind of operators may play a role in the further study of WZW models[13]. In this paper, we only make use of those operators associated to Wilson lines.

We prove the Verlinde conjecture as follows. Consider the Dehn twist along b^{-1} cycle. Under this twist, we have $a \to ab$ and $b \to b$. Note that the basis we choose for the Hilbert space depends on the choice of the homology basis. So in the new basis, $a' = ab, b' = b$, the matrix elements of $T_q(a_i')$, for example, will be the same as those of $T_q(a_i)$ in the old basis. So in the case of the Dehn twist along b^{-1}, we must have

$$T_q(a')'_{mn} = \langle m|' T_q(ab)|n\rangle' = \langle m|T_q(a)|n\rangle = T_q(a)_{mn}, \tag{2.4}$$

where $|n\rangle' = \sum_m |m\rangle S_{mn}(b^{-1})$ is the new basis after performing the Dehn twist. We calculate $\langle m|T_q(ab)|n\rangle$ as shown in fig.6. We do two operations. First, ab and b braiding twice, so there is a factor $exp(2\pi i(h_m + h_q - h_n)$ after resloving the bariding. Secondly, since the framing of ab is twisted, there is an addtional factor $exp(-2\pi i h_q)$ which contributes. We

(3) we note that these operatos are denoted as $\phi_q(a)$ and $\phi_q(b)$ by Verlinde

thus have $\langle m|T_q(ab)|n\rangle = exp(2\pi i(h_m - h_n)\langle m|T_q(b)|n\rangle$. Suppose the modular transformation of the Dehn twist is $S(b^{-1})_{m,n}$ and its inverse $S^{-1}_{mn}(b^{-1}) = S_{mn}(b)$, then we have from (2.4):

$$\sum_{m_1,n_1} S(b)_{m,m_1} S(b^{-1})_{n_1,n} exp(2\pi i(h_{m_1} - h_{n_1})\langle m_1|T_q(b)|n_1\rangle = \langle m|T_q(a)|n\rangle, \qquad (2.5)$$

namely, the matrix $C_{m,n} = S(b)_{m,n} exp(2\pi i h_n)$ diagonalizes $T_q(b)$, the fusion rules, to the matrix $T_q(a)$. This is the statement equivalent to the Verlinde's conjecture, noting that $S(b) = T^{-1}S^{-1}T^{-1}$. Here we use the notation that T is the modular transformation corresponding to the Dehn twist along a, or in matrix elements, $T_{mn} = \delta_{mn}\exp(2\pi i(h_m - c/24))$, and S is the one correponding to $\tau \to -1/\tau$, $\chi_m \to \tilde{\chi}_m = \sum_n S_{mn}\chi_n$.

Another condition implies that the matrix $S(b^{-1})$ commutes with $T_q(b)$. The fact that matrices $T_q(b)$ can be simultaneously diagonalized shows that all these matrices commute among themselves.

We derive $S(b^{-1})$ by using the surgery procedure[6]. This is one way we will use in the next section in deriving $S(b_i^{-1})$ on higher genus. It is obvious that $S(b^{-1})_{m,n} = \langle m|n\rangle'$, equals the partition function of the manifold obtained by gluing together two solid tori with the presence of two unknoted Wilson lines W_n and W_{m^*}, after performing the Dehn twist along b^{-1} with the first one, where m^* is the conjugate of m. To calculate this partition function, we do a surgery along a line parallel to these two lines. Gluing back again the two solid tori, we have again the manifold $S^2 \times S^1$, but now with an additional Wilson line along b. By the method in [6], we find $S(b^{-1})_{n,m} = \sum_q S(b)_{0,q}\langle n|T_q(b)|m\rangle$. Notice that $S(b^{-1})_{0,q} = (TST)_{0q} = exp(2\pi i h_q)S_{0,q}$ in our notation. Here we neglect the common phase depending on the central charge. We write this more compactly as

$$S(b^{-1}) = \sum_q e^{2\pi i h_q} S_{0,q} T_q(b). \qquad (2.6A)$$

Similarly, we have

$$S(b) = \sum_q e^{-2\pi i h_q} S^{-1}_{0,q} T_q(b). \qquad (2.6B)$$

Using the matrix C defined above, we can diagonalize $S(b^{-1})$, and hence prove that $S(b^{-1})$ is unitary, where $(ST)^3 = C$ has been used. C is the charge conjugation operator.

Our formula (2.6) seems trivial in the case of genus one. But it is important in the case of higher genus.

3. Modular transformations in higher genus

The formulas of the kind of (2.6) will play the central role in this section. In fact, Let C be a cycle picked from the canonical homology basis. Again by the surgery argument and taking into account of the central charge, we find

$$S(C^{-1}) = \sum_q e^{2\pi i(\tilde{h}_q + \tilde{h}_0)} S_{0,q} T_q(C), \qquad (3.1A)$$

$$S(C) = \sum_q e^{-2\pi i(\tilde{h}_q + \tilde{h}_0)} S^{-1}_{0,q} T_q(C), \qquad (3.1B)$$

where by $\tilde{h}_q$ we mean $h_q - c/24$, c is the central charge. This formula enables us to convert the problem of calculating modular transformations into the one of calculating matrices T_q. The formula can be proven in another way we will shortly demonstrate. It is a well-known fact that Dehn twists along nontrivial loops generate the modular group, that is why we only consider Dehn twists here.

We denote the vector generated by the graph in fig.7 as $\Psi(q_i, \tilde{q}_i, w_i, c_i, \tilde{c}_i)$. It is easy to show that this basis is orthogonal:

$$\langle\Psi(q_i', \tilde{q}_i', w_i', c_i', \tilde{c}_i')|\Psi(q_i, \tilde{q}_i, w_i, c_i, \tilde{c}_i)\rangle = \frac{1}{S_{0,0}^{g-1}} \prod_i \delta_{q_i,q_i'}\delta_{\tilde{q}_i,\tilde{q}_i'}\delta_{w_i,w_i'}\delta_{c_i,c_i'}\delta_{\tilde{c}_i,\tilde{c}_i'}. \qquad (3.2)$$

Modular transformations are global diffeomorphisms of the Riemann surfaces. Under a modular transformation, closed curves are mapped to closed curves. Consider a modular transformation M which maps a homotopy cycle C to another cycle $\tilde{C}$. Certainly it will change Wilson line operator $T_q(C)$ to $T_q(\tilde{C})$. Let $S(C)$ and $S(\tilde{C})$ be modular transformations generated by Dehn twists along C and $\tilde{C}$, respectively. We shall consider the transformation law

$$|\Psi\rangle \to |\tilde{\Psi}\rangle = M|\Psi\rangle$$

$$T_q(C) \to T_q(\tilde{C}), \qquad S(C) \to S(\tilde{C}),$$

Simply by the argument we used before, we must have

$$\begin{aligned}\langle\Phi|T_q(C)|\Psi\rangle &= \langle\tilde{\Phi}|T_q(\tilde{C})|\tilde{\Psi}\rangle = \langle\Phi|M^{-1}T_q(\tilde{C})M|\Psi\rangle \\ \langle\Phi|S(C)|\Psi\rangle &= \langle\tilde{\Phi}|S(\tilde{C})|\tilde{\Psi}\rangle = \langle\Phi|M^{-1}S(\tilde{C})M|\Psi\rangle,\end{aligned} \tag{3.3}$$

Since the states $|\Phi\rangle$ and $|\Psi\rangle$ are arbitrary, we deduce

$$T_q(\tilde{C}) = MT_q(C)M^{-1}, \qquad S(\tilde{C}) = MS(C)M^{-1}. \tag{3.4}$$

Let us consider first a single Wilson line operator associated with the a_i homotopy cycle on a Riemann surface of genus g. Let the vector be given as in fig.7. By some simple manipulation, we find

$$T_q(a_i)|\Psi\rangle = \frac{S_{q,q_i}}{S_{0,q_i}}|\Psi\rangle, \tag{3.5}$$

namely, by use of this basis, $T_q(a_i)$ is diagonal. Similarly we have $T_q(a_ia_{i+1}^{-1})|\Psi\rangle = S_{q,w_i}/S_{0,w_i}|\Psi\rangle$. We conclude from these facts:

$$[T_q(a_i), T_w(a_j)] = 0, \qquad [T_q(a_i), T_w(a_ja_{j+1}^{-1}] = 0, \tag{3.6}$$

and some other similar relations.

Now let S be the modular transformation which maps a cycles to b cycles, namely, $S:\quad a_i \to b_i, b_i \to b_i^{-1}a_i^{-1}b_i$. Clearly, $T_q(b_i) = ST_q(a_i)S^{-1}$. We then have

$$[T_q(b_i), T_w(b_j)] = 0, \qquad [T_q(b_i), T_w(b_jb_{j+1}^{-1}] = 0, \tag{3.7}$$

which is similar to (3.6). Consider the Dehn twist along a_i, which keeps a_j invariant, so $S(a_i)$ commutes with $T_q(a_j)$. Applying the modular transformation S, we deduce that $S(b_i)$ commutes with $T_q(b_j)$.

When we perform the Dehn twist along a_i or $a_ia_{i+1}^{-1}$, the line q_i or w_i is twisted, so

$$\begin{aligned}S(a_i)|\Psi\rangle &= exp(2\pi i\tilde{h}_{q_i})|\Psi\rangle \\ S(a_ia_{i+1}^{-1})|\Psi\rangle &= exp(2\pi i\tilde{h}_{w_i})|\Psi\rangle.\end{aligned} \tag{3.8}$$

Again by the relation $(ST)^3 = C$ for $g = 1$, we can prove the following formula

$$S(a_i^{-1}) = \sum_q exp(2\pi i(h_q - c/12))S_{0,q}T_q(a_i). \tag{3.9}$$

Then by the modular transformation S, we find that the above formula is valid also for $S(b_i^{-1})$. This proves again (3.1).

We now proceed to use this formula to calculate the modular transformation of the Dehn twist along b_i^{-1}. Here for simplicity, we consider the $SU(2)$ case, since the vertex c is unique if the fusion rule is not zero, and we need not distinguish between the represenation m and its conjugate m^*. As the first example, let us calculate $S(b_1)$ in the case of $g = 2$. To calculate $S(b_1)$ we need to calculate $T_q(b_1)$, the given entry of the latter is the partition function of one graph in fig.9 in the manifold $S^2\times S^1 + S^2\times S^1$. We separate this manifold in to two parts, each one is a manifold $S^2\times S^1$ with boundary S^2. On each boundary there are four punctures, marked by two pairs (q_1, q_1^*) and $(q_1', q_1'^*)$. We can closed these two manifolds by gluing a ball B^3, in which there is a graph as in fig.10. In fact, path integrals of those graphs give a complete basis for the Hilbert space of S^2 with such four punctures, when m varies. We then obtain two manifolds $S^2\times S^1$ with graphs shown in figs.11. We consider fig.11(a) first. Cuting $S^2\times S^1$ to result in a cylinder shows that $m = q$. Note that in figs.11 we still mark the orientations of lines, even though in the case of the group $SU(2)$, this is not necessary. After perform cuting and gluing procedure, fig.11(a) is changed to fig.12, where the simple baryon graph resides in the manifold S^3. Now we turn to fig.11(b). Still using cutting and gluing, we obtained fig.13, which in turn is equivalent to fig.11, we find $q_2' = q_2$, otherwise we have the vanishing result. Separating the graph in fig.14 into two halves, we finally reach fig.15, with two tetrahedrons. A given tetrahedron is related to the braiding matrix, as Witten pointed out in [7]. Here we simply write down what we have obtained by the operations described above:

$$\begin{aligned}\frac{\langle q_1', q_2', w_1'|T_q(b_1)|q_1, q_2, w_1\rangle}{\langle q_1, q_2, w_1|q_1, q_2, w_1\rangle} &= \exp[2\pi i(h_q + h_{q_2} - h_{w_1} - h_{q_1'})] \\ &\quad \sqrt{\frac{S_{0,q_1'}S_{0,w_1}}{S_{0,q_1}S_{0,w_1'}}}B^2_{q_1w_1'}\begin{pmatrix} q & q_2 \\ w_1 & q_1' \end{pmatrix}.\end{aligned} \tag{3.10A}$$

Substituting eq.(3.10A) into eq.(3.1) we obtain

$$\frac{\langle q_1', q_2', w_1' | S(b_1^{-1}) | q_1, q_2, w_1 \rangle}{\langle q_1, q_2, w_1 | q_1, q_2, w_1 \rangle} = \sum_q \exp[2\pi i(2h_q + h_{q_2} - h_{w_1} - h_{q_1'})] S_{0,q} \sqrt{\frac{S_{0,q_1'} S_{0,w_1}}{S_{0,q_1} S_{0,w_1'}}} B^2_{q_1,w_1'} \begin{pmatrix} q & q_2 \\ w_1 & q_1' \end{pmatrix} \tag{3.10B}$$

There is another simpler way to achieve this result. The operations involved are similar to those indicated in fig.23 in [7]. But in that way it is hard to manipulate the phases appearing in the graphs.

We thus see that modular transformations generated by the Dehn twists along b cycles are expressed in terms of braiding matrices. Similarly one can calculate $T_q(b_2)$. Proceeding further, we can prove that the fundamental elements we need to know are $T_q(b_i)$ on the Riemann surfaces of genus three. When $g > 3$, one can prove that all $T_q(b_i)$ can be written in the forms of those on $g = 3$. This in trun implies that modular transformations for genus greater than 3 can be written in terms of modular transformations on genus 3. Our result then indicates that the independent data for modular transformations are those on genus 1, 2 and 3. Modular invariance on higher genus are ensured by modular invariance on genus 1, 2 and 3. This is an interesting result.

For completeness, we write down for two relevant cases on genus 3. Again the group is $SU(2)$. First

$$\langle q_1', q_2', q_3', \tilde{q}_2', w_1', w_2' | T_q(b_1) | q_1, q_2, q_3, \tilde{q}_2, w_1, w_2 \rangle = exp(2\pi i(h_q + h_{\tilde{q}_2} - h_{w_1} - h_{q_1'})) \frac{1}{S^2_{0,0}} \sqrt{\frac{S_{0,w_1} S_{0,q_1'}}{S_{0,w_1'} S_{0,q_1}}} B_{q_1,w_1'} \begin{pmatrix} q & q_2 \\ w_1 & q_1' \end{pmatrix} B_{q_1,w_1'} \begin{pmatrix} q & \tilde{q}_2 \\ w_1 & q_1' \end{pmatrix} \delta_{q_2,q_2'} \delta_{q_3,q_3'} \delta_{\tilde{q}_2, \tilde{q}_2'} \delta_{w_2,w_2'} \tag{3.11}$$

Secondly, we calculate $T_q(b_2)$, which is quartic in braiding matrices:

$$\begin{aligned} &\langle q_1', q_2', q_3', \tilde{q}_2', w_1', w_2' | T_q(b_2) | q_1, q_2, q_3, \tilde{q}_2, w_1, w_2 \rangle \\ &= exp(2\pi i(2h_q + h_{q_3} + h_{q_1} - h_{w_1} - h_{w_2} - h_{q_2'} - h_{\tilde{q}_2'})) \frac{1}{S^2_{0,0}} \sqrt{\frac{S_{0,w_1} S_{0,w_2} S_{0,q_2'} S_{0,\tilde{q}_2'}}{S_{0,w_1'} S_{0,w_2'} S_{0,q_2} S_{0,\tilde{q}_2}}} \\ &B_{q_2,w_1'} \begin{pmatrix} q & q_1 \\ w_1 & q_2' \end{pmatrix} B_{\tilde{q}_2,w_1'} \begin{pmatrix} q & q_1 \\ w_1 & \tilde{q}_2' \end{pmatrix} B_{q_2,w_2'} \begin{pmatrix} q & q_3 \\ w_2 & q_2' \end{pmatrix} B_{\tilde{q}_2,w_2'} \begin{pmatrix} q & q_3 \\ w_2 & \tilde{q}_2' \end{pmatrix} \delta_{q_1,q_1'} \delta_{q_3,q_3'} \end{aligned} \tag{3.12}$$

When $g > 3$, $T_q(b_i)$ can be expressed in either $T_q(b_1)$ or $T_q(b_2)$ on genus 3. For example we have $T_q(b_1) \propto T_q(b_1)|_{g=3}$ and $T_q(b_j) \propto T_q(b_2)|_{g=3}$, when $j \geq 2$. Therefore, modular transformations for $g > 3$ can be expressed in terms of those for $g = 3$.

4. Some application and conclusion

We discuss here only one application of our result. Further applications will be considered in [13]. Dijkgraaf and Verlinde proved in [3] that if there is a nondiagonal version of the modular invariant CFT, there is a nontrivial automorphism of the fusion algebra. We can extend their result to include the braiding matrices. Suppose we have a theory in which the total Hilbert space is

$$\mathcal{H} = \oplus_{i,\bar{i}} [\phi_i] \otimes [\bar{\phi}_{\bar{i}}]. \tag{4.1}$$

This means that, the one-loop partition function can be in general written as

$$Z = \bar{\chi} \cdot \mathrm{II} \cdot \chi, \tag{4.2}$$

where II is an integer matrix. Dijkgraaf and Verlinde showed that II viewed as a map, maps i to a unique $\bar{i}$ in the right sector. Then it is not hard to show that $N_{ijl} = N_{\bar{i}\bar{j}\bar{l}}$. If the matrix II is nondiagonal, then this means there exists a nontrivial automorphism of the fusion algebra.

Our consideration in this paper so far is restricted to the WZW models. We believe that our method used here can be extended to any rational CFT. So the results obtained in the last section should be valid in the general case. We apply our results to some consideration on genus 2 similar to that by Dijkgraaf and Verlinde.

On higher genus, one can use the factorization argument to show that the partition function is the sum of products of the generalized characters in both sectors, each pair are coupled according to $i \to \bar{i}$. For genus 2, we use $\chi_{(q_1,q_2,w_1)}$ to denote the character corresponding to $|\Psi(q_1, q_2, w_1)\rangle$ we considered in the last section. Now we have a generalized matrix II(2) such that the partition function on the Riemann surface of genus 2 can be written as $Z(2) = \bar{\chi} \cdot \Pi(2) \cdot \chi$. Now any modular transformation must commute with $\Pi(2)$, if our theory is modular invariant.

Note that what were used in proving the automorphism of the fusion algebra are the following facts, first Π is symmetric. Second, we have $S_{\bar{i},j}/S_{0,j} = S_{i,\bar{j}}/S_{0,\bar{j}}$[3], this is equivalent to $S_{i,j}/S_{0,j} = S_{\bar{i},\bar{j}}/S_{0,\bar{j}}$. Let Π be an operator such that $|\bar{\Phi}\rangle = \Pi|\Phi\rangle$. Now the above statement amounts to $\Pi T_{\bar{q}}(a)\Pi = T_q(a)$. Similarly we have in the genus 2 case $\Pi(2)T_{\bar{q}}(a_1)\Pi(2) = T_q(a_1)$. Consider the modular transformation S we discussed before. We have $T_q(b_1) = ST_q(a_1)S^+$, thus we have

$$\begin{aligned} \langle\bar{\Phi}|T_{\bar{q}}(b_1)|\bar{\Psi}\rangle &= \langle\Phi|\Pi(2)T_{\bar{q}}(b_1)\Pi(2)|\Psi\rangle \\ &= \langle\Phi(2)|\Pi(2)ST_{\bar{q}}(a_1)S^+\Pi|\Psi\rangle = \langle\Phi|T_q(b_1)|\Psi\rangle \end{aligned} \tag{4.3}$$

where the fact that S commutes with $\Pi(2)$ is used. We note that $\Pi(2)$ is also symmetric, since the partition function $Z(2)$ must be real. This is also used in the above derivation.

Substitute what we have obtained for $T_q(b_1)$ in the last section, and note that actually we can neglect the exatra factors besides the square of the braiding matrix. This is because $h_{\bar{i}} - h_i =$ integer, and of what we stated in the last paragraph. We prove that the squared terms are equal to each other more concretely. For $i = 0$ we find $S_{\bar{0},\bar{j}}/S_{0,\bar{j}} = S_{0,j}/S_{0,j} = 1$. So $S_{0,\bar{j}}/S_{0,\bar{0}} = S_{\bar{0},\bar{j}}/S_{0,\bar{0}} = S_{0,j}/S_{0,0}$, this proves

$$\sqrt{\frac{S_{0,\bar{i}}S_{0,\bar{j}}}{S_{0,\bar{l}}S_{0,\bar{m}}}} = \sqrt{\frac{S_{0,i}S_{0,j}}{S_{0,l}S_{0,m}}}.$$

So we find that, there are also nontrivial automorphisms of squares of braiding matrices. It is simply the generalization of the result obtained in [3]. Similar consideration applies to genus 3 case.

In conclusion, in this paper we showed that all modular transformations can be expressed in terms of braiding matrices and modular transformations of genus one characters. This is similar to what Moore and Seiberg suggested[4]. Indeed, by the use of Witten's Chern-Simons theory, we find explicitly the relations. However, in contrast to Moore and Seiberg's work, our method shows that the $S(j)$ modular transformations of the one point

functions on the torus are not independent data in reconstruction of the projective flat holomorphic vector bundles over moduli space of the Riemann surfaces. Rather, we expect that they can also be expressed in terms of braiding matrices [13]. Also what we find shows that the theory is guaranteed to be modular invariant for every genus provided it is modular invariant for genus 1, 2 and 3.

The polynomial equations are fundamental as the basic ingredients to ensure the modular invariance. One can in principle construct modular transformations by starting from these equations. But things are not so clear as much as in the case when we used Chern-Simons theory. For example, in proving the second equation in (2.3), one has to carry many operations[3,4]. Our results obtained here provide many relations among the braiding matrices. These relations are not easy to prove by using directly the polynomial equations. In addition, we would like to point out that the relation $\prod_i a_i b_i a_i^{-1} b_i^{-1} = 1$, will provide a relation among braiding matrices. For example, the relation will be of quartic polynomial equation in the genus 2 case.

We believe further developments along the line in this paper are possible.

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Figure captions

Fig.1 A solid torus with a Wilson line inside it, represents a vector in the Hilbert space on the genus one Riemann surface.

Fig.2 The baryon graph, with two vertices.

Fig.3 A state generated by the graph inside the handlebody.

Fig.4 Two Wilson lines reside in $S^2 \times S^1$, parallel to the nontrivial cycle, with an additional Wilson line linking one of them.

Fig.5 Three parallel Wilson line, the partition function is not zero only when $N_{qn}^{m} \neq 0$.

Fig.6 Three Wilson lines, among them two are braided.

Fig.7 A general graph in constructing a basis of Hilbert space on the Riemann surface of genus g.

Fig.8 The action of $T_q(a_i)$.

Fig.9 The partition function of this graph is an entry of the operator $T_q(b_1)$ when $g = 2$.

Fig.10 A basis for four punctures of two conjugate pairs.

Fig.11 By use of the basis of fig.10, we separate the graph in fig.9 into two ones.

Fig.12 Fig.11(a) can be cast into the baryon diagram.

Fig.13 This is the graph obtained from the one in fig.11(b), by cuting q_2 and q_2', and gluing them in other way.

Fig.14 This graph is equivalent to fig.13.

Fig.15 Separate the graph in fig.14 into two halves, then attach each one a vertex, we obtain two tetrahedrons, which in turn can be written in terms of braiding matrices.

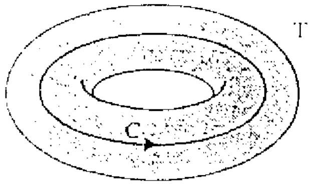

FIGURE 1

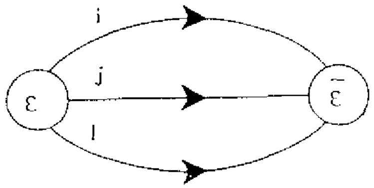

FIGURE 2

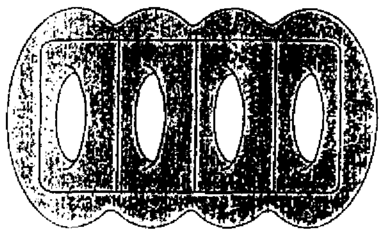

FIGURE 3

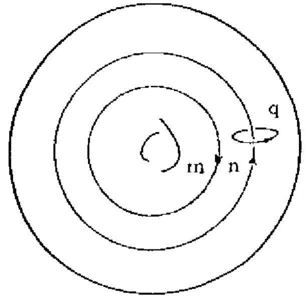

FIGURE 4

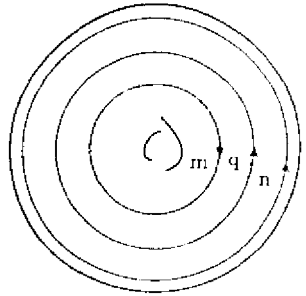

FIGURE 5

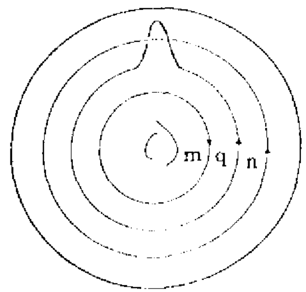

FIGURE 6

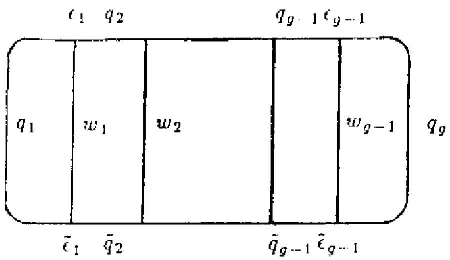

FIGURE 7

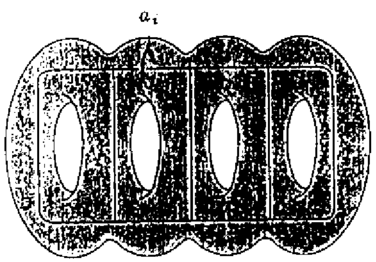

FIGURE 8

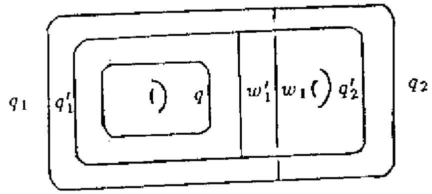

FIGURE 9

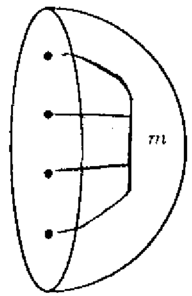

FIGURE 10

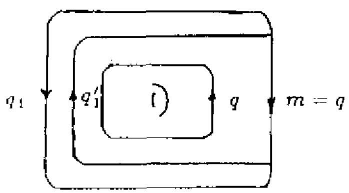

FIGURE 11(a)

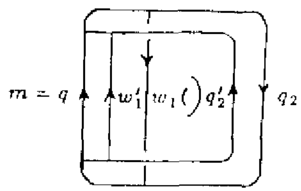

FIGURE 11(b)

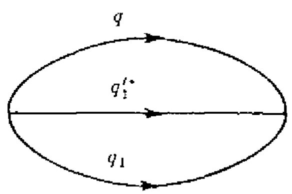

FIGURE 12

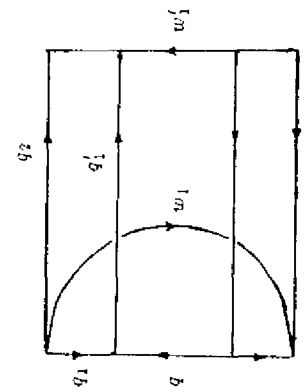

FIGURE 14

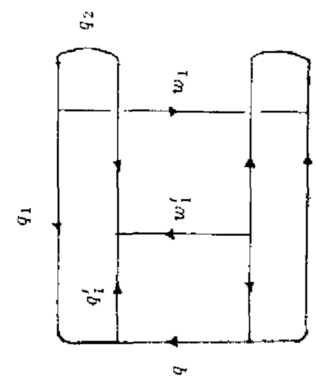

FIGURE 13

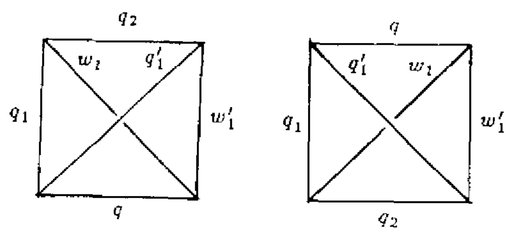

FIGURE 15

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