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$E = \hbar\omega$

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ABSTRACT

We construct exact localized oscillating *finite energy* solutions of the *massless* wave equation which move like massive relativistic particles with energy $E = \gamma\omega$ and momentum $\vec{p} = \gamma\vec{k}$ where ω is the frequency of the oscillating lump and $\vec{k}$ its wave vector. It is a construction of a *single* massive "quantum particle" from "light".

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The relations $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$ between the particle properties and wave properties of the electron, or photon, can be considered to be the foremost principle of quantum theory [1]. For then the wave equation, and everything else, follows. As new hypotheses these relations cannot be "derived" – they are the new properties of quantum particles. But we can try to make models of quantum particles incorporating this particle–wave duality. The earliest of such attempts go back to the originators of these relations. Einstein thought of light quantum as a localized lump of energy [2]. L. de Broglie envisaged the electron to be a singular localized solution of the wave equation quided by a wave with the same phase [3].

After the advent of the probability interpretation of quantum theory [4], one generally stopped making models of a single particle or single event. For, according to Born, a single event either did not have any meaning, or it was impossible to describe it. A free particle in this probability interpretation is a plane wave solution of the Schrödinger (or Dirac) equation which is not localized. It is important to emphasize the difference between the description of a single event and the regularities in repeated experiments, the latter calculated using plane waves [5]. Experimentally we can now observe single dots on a screen in a double slit experiment corresponding to single events, and when enough of these dots are collected in a memory device, the regularities of interference pattern slowly emerge [6]. One may in fact use, to avoid confusion, two different wave functions, a localized individual single event wave function $\phi(x)$, and a nonlocalized plane wave function $\Psi(x)$, the latter to describe the typical probabilistic behaviour, again of single events, but in repeated experiments [7].

In this paper we construct explicitly new localized solutions of the massless wave equation which behave like a single particle with mass m, thereby providing also a model for the concept of mass. These solutions do not spread and have *finite* field energy. They incorporate the basic relations "E proportional to $\omega, \vec{p}$ proportional to $\vec{k}$" and the relativistic relations $E^2 = p^2 + m^2$. Moreover, it is a construction of a three–dimensional soliton–like oscillating lump. This may come as a surprise because finite–energy, soliton–like nonspreading solutions are ususally associated with nonlinear wave equation. We are talking here about the usual linear massless wave equation.

The wave equation $\Box \phi = 0$, has well–known solutions of the form $\phi = f(x - ct) + g(x + ct)$, localized wave moving with group velocity c (but only in one–dimension), and plane wave solutions or wave packets of the form

$$\phi = \int f(\vec{k}_0)\, e^{i(\vec{k}_0 \cdot \vec{x} - \omega_0 t)}\, d\vec{k}_0$$

with phase velocity $c = \omega_0 / k_0$, or dispersion relations $\omega_0 / c = k_0$.

Our solutions are of a novel type: A lump moving with group velocity $v < c$ and phase velocity $u > c$ such that $uv = c^2$. Because of the wave equation $\Box \phi = 0$ is covariant, we can solve it first in the rest frame of the lump and transform it to the laboratory frame. In the comoving frame we write the solution as

$$\phi(r', t') = \int_0^\infty F(r')\, e^{-i\Omega t'}\, f(\Omega)\, d\Omega \qquad (1)$$

There are two essential features here: (1°) the rest frame solution is not static but oscillating with a frequency Ω, $\Omega > 0$, (2°) we allow a frequency spread $f(\Omega) d\Omega$ sharply peaked around some frequency Ω_0 to which we shall make a limit later. The frequency Ω in the rest frame is significant as it determines the mass – the kinetic energy of oscillations of the lump – and it is the cause that the lump moves with $v < c$.

The function $F(r')$ in (1) satisfies clearly the equation $\Delta F + (\Omega/c)^2 F = 0$, hence is of the form

$$F(r') = \sum_{\ell m} C_{\ell m} \frac{1}{\sqrt{r'}} J_{\ell + 1/2}\left(\frac{\Omega}{c} r'\right) P_\ell^m(\cos\theta)\, e^{im\varphi} \qquad (2)$$

In the laboratory frame with $t' = \gamma(t - \frac{1}{c}\vec{\beta} \cdot \vec{x})$, $\vec{\beta} = \vec{v}/c$, $\gamma = 1/\sqrt{1 - \beta^2}$ we get

$$e^{-i\Omega t'} = e^{-i\Omega\gamma(t - \frac{1}{c}\vec{\beta} \cdot \vec{x})} = e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

with

$$\vec{k} = \frac{1}{c}\Omega\gamma\vec{\beta} \qquad \text{and} \qquad \omega = \Omega\gamma. \qquad (3)$$

Hence, the new dispersion relation

$$(\omega/c)^2 - \vec{k}^2 = \frac{\Omega^2}{c^2}\gamma^2(1 - \beta^2) = (\Omega/c)^2, \qquad (4)$$

is the dispersion realtion for a massive particle, although it comes from a solution of the massless equation.

Historically, when dispersion relations for electron (4) were discovered, one added phenomenologically a mass term to the wave equation, Klein–Gordon equation, $(\Box + \frac{m^2 c^2}{\hbar^2})\Phi = 0$ which have *plane wave* solutions, $\Phi = A e^{i(\vec{k} \cdot \vec{x} - \omega t)}$ with dispersion relations (4). We see now that massless wave equation has solutions satisfying (4) but *necessarily* with localized envelope $F(r)$.

In the moving laboratory frame, the solution (1) becomes

$$\phi(\vec{r}, t) = \int_0^\infty F(\rho(\vec{r}, t))\, e^{i(\vec{k} \cdot \vec{x} - \omega t)}\, f\left(\frac{\omega}{\gamma}\right) \frac{d\omega}{\gamma} \qquad (5)$$

where $\vec{\rho}$ is the Lorentz transformed distance

$$\rho = \left[\vec{r}^2 + \beta^2\gamma^2 t^2 - 2\gamma^2 t(\vec{\beta} \cdot \vec{r}) + \gamma^2(\vec{\beta} \cdot \vec{r})^2\right]^{1/2}, \quad \vec{r} = \vec{x} - \vec{a}. \qquad (6)$$

The phase velocity is given by $u = \omega/k = c^2/v$, where v is the group velocity. This is a single event whose maximum is located at $\vec{a}$ at $t = 0$. In fact, the position of the lump at $t = 0$ and the shape parameters $C_{\ell m}$ in Eq.(2) play the role of hidden variables.

We note here, en passant, that we can also construct a tachionic solution of the wave equation with group velocity $v > c$ but then the phase velocity $u < c$. The original wave equation has a single parameter c. The solution (5) has the new parameters, Ω and $\vec{v}$.

A special case of this solution (1–6) for $\ell = 0$, $m = 0$ moving in x–direction is

$$\phi_0 = C_{00} \int \frac{1}{\sqrt{\rho}} J_{1/2}\left(\frac{\omega}{\gamma}\rho\right) e^{i(\vec{k} \cdot \vec{x} - \omega t)}\, f\left(\frac{\omega}{\gamma}\right) \frac{d\omega}{\gamma} \qquad (7)$$

with

$$\rho = [\gamma^2(x - vt)^2 + y^2 + z^2]^{1/2}, \quad J_{1/2}(\xi) = \sqrt{\frac{\pi}{2}} \frac{\sin(\xi)}{\sqrt{\xi}}$$

This solution, but without the factor $f(\frac{\omega}{\gamma})$ and the integral over $d\omega$, was already given by L. de Broglie [3] and further discussed by Machinnon [4]. But without the factor $f(\frac{\omega}{\gamma})$ the solution has infinite energy.

Now we come to the crucial question of the finiteness and the form of the energy. Again we calculate the field energy in the rest frame and we can then transform it. It is given by

$$\mathcal{E} = N_E \int \left\{ \left|\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right|^2 + (\nabla\phi)^2 \right\} d^3x \tag{8}$$

where N_E is a normalization constant which cannot be determined because the wave equation is linear and hence has no scale yet. Using (1), we have – omitting now the primes on r and t –

$$\mathcal{E} = N_\ell \int \left\{ [F^*(\Omega,r)F(\Omega',r)\Omega\Omega' + \nabla F^*(\Omega,r)\cdot\nabla F(\Omega',r)] \right. \\ \left. \times f^*(\Omega)f(\Omega')e^{i(\Omega-\Omega')t}\, d\Omega\, d\Omega' \right\} d^3x. \tag{9}$$

We use the explicit form of F given in (2) and perform the angular integrations with $d^3x = r^2 dr \sin\theta d\theta d\varphi$, in the radial integration we use the crucial relation

$$\int_0^\infty J_{\ell+1/2}\left(\frac{\Omega}{c}r\right) J_{\ell+1/2}\left(\frac{\Omega'}{c}r\right) r dr = \frac{c}{\Omega}\delta(\Omega-\Omega'). \tag{10}$$

Then, Eq.(9) gives

$$\mathcal{E} = N_E \cdot 2\sum_{\ell m} |C_{\ell m}|^2 \frac{4\pi}{2\ell-1}\frac{(\ell+m)!}{(\ell-m)!}\int_0^\infty \Omega|f(\Omega)|^2 d\Omega \\ = N_E\cdot H\int \Omega|f(\Omega)|^2\, d\Omega; \quad H = 2\sum_{\ell m}|C_{\ell m}|^2\frac{4\pi}{2\ell-1}\frac{(\ell+m)!}{(\ell-m)!}. \tag{11}$$

In a similar way, we evaluate the "charge" Q for the scalar field ϕ using the current $j_\mu = \phi^* \overleftrightarrow{\partial}_\mu \phi$.

$$Q = N_Q \int j_0\, d^3x = N_Q \int \left(\frac{\partial\phi^*}{\partial t}\phi - \phi^*\frac{\partial\phi}{\partial t}\right) d^3x. \tag{12}$$

For our solution (1), this turns out to be

$$Q = N_Q\, H \int |f(\Omega)|^2\, d\Omega. \tag{13}$$

Hence

$$\mathcal{E} = \frac{N_E}{N_Q} Q \frac{\int_0^\infty \Omega|f(\Omega)|^2 d\Omega}{\int_0^\infty |f(\Omega)|^2 d\Omega} = \frac{N_E}{N_Q} Q\,\overline{\Omega} \tag{14}$$

where $\overline{\Omega}$ is the expectation value of Ω in the distribution $|f(\Omega)|^2$. We can choose now $|f(\Omega)|^2$ to be the distribution $\delta_+(\Omega-\Omega_0)$, so that $f(\Omega) = e^{i\alpha}\sqrt{\delta_+(\Omega-\Omega_0)}$ is another distribution. Note

that $\sqrt{\delta(\Omega-\Omega_0)}$ does not exist, but $\sqrt{\delta_+(\Omega-\Omega_0)}$ does exist. Our lump has physically only positive frequencies in the rest frame and the integral in (1) extend over positive frequencies. With this choice we finally have

$$\mathcal{E} = \frac{N_E}{N_Q} Q\,\Omega_0. \tag{15}$$

We have here a nonspreading packet of fixed energy whereas the usual "wave packet" is a superposition of different energies and therefore spreads. But it may be advantageous to allow a certain spread in Ω_0 and test for this spread in "mass".

If we now go back to the laboratory frame, i.e. to the boosted state (5), we clearly get $\mathcal{E} = \frac{N_E}{N_Q} Q\,\omega$ and the field momentum

$$\vec{P} = N_E \int d^3x \left[\frac{\partial\phi^*}{\partial t}\nabla\phi\right] \tag{16}$$

gives

$$\vec{P} = \frac{N_E}{N_Q} Q\,\vec{k}$$

with

$$\mathcal{E}^2 = c^2\vec{P}^2 + (\Omega^2)\left(\frac{N}{N_Q}Q\right)^2 \tag{17}$$

Thus $\left(\frac{N}{N_Q}Q\right)\Omega$ can be identified with the rest mass energy mc^2.

We have extended this idea to the more physical electromagnetic case. We found localized finite energy configuration of the $\vec{E}-$ and $\vec{B}-$ fields satisfying free Maxwell's equations and moving like massive particles, and determined their mutual interaction [9].

The solution (1) or (5) to the wave equation – up to changing some constants – persists in the presence of certain special nonlinearities. For the special case (7), again *without* the crucial frequency integral, such nonlinear equations have been considered recently by Guéret and Vigier [10] and some related singular solutions of nonlinear equations by Vigier [11]. But the frequency integral changes the situation completely.

The nonrelativistic limit of the dispersion relation (4) is

$$k = \left(\frac{\Omega}{c}\right)\beta + O(\beta^3) = \frac{mv}{\hbar} + O((v/c)^3)$$

and the solution (5) goes over to

$$\psi = F(\vec{x} - \vec{v}t)e^{i(\vec{k}\cdot\vec{x} - \omega t)} = F(\vec{x} - \vec{v}t)e^{i(\vec{k}\cdot\vec{x} - \frac{\hbar k^2}{2m}t)}\, e^{-i\Omega t}. \tag{18}$$

Our original wave equation $\Box\, \phi = 0$ has of course no nonrelativistic limit. But after separating the rapid rest mass oscillations $e^{i\Omega t}$ in the above equation, the remaining phase, $e^{i(\vec{k}\cdot\vec{x} - \frac{\hbar k^2}{2m}t)}$, is a solution of the Schrödinger equation $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla\psi$. But the full localized nonrelativistic lump (18) is a solution of the Schrödinger equation with the dispersion relation $\omega - \frac{\hbar k^2}{2m} = \Omega$, with F being a solution of $\Delta F + \frac{2m\Omega}{\hbar}F = 0$.

We summarize the main points. Massless ordinary wave equation have localized, non-spreading, finite energy solutions with the de Broglie phase moving like a relativistic particle, if the lump in the rest frame oscillates with a frequency Ω and has a singular distribution in Ω. The case $\Omega = 0$, i.e. static solution in the rest frame, cannot lead to finite energy solutions with the de Broglie phase. This is of course connected with the fact that massless particles have no rest frames. We find also that field energy is proportional to frequency ω and field momentum proportional to the wave vector $\vec{k}$, but the proportionality constant ($\hbar$) cannot be determined in the linear theory because of lack of a scale. It should be determined later by interactions.

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