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# FRACTAL DIMENSION OF A QUASI-TWO-DIMENSIONAL SIERPINSKI GASKET AS A CONDUCTOR*

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## ABSTRACT

It is shown that the fractal dimension of Sierpinski gasket as a conductor is different from that for its geometry. The fractal dimension of Sierpinski gasket can be defined as

$$D = D_\Sigma + iD_R$$

$D_\Sigma$ and $D_R$ are the fractal dimensions determined from its conductance and resistance, respectively. For 2-d Sierpinski gasket $D_R = 3.8$ and $D_\Sigma = 1/D_R = 0.263$, while for 3-d Sierpinski tetrahedron $D_R = 2.41$ and $D_\Sigma = 0.415$.

MIRAMARE - TRIESTE
June 1989

* To be submitted for publication.

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Since Sierpinski gasket provides a well-known simplest model of a distorted structure [1] and its fractal dimension of the geometry D equals $\ln 3/\ln 2 = 1.585$, it is interesting to find its fractal dimension as an electric conductor. This problem has been studied by several authors [1], [2], [3]. In the following we shall not confine ourselves only to the simplest between-two-terminals case, but also extend our study to some other typical cases, we shall show that if we start our derivation using an expression similar to that used to define the Hausdorff-Besicovitch dimension, the numerical results differ from the results of other authors, which were derived by simply relating the resistance to the change of length scale, though they are related by a simple relation.

The initiator and generator of a two-dimensional Sierpinski gasket are equilateral triangles. The first three generations of the gasket are shown in Fig.1.

The resistance between two end points (say B and C) of the $0^{th}$ generator $R_0$ is equivalent to that measured on a modelling triangle constructed of three resistance bars each of length $\ell_0$ with cross-section $a_0$ and resistivity $\rho$ as shown in Fig.2.

Then $R_0 = 2\rho\ell_0/3a_0$ and $3a_0/2$ can be viewed as effective cross-section of the $0^{th}$ generation as a conductor between these two end points. It is to be noted that the effective cross-section should not be confused with the mean cross-section, which always equals $\sqrt{3}\ell_0 d/4$, where d is the thickness.

We shall calculate the electric resistance of the $n^{th}$ generation for typical cases.

Case I: The resistance between two end points $B_nC_n$

It can easily be shown by Kirchhoff's theorem concerning the resistance of a network that for the first few generations

$$\frac{a_0R_0}{\ell_0\rho} = \frac{2}{3} \tag{1}$$

$$\frac{a_0R_1}{\ell_0\rho} = \frac{4}{3} - \frac{\left(\frac{2}{3}\right)^2}{\frac{4}{3} + \frac{a_0R_0}{\ell_0\rho}} = \frac{10}{9} = \frac{a_0R_0}{\ell_0\rho}\left(\frac{5}{3}\right) \tag{2}$$

$$\frac{a_0R_2}{\ell_0\rho} = \frac{20}{9} - \frac{\left(\frac{10}{9}\right)^2}{\frac{20}{9} + \frac{a_0R_1}{\ell_0\rho}} = \frac{50}{27} = \left(\frac{a_0R_1}{\ell_0\rho}\right)\left(\frac{5}{3}\right) \tag{3}$$

It can be shown by using the mathematical induction method in connection with the principle of equivalent circuits, that

$$\frac{a_0 R_n}{\ell_0 \rho} = \frac{a_0 R_{n-1}}{\ell_0 \rho} \left(\frac{5}{3}\right) \qquad (4)$$

i.e. we have a sequence

$$R_0 = \frac{2}{3} \frac{\ell_0 \rho}{a_0}$$

$$R_n = R_{n-1} \left(\frac{5}{3}\right) \qquad (n = 1,2,3,\ldots)$$

and

$$R_n = R_0 \left(\frac{5}{3}\right)^n = \frac{2}{3} \frac{\ell_0 \rho}{a_0} \left(\frac{5}{3}\right)^n = \frac{2}{3} \frac{\ell_0 \rho}{a_0} \left(\frac{5}{6}\right)^n 2^n \quad . \qquad (5)$$

The conductance will be

$$\Sigma_n = \frac{1}{R_n} = \frac{3a_0}{2\ell_0} \sigma \left(\frac{6}{5}\right)^n \frac{1}{2^n} \qquad (6)$$

where $\sigma$ is the conductivity of the solid material. As $n \to \infty$, $R \to \infty$, $\Sigma_n \to 0$.

If $R_n$ is expressed as

$$R_n = \frac{2\ell_n \rho}{3a_n} \qquad (7)$$

where $\ell_n$ *is the gauge length or the* distance between $B_n$ *and* $C_n$, $3a_n/2$ is the "effective" cross-section of the $n^{th}$ generation between measuring points, then from Eq.(5)

$$R_n = \frac{2\ell_n \rho}{3a_n} = \left(\frac{5}{3}\right) R_{n-1} = \frac{5}{3} \cdot \frac{2}{3} \frac{\ell_{n-1} \rho}{a_{n-1}} \qquad (8)$$

As $\ell_n = 2\ell_{n-1}$, it follows from (8) that

$$a_n = \frac{6}{5} a_{n-1} \qquad (9)$$

i.e. when the linear size doubles, the effective cross-section of the $n^{th}$ generation is magnified to 6/5 times that of the $(n-1)^{th}$ and the resistance is magnified to 5/3 times correspondingly.

Sierpinski gasket can also be constructed by section and deletion method. Starting from the $0^{th}$ generation, the first few generations are shown in Fig.3.

By the same argument as mentioned above, we can show that

$$R_{-N} = \frac{5}{3} R_{-n+1} \qquad (n = 1,2,3,\ldots) \qquad (10)$$



then

$$R_{-n} = \left(\frac{5}{3}\right)^n R_0' \qquad (10')$$

an equation which is identical with Eq.(5). The conductance is

$$\Sigma_{-n} = \left(\frac{3}{5}\right)^n \Sigma_0' \qquad (11)$$

which is identical with Eq.(6), but if $R_0'$ is finite, then as $n \to \infty$, $R_{-n} \to \infty$, $\Sigma_{-n} \to 0$. Expressing $R$ as

$$R_{-n} = \frac{2\ell_{-n}}{a_{-n}} \rho \qquad (12)$$

since

$$\ell_n = 2^n \ell_0' \quad , \quad \text{or} \quad \ell_{-n} = 2\ell_{-n+1} \qquad (13)$$

we have

$$a_{-n} = \frac{6}{5} a_{-n+1} \qquad (13)$$

or

$$a_{-n} = \left(\frac{6}{5}\right)^n a_0' \qquad (14')$$

This is the same result as expressed by Eq.(9).

The physical meaning of Eqs(10), (13) and (14') is that if we proceed by section and deletion method to construct the Sierpinski gasket, then the $(-n)^{th}$ generation can be formed by the $(-n+1)^{th}$ generation with the linear size of the latter equal to half the linear size of the former, at the same time, the effective cross-section of the $(-n)^{th}$ generation is equal to $\frac{6}{5}$ that of the $(-n+1)^{th}$.

Fractal dimension of Sierpinski gasket as an electric conductor can be determined in two different ways as follows [4].

(1) From Eq.(6), let $\delta = 1/2^n$, i.e. taking the length as measuring scale which tends to zero as $n \to \infty$, then

$$\Sigma_n = M_\Sigma(\delta)\, \delta^{1-D} = \frac{3a_0}{2\ell_0} \sigma \cdot \delta \left(\frac{6}{5}\right)^{-\frac{\ell n\delta}{\ell n2}} = \frac{3a_0}{2\ell_0} \sigma \delta^{1-\frac{\ell nb - \ell n5}{\ell n2}} \qquad (15)$$

$$D_\Sigma = (\ell n6 - \ell n5)/\ell n2 = 0.263 \qquad (16)$$

(2) From Eq.(5), let $\delta = (5/6)^n$, i.e. taking effective cross-section as measuring scale, then as $n \to \infty$, $\delta \to 0$.

$$R_n = M_R(\delta)\,\delta^{1-D} = \frac{2\ell_0}{3a_0}\,\rho\cdot\delta\cdot 2^{\frac{\ell n \delta}{\ell n5-\ell n6}} = \frac{2\ell_0}{3a_0}\,\rho\cdot\delta^{1-\frac{\ell n2}{\ell n6-\ell n5}}$$

$$D_R = \ell n2/(\ell n6-\ell n5) = 0.693/0.1823 = 3.8 \tag{17}$$

$D_\Sigma$ and $D_R$ are related by the relation

$$D_R = D_\Sigma^{-1} \tag{18}$$

Though they can be derived from each other, $D_R$ and $D_\Sigma$ have different physical meanings, i.e. if we pass a constant current through two end points of the gauge length, the potential diffference measured between these points would vary with the gauge length according to a power law with an exponent $D_\Sigma$. On the other hand, if we apply a constant potential difference on two end points of a Sierpinski gasket the current passing through would vary with an exponent $D_R$ as the section order increases.

Thus we can define the fractal dimension of Sierpinski gasket in a combined form

$$D = D_\Sigma + iD_R \tag{19}$$

where $D_R = 3.8$ and $D_\Sigma = 0.263$.

<u>Case II</u>: <u>The resistance between $A_nD_n$</u>

It can easily be shown that the resistance between end points $A_n$ and the mid point of the opposite side of the $n^{th}$ generation $D_n$ is

$$\begin{aligned} R_n' &= \frac{1}{2}\left(\frac{\ell_0\rho}{a_0} + R_0 + R_1 + R_2 + \dots + R_{n-1}\right) \\ &= \frac{1}{2}\left(\frac{\ell_0\rho}{a_0} + \frac{2\ell_0\rho}{3a_0}(1+\alpha+\alpha^2+\dots\alpha^{n-1})\right) \\ &= \frac{1}{2}\left(\frac{\ell_0\rho}{a_0} + \frac{2}{3}\frac{\ell_0\rho}{a_0}\alpha^{n-1}(1+\beta+\beta^2+\dots+\beta^{n-1})\right) \\ &= \frac{1}{2}\left(\frac{\ell_0\rho}{a_0} + \frac{2}{3}\frac{\ell_0\rho}{a_0}\alpha^{n-1}\frac{1-\beta^n}{1-\beta}\right) \end{aligned} \tag{20}$$

where $\beta = \alpha^{-1} = 3/5$.



Then we have

$$\frac{R_n'}{R_{n-1}'} = \frac{1-\beta+\frac{2}{3}\alpha^{n-1}(1-\beta^n)}{1-\beta+\frac{2}{3}\alpha^{n-2}(1-\beta^{n-1})} \tag{21}$$

as $n \to \infty$

$$\lim_{n\to\infty}\frac{R_n'}{R_{n-1}'} = \alpha = \frac{5}{3} \tag{22}$$

The same argument as in Case-I leads to $D_R = 3.8$ and $D_\Sigma = 0.263$.

<u>Case-III</u>: <u>The resistance between $C_nD_n$</u>

It can be shown by equivalent circuit principle that the resistance between an end point and its neighbouring midpoint of an edge of the $n^{th}$ generation is

$$R_n'' = \frac{11}{12}R_{n-1} \qquad \text{for } n > 1 \tag{23}$$

then

$$\frac{R_n''}{R_{n-1}''} = \alpha = \frac{5}{3} \tag{24}$$

The calculation for Case-I is also applicable to this case, and the dimension $D_R$ still equals 3.8 and $D_\Sigma = 0.263$.

The 3-d Sierpinski gasket can be discussed along a general line similar to that mentioned above, the initiator and generator of three-dimensional Sierpinski gasket are tetrahedra, the first few generations are shown in Fig.4.

The resistance measured between two end points of the $0^{th}$ generation can be modelled by a tetrahedron constructed of four bars, each of length $\ell_0$ with cross-section $a_0$ (Fig.5).

If $R_0$ is the resistance measured between two end points of the $0^{th}$ generation (say $B_0$ and $C_0$), then the resistance of the modelling bars will be $r_0 = 2R_0$

$$R_0 = \frac{r_0}{2} = \frac{\ell_0\rho}{2a_0}. \tag{25}$$

It can be shown by Kirchhoff's theorem and equivalent circuit principle that

$$R_1 = \frac{3}{2}R_0 \tag{26}$$

and by using the mathematical induction method we obtain

$$R_n = \frac{3}{2} R_{n-1} \qquad (n = 1,2,3,\ldots) \tag{27}$$

and

$$R_n = \left(\frac{3}{2}\right)^n R_0 = \frac{\ell_0 \rho}{2a_0} \left(\frac{3}{2}\right)^n \tag{28}$$

Expressing

$$R_n = \frac{\ell_n \rho}{2a_n} \tag{29}$$

then

$$R_n = \left(\frac{3}{2}\right) \frac{\ell_{n-1} \rho}{2\, a_{n-1}} \tag{30}$$

where $\ell_n$ is the gauge length of the $n^{th}$ generation, $a_n$ the effective cross-section. Since $\ell_n = 2\ell_{n-1}$, then $a_n = (4/3)\, a_{n-1}$, i.e. when the linear size of Sierpinski tetrahedron doubles, the effective cross-section is magnified to 4/3 times and the resistance magnified to 3/2 times.

From Eq.(28) we have

$$R_n = \frac{\ell_0 \rho}{2a_0} \left(\frac{3}{4}\right)^n \cdot 2^n \tag{31}$$

and

$$\Sigma_n = \frac{1}{R_n} = \frac{2a_0 \sigma}{\ell_0} \left(\frac{4}{3}\right)^n \frac{1}{2^n} \tag{32}$$

when $n \to \infty$, $R_n \to \infty$, $\Sigma_n \to 0$.

If constructing Sierpinski tetrahedron by section and deletion method, we will have

$$R_{-n} = \frac{\ell_0'}{2a_0'} \left(\frac{3}{4}\right)^n \cdot 2^n \tag{33}$$

and

$$\Sigma_{-n} = \frac{2a_0'}{\ell_0'} \sigma \left(\frac{4}{3}\right)^n \cdot \frac{1}{2^n} \tag{34}$$

which are identical with Eq.(31) and Eq.(32). When $n \to \infty$, $R_{-n} \to \infty$, $\Sigma_{-n} \to 0$.

Taking length as scale, let $\delta = 1/2^n$, from Eq.(32) we obtain

$$\Sigma_n = \frac{2a_0 \sigma}{\ell_0} \left(\frac{4}{3}\right)^{-\frac{\ln \delta}{\ln 2}} = M_\Sigma(\delta)\, \delta^{1-D}$$

thus

$$D_\Sigma = (\ln 4 - \ln 3)/\ln 2 = 0.2877/0.6931 = 0.415 \tag{35}$$



If taking effective cross-section as scale, from Eq.(32) let $\delta = (3/4)^n$, then we have

$$D_R = 1/D_\Sigma = 2.41 \; . \tag{36}$$

In a combined form, the fractal dimension of Sierpinski tetrahedron is

$$D = D_\Sigma + iD_R \tag{37}$$

where $D_\Sigma = 0.415$, $D_R = 2.41$.

In principle, it can be proved that Eq.(27) holds for other typical cases, only tedious algebraic work has to be involved, so it is omitted here.

We have shown that for different typical cases Sierpinski gasket as a conductor has the same fractal dimension that can be expressed in a combined form. The same problem of the fractal behaviour of Sierpinski gasket and its d-dimensional version has been discussed by Liu [1], Clerc et al. [3], Gefen et al. [4]. The admittance and conductivity, as suggested by Liu [1], follow the same exponential law and the exponent $\alpha$ for both these quantities in two dimensional case equals $(\ln 5-\ln 3)/\ln 2 = 0.737$. For the generalized SG of dimension $d$, the exponent relating the admittance $Y_n$ to $L_n$ of the $n^{th}$ generation is

$$\alpha = [\ln(d+3) - \ln(d+1)]/\ln 2 \tag{38}$$

and the exponent of the conductivity is

$$\alpha' = \alpha + d - 2 \tag{39}$$

Our results expressed by (19) and (16), (17), (35) and (36) differ from these authors. But in reality our $D_\Sigma$ is related to $\alpha$ by a relation $D_\Sigma = 1 - \alpha$. The reason for this difference lies in the fact that our derivation starts from a expression similar to the definition of Hausdorff dimension, not simply from relating the physical quantities to the change of the length scale; and physically, the effective cross-section of a quasi-two-dimensional Sierpinski gasket as a conductor varies with its linear size.

ACKNOWLEDGMENTS

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, and to Dr. Zhi-Yin Wen, Mathematical Department, Wuhan University, for very helpful and stimulating discussions.

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## FIGURE CAPTIONS

Fig.1 The first few generations of Sierpinski gaskets.

Fig.2 The initiator and its equivalent circuit.

Fig.3 Sierpinski gaskets produced by section and deletion.

Fig.4 The frist few generations of Sierpinski tetrahedra.

Fig.5 Initiator tetrahedron and its equivalent circuit.

Fig.1

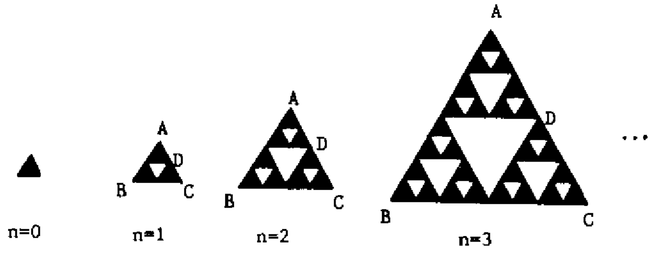



Fig.2


Fig.3

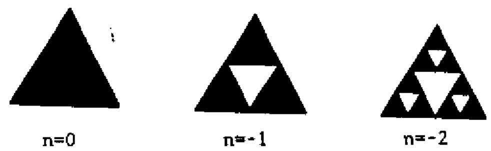



Fig.4

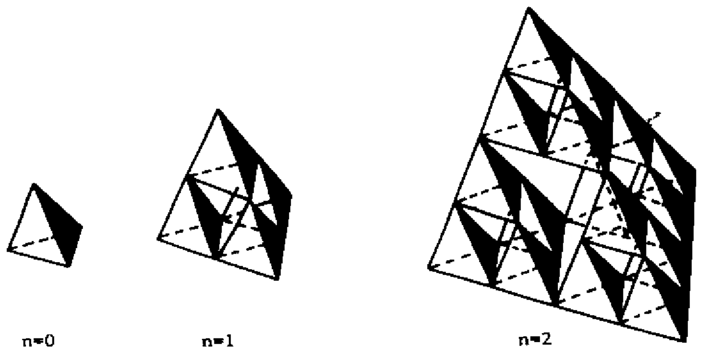
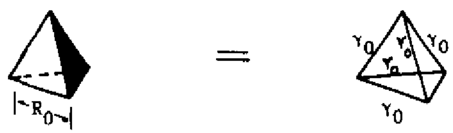



Fig.5



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