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**Off-Shell Superspace D=10 Super-Yang-Mills from  
Covariantly Quantized Green-Schwarz Superstring**

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**ABSTRACT**

We construct a gauge invariant superspace action in terms of unconstrained off-shell superfields for the D=10 Supersymmetric Yang-Mills (SYM) theory. We use to this effect:

- (i) the point particle limit of the BRST charge of the covariantly quantized harmonic Green-Schwarz superstring,
- (ii) a general covariant action principle for overdetermined systems of nonlinear field equations of motion.

One obtains gauge and super-Poincare invariant equations of motion equivalent to the Nilsson's constraints for D=10 SYM.

In the previous approaches (light-cone-gauge, component-fields) one would have to sacrifice either explicit Lorentz invariance or explicit supersymmetry while in the present approach they are both manifest.

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## 1. Introduction

It is hoped that a relativistic quantum theory of supersymmetric strings [1] can describe in a consistent way the quantum theory of space-time i.e. quantum gravity [2]. Unfortunately the proof (or disproof) of this conjecture was not completed to this date [3] because it was impossible until recently to express the quantum theory of *superstrings* in a form which displays explicitly the super-Poincare invariance.

In fact the form of the theory which forsakes the space-time supersymmetry (SUSY) - the Ramond-Neveu-Schwarz (RNS) formalism [4], is also very awkward in expressing the super-Yang-Mills (SYM) and supergravity (SUGRA) field theories in the massless sector of the superstring.

Indeed, the zero-mass bosons, described by these theories are in fact appearing in the RNS formulation as string-excitations and are recognized as the physical (degenerate) string ground states only after the GSO [5] truncation which eliminates the Fock-space ground state together with half of the spectrum.

It is strange that these gauge particles should appear at the excited string level while in fact describing infrared properties of the model and string ground states.

This way of appearance of the SUGRA and SYM gauge particles greatly obscures their relation with the geometry of supersymmetric space-time which is believed to be the physical fundamental reason behind their very existence [6].

This relation of the superstring with the supersymmetric space-time geometry is more naturally expressed within the the explicitly space-time supersymmetric formalisms of the Green-Schwarz (GS) superstring [7] and its point-particle limit- the Brink-Schwarz (BS) superparticle [8]. However, until recently, it seemed impossible [9] to quantize the GS and

BS systems while preserving the manifest super-Poincare invariance.

In a series of papers [10-15] this obstacle was overcome through the introduction of appropriate "spinorial vielbein" variables called "harmonic variables".

It becomes now appropriate to address the question of how the SYM and SUGRA theories do appear in our explicitly covariant quantum superstring formalism, i.e in which sense is the GS superstring theory a generalization of the SYM and SUGRA theories. After all, the very purpose of the superstring theory is to provide a generalization of these theories which is finite through the regularizing effect of its infinite tower of excited modes.

The space-time SUSY and YM gauge invariance, are important also technically, as tools in finiteness proofs [16]. For the superstring theory such a proof is necessary for the very consistency of the theory. In the RNS formalism there is little hope that such a proof can be performed as the quantum perturbation expansion in higher than two loops is plagued presently by severe complications [17].

The explicit space-time supersymmetric superfield approach is also a very desirable tool in string phenomenology. Some successes were already obtained in the study of the spontaneous symmetry breaking mechanisms which govern the "low" energy spectrum and dynamics of the string theory using a superfield effective action for the low-energy states. [18]. The direct use of the superstring field action might improve these studies solving in particular the problems of CP violation, cosmological constant etc. One may hope to explain some of the "miracles" appearing in the RNS formalism [19].

In ref. [15] it was shown how to describe the interactions of the covariant quantum GS superstring in terms of explicitly supercovariant dual-model massless vertices.

The long range objective of the string-field theory approach is the

systematic study of the nonperturbative quantum ground state of space-time in a way quite similar to the study of the ground state of any other theory (QCD, electro-weak, etc). The practical use of the second quantized formalism requires that we describe interactions also in the framework of the resulting super-string-field-theory.

Since in the GS formalism there is no GSO truncation, the field theories characterizing the zero-mass sector are expected to appear in a natural way already at the point-limit level.

Therefore, before attacking the issue of the super-Poincare invariant superstring field theory it is appropriate to study its massless limit.

In the present paper we use the BRST charge of the super-Poincare covariant first quantized GS superstring with  $N=1$  space-time SUSY computed in [15] to construct the gauge and super-Poincare covariant field theory corresponding to its zero-mass sector (i.e. the  $D=10$  SYM) in terms of unconstrained (off-shell) superfields.

To this end we employ (and review below) the general covariant action principle for arbitrary consistent overdetermined systems of nonlinear field equations proposed in our preceding paper [20].

The actions obtained this way resemble the Siegel-Zwiebach-Witten-Neveu-West [21] construction of (super)string field actions but do not involve the peculiarities (star products, Chern-Simons forms etc.) specific to the field theory of RNS strings. The main tool on which our covariant action principle is based is the BFV-BRST ghost formalism [22].

The use of this action principle to derive the superspace action for  $D=10$  SYM in terms of unconstrained superfields is inevitable since, as we show in section 5, the  $D=10$  SYM on shell equations (the so-called Nilsen constraints [23], see equations (1.9) and (5.1),(5.3-7) below), can be reformulated with the help of the auxiliary harmonic variables [10-15] as a consistent **overdetermined** system of nonlinear superfield equations meeting all the requirements for application of our covariant action principle.

ple.

The D=10 SYM is a fascinating system in his own right [23,24,6,25] as it has deep connections with integrable systems (i.e. admitting Lax representation), super-twistors, light-like geometry of space-time etc. By dimensional reduction to D=4 it yields the N=4 SYM which has the remarkable property to be a finite quantum field theory. The supersymmetric D=10 SYM field theory was discovered in the component formalism by [5,26] and has an on-shell supersymmetry due to the celebrated Fierz identity for D=10  $\sigma$ -matrices:

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} + (\sigma^\mu)_{\beta\gamma}(\sigma_\mu)_{\alpha\delta} + (\sigma^\mu)_{\gamma\alpha}(\sigma_\mu)_{\beta\delta} = 0 \quad (1.1)$$

It is this identity (and its analog in D=3,4,6) that is responsible for the manifest (classical) SUSY of the GS superstring [7]. (1.1) stands behind many beautiful mathematical constructions (division algebras, triality relations, etc.) which are accumulating more and more evidence to be in deep connection with D=3,4,6 and 10 SUSY [27].

Unfortunately, in the form in which it was discovered, the SYM Lagrangian:

$$L = -\frac{1}{4}tr(f_{\mu\nu}f^{\mu\nu}) - \frac{i}{2}tr(w \not{\partial} w) \quad (1.2)$$

was not explicitly supersymmetric. In (1.2)  $w_\alpha$  is a left handed Majorana-Weyl D=10 spinor while  $f_{\mu\nu}$  and  $\nabla_\mu$  are gauge covariant expressions in terms of a gauge vector field  $a_\mu(x)$ :

$$igf_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \quad (1.3)$$

$$\nabla_\mu = \partial_\mu + ig[a_\mu, \cdot] \quad (1.4)$$

We use here lower case characters  $a, \nabla, f, w, \phi$  etc... in order to emphasize that the respective expressions are ordinary fields and *not* superfields as

we will use in the rest of the paper and denote by capitals:  $A, \nabla, F, W, \phi$ . Later we will also introduce ghost-haunted superfields which we will denote by  $\mathcal{A}, \mathcal{F}, \mathcal{W}, \mathcal{\Phi}$  etc.

The field equations of motion which this field action generates by varying with respect to  $\alpha$  and respectively  $w$  are: generalizations of the Maxwell

$$\nabla^\mu f_{\mu\nu} = \frac{g}{2} (\sigma_\nu)^{\alpha\beta} \{w_\alpha, w_\beta\} \quad (1.5)$$

and respectively Dirac

$$\not{\partial} w \equiv (\sigma^\mu)^{\alpha\beta} \nabla_\mu w_\beta = 0 \quad (1.6)$$

equations.

The lack of off-shell superspace formulation of (1.2) hampered its study and the extension of its successes to other models (may be even superstrings).

In order to obtain an explicitly supersymmetric theory it was tried to formulate the theory in terms of superfields:

$$\phi(x, \theta) = \phi(x) + \sum_{r=1}^{16} \frac{1}{r!} \theta_{\alpha_1} \dots \theta_{\alpha_r} \phi^{\alpha_1 \dots \alpha_r}(x) \quad (1.7)$$

These attempts to formulate the theory in superspace lead to more tantalizing discoveries. The general 1-form gauge superfield in the  $D=10$   $N=1$  superspace:

$$A(x, \theta) = dx^\mu A_\mu(x, \theta) + d\theta_\alpha A^\alpha(x, \theta) \quad (1.8)$$

describes too many degrees of freedom and in order to describe just the on-shell SYM it has to be submitted to certain constraints (the Nilsson

constraints) [23]:

$$gF^{\alpha\beta} \equiv \{\nabla^\alpha, \nabla^\beta\} - 2i(\sigma^\mu)^{\alpha\beta}\nabla_\mu = 0 \quad (1.9)$$

where:

$$\nabla_\mu = \partial_\mu + ig[A_\mu, \cdot] \quad (1.10)$$

$$\nabla^\alpha = D^\alpha + g[A^\alpha, \cdot] \quad (1.11)$$

$$D^\alpha = \frac{\partial}{\partial\theta_\alpha} + i(\sigma^\mu)^{\alpha\beta}\theta_\beta\partial_\mu \quad (1.12)$$

The structure of the constraints which the superfield has to fulfill in order to describe on shell the SYM spectrum turned out to have deep geometrical meaning in the sense that it can be interpreted as the integrability conditions [6] for certain systems of linear partial differential equations. In fact in the case of SYM, these partial differential equations express the pure gauge character of the field along arbitrary light-like superspace directions  $\lambda^\mu$ :

$$x^\mu = x_0^\mu + t\lambda^\mu - (\sigma^{\mu\nu})^\alpha{}_\beta\lambda_\nu\theta_\alpha^0\epsilon^\beta$$

$$\theta_\alpha = \theta_\alpha^0 + \lambda_\mu\sigma_{\alpha\beta}^\mu\epsilon^\beta$$

where  $(x_0, \theta^0)$  parametrize the offset of an arbitrary super-light-like ray while  $(t, \epsilon^\alpha)$  parametrize the position within the super-ray.

This pure gauge character of some of the field-theoretic degrees of freedom is in turn related to the fermionic  $\kappa$ -gauge invariance [28] essential for the consistency of the GS superstring [7]. On the other hand the integrability conditions (1.9) together with their consequences from the Bianchi identities for the covariant derivatives (1.10), (1.11) can be shown to be equivalent to the ordinary field equations of motion obtained from the (1.2) Lagrangian [6, 24, 25].

The great puzzle was the fact that it was still impossible to find an explicitly super-Poincare invariant action from which the Nilsson's constraints (1.9) would appear as field equations of motion when the action is varied with respect to the superfields. This impossibility was codified in the statement [29] that it is impossible (within certain assumptions) to write a D=10 SYM action in terms of unconstrained superfields (i.e. superfields which vary freely off-shell and the only constraints appear on-shell as a result of the free variation of the action). The existence of a covariant unconstrained superfield action is the necessary basis for a quantum field theory which explicitly displays all the super-space-time invariances of the model.

Similar properties are shared by N=2 type B D=10 supergravity [30] and of course, at least in some sense, by the superstrings.

The harmonic "spinor-vielbein-like" variables avoid the above no-go theorems [10-15]. This is not completely unexpected in view of similar successes obtained by the harmonic superspace approach in different contexts [31]. Moreover the apparent relation of certain "vielbein-like" auxiliary variables with supertwistors [32,33,34] renders natural their usefulness in describing massless systems [6,25].

In the present paper we show explicitly that our gauge and super-Poincare invariant unconstrained superfield action based on the point-particle limit of the BRST charge  $Q_{BRST}$  of the Super-Poincare covariant GS superstring gives on-shell the Nilsson constraint equations of D=10 SYM.

The plan of the paper is as follows.

In section 2 we review pedagogically the developments [10-15] which lead us to the super-Poincare covariant  $Q_{BRST}$  of the GS superstring. In particular we explain the origin of the auxiliary variables and of the additional gauge invariances. Also the statement in the recent paper by Kallosh and Rahmanov [35] claiming "nonunitarity" of our formalism is



shown to be incorrect.

In section 3 we describe the super-Poincare covariant first quantization of the  $N=1$  BS superparticle (the zero-mode of the GS superstring) in the Dirac canonical formalism.

Section 4 is devoted to the covariant first- and second- quantization of the  $D=10$ ,  $N=1$  BS superparticle in the BFV-BRST formalism.

In section 5 we derive a harmonic superfield representation of the Nilsson constraints for  $D=10$   $N=1$  SYM and prove its equivalence to the original Nilsson constraints. In particular, the linearized form of these harmonic superfield equations is shown to exactly coincide with the Dirac constraint equations for the superfield wave function of the covariantly quantized  $D=10$   $N=1$  BS superparticle.

In section 6 we review our general covariant action principle for arbitrary overdetermined systems of nonlinear field equations and apply it to construct a superspace action for  $D=10$   $N=1$  SYM in terms of unconstrained (off-shell) superfields.

In section 7 we discuss the implications of the present results and the directions for further developments. Appendix A summarizes the spinor conventions while appendix B describes a remarkable Lorentz- $SO(1,9)$ -invariant  $SO(8)$  algebra. Appendix C supplies the general proof of the pure gauge nature of the auxiliary harmonic variables, needed to perform super-Poincare covariant quantization.

## 2. The Super-Poincare covariant quantization of the GS superstring

The present work constructs the unconstrained superfield action of the D=10 SYM making use crucially of the point-particle limit of the explicitly super-Poincare invariant BRST charge of the GS superstring.

Such a BRST construction was possible as a consequence of the harmonic superstring program for a manifestly super-Poincare covariant quantization of the GS superstring which we developed during the last year [10-15].

In order to make the structure and the origin of the BRST charge construction clear, we will describe in this section in a sketchy but hopefully pedagogical way the main ideas and concepts of the harmonic superstring program.

The GS superstring is formulated as a superspace generalization of the bosonic string.

Indeed, it substitutes in the bosonic string action

$$S_B = \int d\tau d\xi \sqrt{-g} \left\{ -\frac{1}{2} g^{mn} \partial_m X^\mu \partial_n X_\mu \right\} \quad (2.1)$$

in place of  $\partial_m X^\mu$  the superspace expression

$$\partial_m X^\mu + i(\theta \sigma^\mu \partial_m \theta) \quad (2.2)$$

The resulting GS action [7] in the Lagrangian form is:

$$S_{GS} = \int d\tau \int_{-\pi}^{\pi} d\xi \sqrt{-g} \left\{ -\frac{1}{2} g^{mn} \Pi_m^\mu \Pi_{n\mu} - i\epsilon^{mn} \Pi_m^\mu \sum_A (-1)^A \theta_A \sigma_\mu \partial_n \theta_A \right. \\ \left. - \epsilon^{mn} (\theta_1 \sigma_\mu \partial_m \theta_1) (\theta_2 \sigma^\mu \partial_n \theta_2) \right\} \quad (2.3)$$

where

$$\Pi_m^\mu \equiv \partial_m X^\mu + i \sum_{A=1,2} (\theta_A \sigma^{\mu} \partial_m \theta_A) \quad (2.4)$$

Here  $g_{mn}(\tau, \xi)$  ( $m, n=0,1$ ) is the 2-D world-sheet metric and  $X^\mu(\tau, \xi)$ ,  $\theta_A(\tau, \xi)$ , ( $A = 1, 2$ ) are the superstring coordinates which are world-sheet scalars.  $\theta_A$  are anticommuting (grassmanian) D=10 Majorana-Weyl (MW) spinors. The last term in the superspace action (2.3) is crucial for the correct physical content of the theory. Classically  $S_{GS}$  (2.3) exists in D=3,4,6 and 10, however, after quantization, due to quantum anomalies it is consistent only in D=10 [7]. From now on we shall always work in the critical space-time dimension D=10.

The superspace formulation of the theory has very important potential advantages in the study of the conceptual foundations of the theory and of its implications.

Unfortunately, these advantages could not be exploited until recently because of severe problems in quantizing this theory.

indeed, the conditions for the application of the covariant Faddeev-Popov procedure in the Lagrangian formalism do not hold for the GS superstring since its local symmetries do not form a Lie group. Thus, one has to return to the techniques of canonical Hamiltonian quantization.

The heart of the passage to the Hamiltonian formulation consists in the introduction of conjugate momenta to the coordinates parametrizing the Lagrangian  $L(q, \dot{q})$ :

$$p_i = \frac{\delta L(q_i, \dot{q}_i)}{\delta \dot{q}_i} \quad (2.5)$$

For nonsingular systems, one can then eliminate the variables  $\dot{q}_i$  by expressing them as functions of  $q_i$ ,  $p_i$  from the system (2.5).

However, if the system (2.5) is singular [36] i.e.:

$$\det\left(\frac{\delta^2 L}{\delta \dot{q}_i \delta \dot{q}_j}\right) = 0$$

(as the GS superstring happens to be) this elimination cannot be effected and two types of situations can arise which can be epitomized by their simplest cases:

$$\text{case 1} \quad : \quad p_i = 0 \text{ for some } i \text{ in (2.5)} \quad (2.6)$$

$$\text{case 2} \quad : \quad p_i \sim q_i \text{ for some } i \text{ in (2.5) ; with } p \text{ and } q \text{ grassmanian.} \quad (2.7)$$

In both cases, since  $\dot{q}_i$  does not appear, it cannot be eliminated. The equations (2.6) and (2.7) are the simplest examples of first and respectively second class constraints [36]. Given a constraint system, the precise definitions are:

- 1) A subset of constraints is first class if their Poisson brackets (PB) with any constraint of the system are 0 modulo the constraints of the system (i.e. "on the constraints shell").
- 2) A subset of constraints is second class if the matrix formed by the Poisson brackets among its elements is nonsingular on the constraints shell.

It turns out that any set of constraints can be reduced to the form (2.6) and/or (2.7) by appropriate canonical coordinate transformations (which however break in general the space-time locality and the global symmetries of the problem).

The GS superstring has constraints of both classes.

In order to concentrate on the essentials we will explain this on the example of the D=10, N=1 BS superparticle [8] where the formulae are simpler.

The BS action in the Hamiltonian formalism is:

$$S = \int d\tau [p_\mu \partial_\tau x^\mu + p_\theta^\alpha \partial_\tau \theta_\alpha - H_T] \quad (2.8)$$

$$H_T = \lambda p^2 + \lambda_\alpha D^\alpha \quad (2.9)$$

In (2.8),  $\theta_\alpha$  is a left-handed D=10 MW spinor<sup>\*</sup>,  $\lambda$  and  $\lambda_\alpha$  are Lagrange multipliers, and the fermionic constraint  $D^\alpha$  reads:

$$D^\alpha \equiv -i p_\theta^\alpha - \not{p}^{\alpha\beta} \theta_\beta \quad (2.10)$$

The system (2.5) is therefore:

$$p_\theta^\alpha = i \not{p}^{\alpha\beta} \theta_\beta \quad (2.11)$$

with

$$p^2 = 0 \quad (2.12)$$

In a particular frame:  $p = (p^+, 0, 0, p^- = 0)$ , and using a noncovariant separation of the D=10 MW spinors  $\theta_\alpha$  (and also  $p_\theta^\alpha$ ) into two SO(8) (s) and (c) spinors  $\theta_a^+, \theta_a^-$ , (2.11) reduces indeed to a mixture of equations of the form (2.6) and (2.7):

$$-i \begin{bmatrix} p_\theta^+ \\ p_\theta^- \end{bmatrix} = \begin{bmatrix} \sqrt{2} p^+ & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta^+ \\ \theta^- \end{bmatrix} \quad (2.13)$$

i.e.:

$$-i \begin{bmatrix} p_\theta^+ \\ p_\theta^- \end{bmatrix} = \begin{bmatrix} \sqrt{2} p^+ \theta^+ \\ 0 \end{bmatrix} \quad (2.14)$$

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\* see Appendix A for our spinor conventions.

which on components reads:

$$-ip_{\theta}^{+} = \sqrt{2}p^{+}\theta^{+} \quad (2.15)$$

$$p_{\theta}^{-} = 0 \quad (2.16)$$

The first line is now explicitly of class 2 (cf. eq. (2.7)) while the second line is explicitly of class 1 (cf. (2.6)).

In order to carry out the quantization procedure, one has to deal separately with the first and second class constraints. In the technical jargon the system of constraints has to be reduced to a covariant, first class and BFV-irreducible one [22].

However, since (2.10) is an irreducible Lorentz expression (a MW spinor), any separation of the type (2.15)-(2.16) has to break the Lorentz invariance.

This was for years the puzzle of covariant quantization of the GS superstring (and the BS superparticle): the spinor objects relevant for the quantization procedure are too small to fit into a spinor representation of the Lorentz group. In fact the structure of the constraints requires objects which transform under the group  $SO(8) \times SO(1, 1)$ .

The solution of the problem lies in an analogy with the vielbein formalism of general relativity. Using vielbein-like objects we reduced the GS superstring to a system whose constraints were covariant, first class, and BFV-irreducible [13-15]. The covariant BFV-BRST quantization of the system is then straightforward.

Let us now comment on the importance of BFV-irreducibility [22] requirement which was often underestimated. Indeed, the BFV-irreducibility is crucial [37,38,39] for the success of the super-Poincare covariant BFV quantization of the GS and BS systems. The BFV-irreducibility means the functional independence of the constraints. The

entire puzzle of the covariant quantization of the GS and BS systems can be formulated as a clash between the concept of irreducibility of Lorentz group representations and the concept of BFV-irreducibility of constraint systems. If part of the constraints of a system are dependent, the BFV procedure requires the introduction of additional ghosts. In particular, it was repeatedly tried [40,38] to quantize the system by expressing the 8 first class constraints (2.16) covariantly in terms of a 16 component MW spinor. This lead into trouble because 8 of the 16 components have to be dependent. Then, the correct quantization procedure requires the introduction of 8 (second generation) ghosts which will again raise the problem of expressing covariantly 8 component objects. One can again try to do it by introducing 16 (second generation) ghosts related by 16 (second generation) constraints out of which only 8 constraints are independent. However the other 8 (dependent) constraints will again require 8 (third generation) ghosts and so on. The procedure never ends unless at some stage one imposes directly and irreducibly 8 independent constraints [38] which however are bound to break Lorentz invariance. In [38] the Lorentz invariance is broken through the introduction of two fixed light-like vectors  $N$  and  $M$  which are not dynamical variables. Even though it can be formally shown that the physics does not depend of the direction of the two fixed vectors  $N, M$ , their presence is no less a breaking of the explicit Lorentz invariance than the imposition of the light-cone gauge. There too one can show formally that the physics remains Lorentz invariant in critical dimensions.

Let us now return to our fundamental vielbein-like construction. To better understand it, let us first see what is the role of the vielbein variables in general relativity and then we will explain the role of the vielbein-like variables in our formalism. In general relativity, the vielbeins are introduced to solve the following problem (see e.g. [41]). One cannot introduce spinors as representations of the general coordinate

reparametrization group. The relevant objects (vector fields etc.) transform as:

$$V^\mu(x) \rightarrow V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x) \quad (2.17)$$

where the matrix  $\frac{\partial x'^\mu}{\partial x^\nu}$  belongs to the  $GL(N, R)$  group. The  $GL(N, R)$  group (2.17) does not have spinor representations.

However, spinors are representations of the  $SO(N)$  group which is a subgroup of  $GL(N, R)$ .

In order to express objects transforming under  $SO(N)$  without breaking the  $GL(N, R)$  invariance, one introduces some auxiliary variables: the vielbeins.

As by-product of conceptual value, the vielbeins constitute an explicit realization of the principle of equivalence which is the underlying feature of the general relativity. Indeed, introducing the vielbeins is in fact defining at each point in space time an arbitrary orthonormal Lorentz frame. The equivalence principle requires that the physics is independent of the particular orientation of the Lorentz-frame in each point.

This requirement is realized by choosing the appearance of the vielbeins in the Lagrangian in such a way that the physics is not modified.

Mathematically, these properties are expressed as follows: The vielbeins have  $N$ -valued  $GL(N, R)$  indices  $\mu$  which identify them as space-time vectors and  $N$ -valued indices  $a$  which index them as the elements of a basis. The fact that the basis is orthonormal is expressed by:

$$e_{\mu a} e^\mu_b = \eta_{ab} \quad (2.18)$$

(where  $\eta_{ab} = \text{diag}(-, +, \dots, +)$ ).

The principle of equivalence is realized by requiring that the physical quantities are invariant under the rotations of the local orthonormal ba-



sis. In other words, the theory possesses a local  $SO(N)$  invariance acting on the internal indices  $a$ .

The presence of the vielbeins allows now to express objects covariant under  $SO(N)$  but not under  $GL(N, R)$  in a way in which  $GL(N, R)$  is not broken.

For instance a spinor transforms as:

$$\psi(x) \rightarrow \Lambda(x)\psi(x)$$

where  $\Lambda$  is an  $SO(N)$  matrix written in the spinor representation. Also a  $GL(N, R)$  vector can be expressed in terms of  $N$   $GL(N, R)$  scalars organized as an  $SO(N)$  vector:

$$V^a = e^a_\mu V^\mu \quad (2.19)$$

Let us now see how these ideas apply to our case. We want to be able to express objects transforming as spinors of  $SO(8)$  without breaking the  $SO(1, 9)$  Lorentz symmetry of the system .

To this end we introduce the auxiliary variables [10-15]:

$$u_\mu^a, v_\alpha^{\pm\frac{1}{2}} \quad (2.20)$$

where the indices  $\mu, \alpha$  transform as vector and MW-spinor under the global Lorentz  $SO(1, 9)$  respectively, while the indices  $a, \pm\frac{1}{2}$  transform respectively under the internal groups  $SO(8)$  and  $SO(1, 1)$ . These auxiliary variables will act as "spinorial vielbeins" bridging covariantly between  $SO(1, 9)$  and  $SO(8)$  spinor objects. Due to the remarkable triality properties of  $SO(8)$ , the indices  $a$  can be chosen to transform in any of the fundamental (s),(c),(v) representations of  $SO(8)$ .

The orthogonality relations analogous to (2.18) are:

$$u_{\mu}^a u^{b\mu} = C^{ab}$$

$$[v_{\alpha}^{\pm\frac{1}{2}}(\sigma^{\mu})^{\alpha\beta} v_{\beta}^{\pm\frac{1}{2}}] u_{\mu}^a = 0 \quad (2.21)$$

$$[v_{\alpha}^{+\frac{1}{2}}(\sigma^{\mu})^{\alpha\beta} v_{\beta}^{+\frac{1}{2}}][v_{\gamma}^{-\frac{1}{2}}(\sigma_{\mu})^{\gamma\delta} v_{\delta}^{-\frac{1}{2}}] = -1$$

In the first line of (2.21)  $C^{ab}$  denotes the invariant metric tensor in the relevant  $SO(8)$  representation space (see appendix A).

In the sequel the following two light-like vectors  $u_{\mu}^{\pm}$  will appear, which are composite variables built out of the elementary variables  $v_{\alpha}^{\pm\frac{1}{2}}$ :

$$u_{\mu}^{\pm} = v_{\alpha}^{\pm\frac{1}{2}}(\sigma_{\mu})^{\alpha\beta} v_{\beta}^{\pm\frac{1}{2}} \quad (2.22)$$

This construction automatically encodes the light-like geometrical character of  $u_{\mu}^{\pm}$  which is due to the  $D=10$  Fierz identity (1.1). There are indications that this fact has deep relations with the twistor light-like geometry of space-time [42,33,34]. Henceforth, we shall use the shorthand notations:

$$u_{\mu}^{\pm} \text{ as in (2.22)}$$

$$A^{\pm} \equiv u_{\mu}^{\pm} A^{\mu} \equiv v^{\pm\frac{1}{2}} \not{A} v^{\pm\frac{1}{2}}$$

$$A^a \equiv u_{\mu}^a A^{\mu} ; \sigma^{a_1 \dots a_n} \equiv u_{\mu_1}^{a_1} \dots u_{\mu_n}^{a_n} \sigma^{[\mu_1 \dots \mu_n]} \quad (2.23)$$

for any Lorentz vector  $A^{\mu}$ . Let us particularly stress that  $A^{\pm}, A^a$  are Lorentz scalars and they should not be confused with the vector components of  $A^{\mu}$  which appear in the non-covariant light-cone formalism.

The gauge invariances insuring that the introduction of the vielbein-like variables  $u, v$  does not affect the physics are expressed in the Hamiltonian formalism by the first class constraints:

$$D^{ab} \equiv u_{\mu}^a \frac{\partial}{\partial u_{\mu b}} - u_{\mu}^b \frac{\partial}{\partial u_{\mu a}} + \frac{1}{2}(v^{+\frac{1}{2}} \sigma^{ab} \frac{\partial}{\partial v^{+\frac{1}{2}}} + v^{-\frac{1}{2}} \sigma^{ab} \frac{\partial}{\partial v^{-\frac{1}{2}}}) \quad (2.24)$$

$$D^{-+} \equiv \frac{1}{2}(v_{\alpha}^{+\frac{1}{2}} \frac{\partial}{\partial v_{\alpha}^{+\frac{1}{2}}} - v_{\alpha}^{-\frac{1}{2}} \frac{\partial}{\partial v_{\alpha}^{-\frac{1}{2}}}) \quad (2.25)$$

$$D^{\pm a} \equiv u_{\mu}^{\pm} \frac{\partial}{\partial u_{\mu a}} + \frac{1}{2} v^{\mp \frac{1}{2}} \sigma^{\pm} \sigma^a \frac{\partial}{\partial v^{\mp \frac{1}{2}}} \quad (2.26)$$

They express the fact (analogous to the principle of equivalence) that the physics is invariant under arbitrary rotations of the "vielbein-like" frame  $(u_{\mu}^a, u_{\mu}^{\pm})$ . The operators (2.24)-(2.26) represent indeed the  $SO(1, 9)$  algebra under commutation:

$$[D^{ab}, D^{cd}] = C^{bc} D^{ad} - C^{ac} D^{bd} + C^{ad} D^{bc} - C^{bd} D^{ac}$$

$$[D^{ab}, D^{\pm c}] = C^{bc} D^{\pm a} - C^{ac} D^{\pm b}; [D^{ab}, D^{-+}] = 0$$

$$[D^{-+}, D^{\pm a}] = \pm D^{\pm a} \quad (2.27)$$

$$[D^{+a}, D^{-b}] = C^{ab} D^{-+} + D^{ab}$$

From (2.27) one immediately recognizes  $D^{ab}, D^{-+}$  as generators of  $SO(8) \times SO(1, 1)$  whereas  $D^{\pm a}$  are recognized as the coset generators corresponding to  $\frac{SO(1, 9)}{SO(8) \times SO(1, 1)}$ .

The second important requirement is the requirement about the **specific** dependence of the wave functions  $\phi(x, \theta, u, v)$  on the auxiliary variables  $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$ . The representation space  $\mathcal{H}_0$  of the quantum algebra (2.27) is spanned by definition by functions of the following general form (here  $\phi$  is taken in the momentum space representation with respect to  $x$ ):

$$\phi(p, \theta, u, v) = \sum_{\{\lambda\}\{\nu\}} \left(\frac{u_{\lambda_1}^+}{p^+}\right) \dots \left(\frac{u_{\lambda_k}^+}{p^+}\right) \left(\frac{u_{\nu_1}^-}{p^-}\right) \dots \left(\frac{u_{\nu_l}^-}{p^-}\right) \phi_{kl}^{\{\lambda\}\{\nu\}}(p, \theta, u, v) \quad (2.28)$$

(recall  $p^\pm \equiv v^{\pm\frac{1}{2}} p v^{\pm\frac{1}{2}}$ ), where  $\phi_{kl}^{\{\lambda\}\{\nu\}}$  are defined by the following **specific** expansion in the auxiliary variables  $u, v$  (recall (2.22)):

$$\begin{aligned} \phi_{kl}^{\{\lambda\}\{\nu\}}(p, \theta, u, v) &= \sum_{\{\mu\}\{\kappa\}} [u_{\mu_1}^{a_1} \dots u_{\mu_n}^{a_n}]_{SO(8)} \text{ singlet } u_{\kappa_1}^+ \dots u_{\kappa_m}^+ u_{\kappa_{m+1}}^- \dots u_{\kappa_{2m}}^- \\ &\times \phi_{klm}^{\{\lambda\}\{\nu\}\{\mu\}\{\kappa\}}(p, \theta) \end{aligned} \quad (2.29)$$

The expansion (2.29) is characterized by the fact that each term is a monomial in the auxiliary variables in which all the  $SO(8) \times SO(1, 1)$  indices are **saturated** among the  $u_\mu^a$ 's and the  $v_\alpha^{\pm\frac{1}{2}}$ 's only, whereas the coefficients  $\phi_{klm}^{\{\lambda\}\{\nu\}\{\mu\}\{\kappa\}}(p, \theta)$  are arbitrary ordinary superfields **inert** under the  $SO(8) \times SO(1, 1)$  internal group, i.e. they do not carry any  $SO(8) \times SO(1, 1)$  indices. In order to have the terms of the expansion (2.28) all independent, the coefficient superfields have to be symmetric in the indices  $\{\lambda\}$  and  $\{\nu\}$ :

$$\begin{aligned} \phi_{kl}^{\{\dots\lambda_i \dots \lambda_j \dots\}\{\nu\}}(p, \theta, u, v) &= \phi_{kl}^{\{\dots\lambda_j \dots \lambda_i \dots\}\{\nu\}}(p, \theta, u, v) \\ \phi_{kl}^{\{\lambda\}\{\dots\nu_i \dots \nu_j \dots\}}(p, \theta, u, v) &= \phi_{kl}^{\{\lambda\}\{\dots\nu_j \dots \nu_i \dots\}}(p, \theta, u, v) \end{aligned} \quad (2.30)$$

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the homogenous space  $\frac{\mathcal{L}}{SO(8) \times SO(1,1)}$  instead of being a function on the original space  $\mathcal{L}$  defined by the kinematical constraints (2.21) on  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a$ . Hence the functions of the form (2.28) will be called **harmonic superfields** whereas the functions (2.29) will be called **analytic harmonic superfields** (because of their analytic dependence on  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a$ ). This also justifies the name "harmonic" for the auxiliary variables  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a$  which effectively enter the present formalism through  $\frac{\mathcal{L}}{SO(8) \times SO(1,1)}$ . Analytic harmonic superfields were first introduced in a different context in [31].

Let us point out, that the harmonic superfields (2.28)(2.29) are also characterized by the fact that they do not carry external overall  $SO(8) \times SO(1,1)$  indices (hence the subscript 0 in the notation  $\mathcal{H}_0$  of their space). In section 3 we shall introduce more general harmonic superfields bearing external  $SO(8) \times SO(1,1)$  indices which are simply expressed in terms of the fields (2.28),(2.29) (see also Appendix C).

Now, one can easily deduce (cf. [11]), that the Dirac constraint equations:

$$(D^{ab}, D^{-+}, D^{+a}, D^{-a})\phi = 0 \quad (2.36)$$

on the space  $\mathcal{H}_0$  (2.28),(2.29) imply (in the notations of (2.28),(2.29)):

$$\begin{aligned} \phi_{0000}(p, \theta) &= \text{arbitrary,} \\ \phi_{klmn}^{\{\lambda\}\{\nu\}\{\mu\}\{\kappa\}}(p, \theta) &= 0, \quad (k, l, m, n) \neq (0, 0, 0, 0) \end{aligned} \quad (2.37)$$

i.e. the Dirac first-class constraints (2.24)-(2.26) together with the specification (2.28),(2.29) of the representation space  $\mathcal{H}_0$  imply that  $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$  are pure gauge degrees of freedom. A simple explanation of this property is that the harmonic superfields from  $\mathcal{H}_0$  (2.28),(2.29) depend only on 45 independent combinations of harmonic variables\*:  $u_\mu^a, u_\mu^\pm = v^{\pm\frac{1}{2}} \sigma_\mu v^{\pm\frac{1}{2}}$

\* The 100 harmonic combinations  $u_\mu^a, u_\mu^\pm$  are subject to 55 kinematical constraints (cf. [46]):  $u_\mu^a u^{b\mu} = C^{ab}, u_\mu^a u^{\pm\mu} = 0, (u^\pm)^2 = 0, u_\mu^+ u^{-\mu} = -1$

which exactly matches the number of equations (2.36).

The restriction of the quantum states of the harmonic formalism  $\phi(p, \theta, u, v)$  to the form (2.28),(2.29), i.e. to the space  $\mathcal{H}_0$  is crucial. It is this restriction which substitutes within the harmonic formalism for the "missing"

$$\begin{aligned}
 & 14 \text{ gauge invariances} = \\
 & 59 \text{ (the number of independent } u_\mu^a, v_\alpha^{\pm\frac{1}{2}} \text{ from (2.21) )} \\
 & - 45 \text{ (the number of } D^{ab}, D^{-+}, D^{\pm a})
 \end{aligned}$$

which would be necessary to gauge away completely all the  $u_\mu^a$ 's and the  $v_\alpha^{\pm\frac{1}{2}}$ 's if the wave functions  $\phi$  were allowed to depend arbitrarily on the  $u_\mu^a$ 's and the  $v_\alpha^{\pm\frac{1}{2}}$ 's:

$$\begin{aligned}
 \phi_{\text{arbitrary}}(p, \theta, u, v) = & \sum_{\{\mu\}\{\alpha\}\{\beta\}} u_{\mu_1}^{a_1} \dots u_{\mu_n}^{a_n} v_{\alpha_1}^{+\frac{1}{2}} \dots v_{\alpha_m}^{+\frac{1}{2}} v_{\beta_1}^{-\frac{1}{2}} \dots v_{\beta_k}^{-\frac{1}{2}} \\
 & \times \phi_{a_1 \dots a_n}^{+\frac{1}{2}(m-k)\{\mu\}\{\alpha\}\{\beta\}}(p, \theta) .
 \end{aligned} \tag{2.38}$$

Overlooking the crucial difference between the naive wave functions (2.38) and the relevant space  $\mathcal{H}_0$  of quantum states defined by (2.28),(2.29) lead to a statement in a recent paper by Kallosh and Rahmanov [35] claiming the "non-unitarity" of our formalism. The above explanation, and the discussion in Appendix C below, shows that their claim is not correct.

From the constraint algebra (2.27) one easily deduces the action describing the pure-gauge "dynamics" of the system of harmonic variables  $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$ :

$$\begin{aligned}
 S_{\text{harmonic}} = & \int d\tau [p_{u_a}^\mu \partial_\tau u_\mu^a + p_v^{-\frac{1}{2}\alpha} \partial_\tau v_\alpha^{+\frac{1}{2}} + p_v^{+\frac{1}{2}\alpha} \partial_\tau v_\alpha^{-\frac{1}{2}} \\
 & - \Lambda_{ab} D^{ab} - \Lambda^{+-} D^{-+} - \Lambda_a^- D^{+a} - \Lambda_a^+ D^{-a}]
 \end{aligned} \tag{2.39}$$

Because of the kinematical constraints on  $(u, v)$  (2.21), their respec-

tive conjugate momenta are similarly kinematically constrained:

$$\begin{aligned}
 p_{u_\mu}^{(a} u^{b)\mu} &= 0 \\
 p_{u_\mu}^a v^{\pm\frac{1}{2}} \sigma^\mu v^{\pm\frac{1}{2}} &= 0 \\
 v_\alpha^{+\frac{1}{2}} p_v^{-\frac{1}{2}\alpha} + v_\alpha^{-\frac{1}{2}} p_v^{+\frac{1}{2}\alpha} &= 0
 \end{aligned} \tag{2.40}$$

In (2.39)  $\Lambda_{ab}, \dots, \Lambda_a^+$  denote Lagrange multipliers for the corresponding first-class constraints  $D^{ab}, \dots, D^{-a}$  which are the classical counterparts of the harmonic differential operators (2.24)-(2.26) and, therefore all constraints are first class.

The classical analog of the requirements (2.28), (2.29) on the representation space  $\mathcal{H}_0$  of quantum states is the requirement on the form of the classical "observables" [11]. The latter are not arbitrary functions of  $(u, v)$  and their conjugate momenta  $(p_u, p_v)$ , but are given as expansions in  $(u, v; p_u, p_v)$  where all internal  $SO(8) \times SO(1, 1)$  are saturated among  $u, v, p_u, p_v$  and, therefore the corresponding coefficients do not carry any  $SO(8) \times SO(1, 1)$  indices.

The BRST charge corresponding to (2.24)-(2.26) reads:

$$\begin{aligned}
 Q_{\text{harmonic}} &= i\eta_{ab} [D^{ab} + \eta^{+a} \frac{\partial}{\partial \eta_b^+} - \eta^{+b} \frac{\partial}{\partial \eta_a^+} \\
 &\quad + \eta^{-a} \frac{\partial}{\partial \eta_b^-} - \eta^{-b} \frac{\partial}{\partial \eta_a^-} + \eta_d^a \frac{\partial}{\partial \eta_{bd}} - \eta_d^b \frac{\partial}{\partial \eta_{ad}}] \\
 &\quad + i\eta^{+-} [D^{-+} + \eta_a^+ \frac{\partial}{\partial \eta_a^+} - \eta_a^- \frac{\partial}{\partial \eta_a^-}] \\
 &\quad + i\eta_a^- D^{+a} \\
 &\quad + i\eta_a^+ [D^{-a} + \eta_a^- \frac{\partial}{\partial \eta^{+-}} - \eta^{-b} \frac{\partial}{\partial \eta^{ab}}]
 \end{aligned} \tag{2.41}$$

where

$$\eta_{ab}, \eta^{+-}, \eta^{+a}, \eta^{-a} \tag{2.42}$$



are the ghosts corresponding to the constraints

$$D^{ab}, D^{-+}, D^{-a}, D^{+a}. \quad (2.43)$$

Let us now see how the presence of the auxiliary variables allows us to express separately the first and second class constraints [11].

Given the constraints (2.10), and using the vielbein-like harmonic variables (2.21) we can express the 16-component 10 D MW spinor constraint (2.10):

$$D^\alpha \equiv -ip_\theta^\alpha - \not{p}^{\alpha\beta} \theta_\beta$$

in terms of 2 sets of 8 Lorentz scalars (organized as two SO(8) spinors):

$$G^{+\frac{1}{2}a} = \frac{1}{2} u_\mu^a v_\alpha^{-\frac{1}{2}} (\sigma^\mu)^{\alpha\beta} (\sigma^+)_{\beta\gamma} D^\gamma \quad (2.44)$$

$$D^{+\frac{1}{2}a} = u_\mu^a v_\alpha^{+\frac{1}{2}} (\sigma^\mu)^{\alpha\beta} \not{p}_{\beta\gamma} D^\gamma \quad (2.45)$$

The projectors in front of  $D^\gamma$  are chosen such that the Lorentz indices are saturated. In particular one sees that without the  $v$ 's it is impossible to saturate the Lorentz-spinor indices. However  $v_\alpha^{\pm\frac{1}{2}}$  together with  $(\sigma^\mu)^{\alpha\beta}$  can convert a spinor index  $\beta$  into a vector index  $\mu$  which is then saturated with an  $u_\mu^a$ . The role of  $\sigma^+$  (cf. (2.23),  $+$  is an internal SO(1,1) index which is inert under Lorentz transformations) in (2.44) is to appropriately raise a spinor index. The role of  $\not{p}$  in (2.45) is to make  $D^{+\frac{1}{2}a}$  first class.

It turns out that indeed  $G^{+\frac{1}{2}a}$  constitute 8 second class constraints (i.e. the matrix of their Poisson brackets

$$\{G^{+\frac{1}{2}a}, G^{+\frac{1}{2}b}\}_{PB} = ip^+ C^{ab} \quad (2.46)$$

is nonsingular) while (2.45) are first class (i.e.:

$$\{D^{+\frac{1}{2}a}, D^{+\frac{1}{2}b}\}_{PB} = -2i(p^+)p^2 \quad (= 0 \text{ modulo constraints}). \quad (2.47)$$

In turn  $D^\alpha$  can be reconstructed out of  $D_a^{+\frac{1}{2}}$  and  $G_a^{+\frac{1}{2}}$ :

$$D^\alpha = (p^+)^{-1}(\sigma^b v^{+\frac{1}{2}})^\alpha D_b^{+\frac{1}{2}} + (p^+)^{-1}(\not{p}\sigma^+ \sigma^b v^{-\frac{1}{2}})^\alpha G_b^{+\frac{1}{2}} \quad (2.48)$$

Once the covariant separation of the constraints is effectuated, one can use a trick invented in [43,44] to transform the second class constraints  $G^{+\frac{1}{2}a}$  (2.44) into first class constraints  $\hat{G}^{+\frac{1}{2}a}$  without changing the physical content of the constrained system\* by introducing auxiliary dynamical real fermionic variables  $\Psi^a$  [13,14] with transformation properties\*\* and Poisson Brackets similar to the second class constraints which they convert into first class ones:

$$\hat{G}^{+\frac{1}{2}a} = G^{+\frac{1}{2}a} + \sqrt{p^+}\Psi^a \quad (2.49)$$

$$\{\Psi^a, \Psi^b\}_{PB} = -iC^{ab} \quad (2.50)$$

One can then use the decomposition (2.48) to reconstruct [13,14] the Lorentz MW-spinor first-class constraint  $\hat{D}^\alpha$  out of the first class constraints  $D^{+\frac{1}{2}a}$  (2.45) and  $\hat{G}^{+\frac{1}{2}a}$  (2.49):

$$\begin{aligned} \hat{D}^\alpha &\equiv (p^+)^{-1}(\sigma^b v^{+\frac{1}{2}})^\alpha D_b^{+\frac{1}{2}} + (p^+)^{-1}(\not{p}\sigma^+ \sigma^b v^{-\frac{1}{2}})^\alpha \hat{G}_b^{+\frac{1}{2}} \\ &\equiv D^\alpha + (p^+)^{-\frac{1}{2}}(\not{p}\sigma^+ \sigma^a v^{-\frac{1}{2}})^\alpha \Psi_a \end{aligned} \quad (2.51)$$

The introduction of the auxiliary fermionic variables  $\Psi^a$  (2.50) into (2.51) necessitates the simultaneous modification of the harmonic first

\* In some special cases [34] the Faddeev-Shatashvili trick works even when separation is not possible. However, the auxiliary vielbein-like variables  $u, v$  are always necessary. It turns out that they are related with certain twistor-like  $D=10$  objects [34]

\*\* The  $\Psi$ 's are related to the grassman-odd components of certain supertwistors [34]

class constraints (2.24)- (2.26):

$$D^{ab} \rightarrow \hat{D}^{ab} = D^{ab} + \tilde{R}^{ab}, \quad (2.52)$$

$$D^{-a} \rightarrow \hat{D}^{-a} = D^{-a} - \frac{p_b}{p^+} \tilde{R}^{ab}, \quad (2.53)$$

with  $D^{-+}, D^{+a}$  remaining the same, where:

$$\tilde{R}^{ab} \equiv \frac{1}{2} (\tilde{S}^{ab})_{cd} \Psi^c \Psi^d \quad (2.54)$$

$$(\tilde{S}^{ab})_{cd} \equiv \frac{1}{2} (\tilde{\gamma}^{ab})_{cd} \equiv \frac{1}{2} v^{-\frac{1}{2}} \sigma_c \sigma^{ab} \sigma^+ \sigma_d v^{-\frac{1}{2}} \quad (2.55)$$

The  $8 \times 8$  matrices  $\tilde{S}^{ab}$  (2.55) are precisely the D=10 Lorentz-invariant generators of the harmonic  $SO(8)$  (c)-spinor representation (see Appendix B).

Also one can easily check, using the explicit expression (2.54) and the anticommutation (upon quantization) relations (2.50), that:

$$[\tilde{R}^{ab}, \tilde{R}^{cd}] = C^{bc} \tilde{R}^{ad} - C^{ac} \tilde{R}^{bd} + C^{ad} \tilde{R}^{bc} - C^{bd} \tilde{R}^{ac} \quad (2.56)$$

$$[D^{ab}, \tilde{R}^{cd}] = 0 \quad (2.57)$$

Thus, both parts  $D^{ab}$  (2.24) and  $\tilde{R}^{ab}$  (2.54) in the modified first-class constraint  $\hat{D}^{ab}$  (2.52) (generating once again the  $SO(8)$  algebra (2.27) under commutation) may be interpreted as harmonic "orbital" and harmonic "spin"  $SO(8)$  rotations respectively. The implications of these will be elaborated upon in Section 3 (see also Appendix C).

The modifications (2.52), (2.53) are needed in order to preserve the first-class property of the new system of covariant and irreducible constraints:

$$p^2, \hat{D}^\alpha(2.51), \hat{D}^{ab}(2.52), D^{-+}, D^{+a}, \hat{D}^{-a}(2.53), \quad (2.58)$$

$$\{\hat{D}^\alpha, \hat{D}^\beta\}_{PB} = [i \frac{(\sigma^+)^{\alpha\beta}}{p^+}] p^2 \quad (2.59)$$

$$\{\hat{D}^{-a}, \hat{D}^{-b}\}_{PB} = [i \frac{1}{(p^+)^2} \tilde{\kappa}^{-b}] p^2 \quad (2.60)$$

$$\{\hat{D}^{-a}, \hat{D}^\beta\}_{PB} = [-\frac{i}{2} (p^+)^{-\frac{1}{2}} (\sigma^+ \sigma^{ab} v^{-\frac{1}{2}})^\alpha \Psi_b] p^2 \quad (2.61)$$

$$i\{D^{+a}, \hat{D}^{-b}\}_{PB} = C^{ab} D^{-+} + \hat{D}^{ab} \quad (2.62)$$

all remaining PB relations being unaltered.

Thus we arrive at the harmonic BS action [14]:

$$\hat{S}_{superparticle} = \hat{S}_{BS} + \hat{S}_{harmonic} \quad (2.63)$$

$$\hat{S}_{BS} = \int d\tau [p_\mu \partial_\tau x^\mu + p_\theta^\alpha \partial_\tau \theta_\alpha + i \Psi^a \partial_\tau \Psi_a - \Lambda p^2 - \Lambda_a \hat{D}^a] \quad (2.64)$$

$$\begin{aligned} \hat{S}_{harmonic} = \int d\tau [ & p_{u_a}^\mu \partial_\tau u_\mu^a + p_v^{-\frac{1}{2}\alpha} \partial_\tau v_\alpha^{+\frac{1}{2}} + p_v^{+\frac{1}{2}\alpha} \partial_\tau v_\alpha^{-\frac{1}{2}} \\ & - \Lambda_{ab} \hat{D}^{ab} - \Lambda^{+-} D^{-+} - \Lambda_a^- D^{+a} - \Lambda_a^+ \hat{D}^{-a}] \end{aligned} \quad (2.65)$$

The new action (2.63)-(2.65) is physically equivalent to the original BS action (2.8), however, it possesses the decisive advantage of having super-Poincare covariant and irreducible first-class constraints only (2.58)-(2.62). Thus, the super-Poincare covariant canonical quantization of the BS superparticle (either *à la* Dirac or in the BFV-BRST formalism) is now straightforward (see Sections 3 and 4).

Let us particularly stress, that all first-class constraints (2.58) and the auxiliary fermionic variables  $\Psi_a$  (2.50) are all real. Therefore, the harmonic BS-action (2.63)-(2.65) is real too. The harmonic BS and GS actions in [13-15,34] are also real. This is to be contrasted with our initial choice of a set of BFV-irreducible first-class constraints in the very first paper on the harmonic superstring program [11]. There, we have used the special constraint structure of the N=2 BS superparticle to form **holomorphic** (rather than real) first-class combinations\* out of the two available fermionic constraints  $D_A^\alpha$ ,  $A = 1, 2$ . Thereby we escaped the necessity to introduce the auxiliary fermionic variables  $\Psi_a$  [13-15] through the Faddeev-Shatashvili trick [43].

The harmonic super-string action in the hamiltonian formulation is a generalization of the above harmonic BS action (2.63)-(2.65) [13-15]:

$$\hat{S} = \hat{S}_{GS} + \hat{S}_{\text{harmonic}} \quad (2.66)$$

$$\begin{aligned} \hat{S}_{GS} = \int d\tau \int_{-\pi}^{\pi} d\xi [ & P_\mu \partial_\tau X^\mu + \sum_{A=1,2} (p_{\hat{\theta}A} \partial_\tau \theta_{A\alpha} + i\Psi_A^\alpha \partial_\tau \Psi_{A\alpha}) \\ & - \sum_{A=1,2} (\Lambda_A \hat{T}_A + \Lambda_{A\alpha} \hat{D}_A^\alpha)] \end{aligned} \quad (2.67)$$

The main characteristics of this harmonic superstring action (2.66)(2.67) are:

- 1) it contains the harmonic space variables  $v_\alpha^{\pm \frac{1}{2}}$ ,  $u_\mu^a$ ;
- 2) it contains new fermionic string variables  $\Psi_A^\alpha(\xi)$ ;

---

\* In references [12,35] the action is also not real because the constraints  $E^{+I}$  in [12] (identical with  $F_r^{I+}$  in [35]) are holomorphic and their complex conjugates  $E^{-I}$  were not included in the action. However, this is not a drawback of refs. [11,12,35] since it is perfectly consistent to quantize in the canonical Dirac formalism systems with holomorphic (instead of real) first-class constraints (see [50] and appendix C of [11]). The corresponding procedure is called the Gupta-Bleuler quantization.

- 3) all its constraints are first class and irreducible;
- 4) the space-time supersymmetry is realized linearly;
- 5) it possesses a larger set of gauge invariances and it reduces, in a particular gauge, to the original GS action.

The term  $\hat{S}_{\text{harmonic}}$  in (2.66) has precisely the same form as  $S_{\text{harmonic}}$  (2.39) with the constraints  $D^{ab}$ ,  $D^{-a}$  appropriately modified due to the introduction of  $\Psi_A^a(\xi)$  (cf (2.52)(2.53)). Accordingly, the new first class, independent and covariant system of constraints is more complicated [15]. The constraints generalizing the harmonic constraints (2.24)-(2.26) are:

$$D^{-+} \equiv \frac{1}{2}(v_\alpha^{+\frac{1}{2}} \frac{\partial}{\partial v_\alpha^{+\frac{1}{2}}} - v_\alpha^{-\frac{1}{2}} \frac{\partial}{\partial v_\alpha^{-\frac{1}{2}}}) \quad (2.68)$$

$$D^{+a} \equiv u_\mu^+ \frac{\partial}{\partial u_{\mu a}} + \frac{1}{2} v^{-\frac{1}{2}} \sigma^+ \sigma^a \frac{\partial}{\partial v^{-\frac{1}{2}}} \quad (2.69)$$

$$\hat{D}^{ab} \equiv D^{ab} + \sum_A \int_{-\pi}^{\pi} d\xi \tilde{R}_A^{ab}, \quad (2.70)$$

$$\begin{aligned} \hat{D}^{-a} \equiv & D^{-a} - \sum_A \int_{-\pi}^{\pi} d\xi (\Pi_A^+)^{-1} \Pi_{Ab} \tilde{R}_A^{ab} \\ & - \frac{i}{3} \sum_A (-1)^A \int_{-\pi}^{\pi} d\xi (\Pi_A^+)^{-\frac{1}{2}} \tilde{R}_A^{cd} (v^{-\frac{1}{2}} \sigma_b \sigma^a \sigma_{cd} \sigma^+ \theta'_A) \Psi_A^b, \end{aligned} \quad (2.71)$$

where (cf. (2.54)):

$$\tilde{R}_A^{ab} \equiv \frac{1}{2} (\tilde{S}^{ab})_{cd} \Psi_A^c \Psi_A^d \quad (2.72)$$

$$\{\Psi_A(\xi), \Psi_B^b(\eta)\}_{PB} = -i \delta_{AB} C^{ab} \delta(\xi - \eta) \quad (cf. (2.50)), \quad (2.73)$$

and  $\tilde{S}^{ab}$  is the same as in (2.55) (cf. appendix B). The bosonic constraints

generalizing  $p^2$  are:

$$\hat{T}_A(\xi) \equiv \Pi_A^2 - 4i(-1)^A \theta'_{A\alpha} D_A^\alpha + 2i(-1)^A \Psi_A^\alpha(\xi) \Psi'_{A\alpha}(\xi), \quad (2.74)$$

with the notation:

$$\Pi_A^\mu \equiv P^\mu + (-1)^A [X'^\mu + 2i\theta_A \sigma^\mu \theta'_A]. \quad (2.75)$$

The fermionic constraints (2.51) are generalized by:

$$\begin{aligned} \hat{D}_A^\alpha(\xi) \equiv & D_A^\alpha(\xi) - i(-1)^A (\Pi_A^+)^{-1} (\sigma^{bc} \sigma^+ \theta'_A)^\alpha \tilde{R}_{Abc} \\ & + (\Pi_A^+)^{-\frac{1}{2}} (\Pi_A \sigma^+ \sigma^b v^{-\frac{1}{2}})^\alpha \Psi_{Ab}, \end{aligned} \quad (2.76)$$

where  $D_A^\alpha$  is the mixture of first and second class constraints appearing in the original GS formulation [7]:

$$D_A^\alpha = -ip_{\theta A}^\alpha - [P^\mu + (-1)^A (X'^\mu + i\theta_A \sigma^\mu \theta'_A)] (\sigma_\mu \theta_A)^\alpha.$$

The PB algebra of these covariant, first class, BFV-irreducible constraints (2.68)-(2.71), (2.73), (2.75) is:

$$\{\hat{T}_A(\xi), \hat{T}_B(\eta)\}_{PB} = 8(-1)^A \delta_{AB} [T_A(\xi) \delta'(\xi - \eta) + \frac{1}{2} T'_A(\xi) \delta(\xi - \eta)]; \quad (2.77)$$

$$\{\hat{T}_A(\xi), \hat{D}_B^\alpha(\eta)\}_{PB} = 4(-1)^A \delta_{AB} \hat{D}_A^\alpha(\xi) \delta'(\xi - \eta); \quad (2.78)$$

$$\{\hat{D}_A^\alpha(\xi), \hat{D}_B^\beta(\eta)\}_{PB} = i\delta_{AB} \delta(\xi - \eta) (\sigma^+)^{\alpha\beta} (\Pi_A^+(\xi))^{-1} \Omega_A(\xi); \quad (2.79)$$

$$-i\{\hat{D}^{-a}, \hat{D}^{-b}\}_{PB} = \sum_A \int_{-\pi}^{\pi} d\xi (\Pi_A^+)^{-2} \tilde{R}_A^{ab} \Omega_A(\xi); \quad (2.80)$$

$$i\{\hat{D}^{ab}, \hat{D}^{-c}\}_{PB} = C^{bc}\hat{D}^{-a} - C^{ac}\hat{D}^{-b},$$

$$i\{\hat{D}^{ab}, \hat{D}^{cd}\}_{PB} = C^{bc}\hat{D}^{ad} - C^{ac}\hat{D}^{bd} + C^{ad}\hat{D}^{bc} - C^{bd}\hat{D}^{ac},$$

$$i\{\hat{D}^{ab}, D^{+c}\}_{PB} = C^{bc}D^{+a} - C^{ac}D^{+b}, \quad \{\hat{D}^{ab}, D^{-+}\}_{PB} = 0; \quad (2.81)$$

$$i\{D^{-+}, D^{+a}\}_{PB} = +D^{+a},$$

$$i\{D^{-+}, \hat{D}^{-a}\}_{PB} = -\hat{D}^{-a}; \quad (2.82)$$

$$i\{D^{+a}, \hat{D}^{-b}\}_{PB} = C^{ab}D^{-+} + \hat{D}^{ab}; \quad (2.83)$$

$$\{\hat{D}^{-a}, \hat{D}_A^\alpha(\xi)\}_{PB} = -\frac{i}{2}(\Pi_A^+(\xi))^{-\frac{1}{2}}(\sigma^+ \sigma^{ab} v^{-\frac{1}{2}})^\alpha \Psi_{Ab}(\xi) \Omega_A(\xi), \quad (2.84)$$

where the following notation is used:

$$\Omega_A \equiv \hat{T}_A + 4i(-1)^A \theta'_{A\alpha} \hat{D}_A^\alpha \quad (2.85)$$

The information of the constraint algebra (2.77)-(2.84) is encoded in the BRST charge [15]:

$$Q_{BRST} = \hat{Q}_{\text{harmonic}} + Q_{\text{string}}; \quad (2.86)$$



$$\begin{aligned}
\hat{Q}_{\text{harmonic}} = & i\eta_{ab}[\hat{D}^{ab} + \eta^{+a} \frac{\partial}{\partial \eta_b^+} - \eta^{+b} \frac{\partial}{\partial \eta_a^+}] \\
& + \eta^{-a} \frac{\partial}{\partial \eta_b^-} - \eta^{-b} \frac{\partial}{\partial \eta_a^-} + \eta_a^a \frac{\partial}{\partial \eta_{bd}} - \eta_a^b \frac{\partial}{\partial \eta_{ad}}] \\
& + i\eta^{+-}[D^{-+} + \eta_a^+ \frac{\partial}{\partial \eta_a^+} - \eta_a^- \frac{\partial}{\partial \eta_a^-}] + i\eta_a^- D^{+a} \\
& + i\eta_a^+ [\hat{D}^{-a} + \eta_a^- \frac{\partial}{\partial \eta^{+-}} - \eta^{-b} \frac{\partial}{\partial \eta^{ab}}] \\
& + \frac{1}{2} \sum_A \int_{-\pi}^{\pi} d\xi (\Pi_A^+)^{-2} (\eta^{+b} \tilde{R}^{Aab} - i(\Pi_A^+)^{\frac{1}{2}} (\chi_A \sigma^+ \sigma_{ab} v^{-\frac{1}{2}}) \Psi_A^b) \\
& \left( \frac{\delta}{\delta c_A} + 4i(-1)^A \theta'_{A\alpha} \frac{\delta}{\delta \chi_{A\alpha}} \right)
\end{aligned} \tag{2.87}$$

$$\begin{aligned}
Q_{\text{string}} = & \sum_A \int_{-\pi}^{\pi} d\xi \{ c_A [\hat{T}_A - 4i(-1)^A (c'_A \frac{\delta}{\delta c_A} + \chi'_{A\alpha} \frac{\delta}{\delta \chi_{A\alpha}})] \\
& + \chi_{A\alpha} \hat{D}_A^\alpha + (2\Pi_A^+)^{-1} (\chi_A \sigma^+ \chi_A) [\frac{\delta}{\delta c_A} + 4i(-1)^A \theta'_{A\alpha} \frac{\delta}{\delta \chi_{A\alpha}}] \}.
\end{aligned} \tag{2.88}$$

The ghosts appearing in (2.87)-(2.88) correspond to the following constraints:

$$\left[ \begin{array}{cc} \text{ghost} & \text{constraint} \\ c_A(\xi) & \hat{T}_A(\xi) \\ \chi_{A\alpha}(\xi) & \hat{D}_A^\alpha(\xi) \\ \eta_{ab} & \hat{D}^{ab} \\ \eta^{+-} & D^{-+} \\ \eta^{-a} & D^{+a} \\ \eta^{+a} & \hat{D}^{-a} \end{array} \right] \tag{2.89}$$

In [15] it was shown that using the harmonic superstring formulation in the dual model framework, one obtains covariant vertices for the

emission of massless states which represent the D=10 SYM multiplet:

$$V_B(\zeta; k) = \zeta_a(k) g_B^{ab}(\mathcal{P}^+; \Psi) [\mathcal{P}_b - \frac{k_b}{k^+} \mathcal{P}^+] e^{ikX}; \quad (2.90)$$

$$V_F(F; k) = -i(\mathcal{P}^+)^{-\frac{1}{2}} (\Psi_d(\gamma_a)^{cd} F_c^{+\frac{1}{2}}) g_F^{ab}(\mathcal{P}^+; \Psi) [\mathcal{P}_b - \frac{k_b}{k^+} \mathcal{P}^+] e^{ikX}, \quad (2.91)$$

with the notations:

$$\mathcal{P}^\mu(\xi) = P^\mu(\xi) - X'^\mu(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{in\xi}, \quad (2.92)$$

$$g_B = \cosh(M^{\frac{1}{2}}),$$

$$g_F = (M^{-\frac{1}{2}}) \sinh(M^{\frac{1}{2}}); \quad (2.93)$$

where  $M = (M^{\mu\nu})$  is a matrix with elements:

$$M^{ab} = 2k^+(\mathcal{P}^+)^{-1} \tilde{R}^{ab},$$

$$M^{a-} = 2(\mathcal{P}^+)^{-1} \tilde{R}^{ac} k_c = -M^{-a}, \quad (2.94)$$

Using these vertices we have shown that one recovers in a covariant way the four-point scattering amplitudes:

$$A_4 = \text{Kinematical factor } (\zeta_1, k_1; \zeta_2, k_2; \zeta_3, k_3; \zeta_4, k_4)_{\text{harmonic scalar}} \\ \times \frac{\Gamma(-\frac{s}{2})\Gamma(-\frac{t}{2})}{\Gamma(1 - \frac{s}{2} - \frac{t}{2})}. \quad (2.95)$$

These vertices were computed in a Lorentz-covariant gauge in which the supersymmetry is explicit but partly nonlinear. Loop computations and

vertices with linearly realized supersymmetry will be presented elsewhere. In the sequel it is the field-theory formalism which will occupy our attention.

We conclude this section with the following remarks concerning the extension [35] of the harmonic superstring program [10-15] to the Lagrangian formalism. This important development might allow for the application of the powerful methods of the two dimensional conformal field theory to the covariantly quantized GS superstring.

Since in the present formalism the harmonic variables  $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$  do not depend on the string world-sheet coordinate  $\xi$ , the action  $\hat{S}_{harmonic}$  in (2.66) does not possess manifest reparametrization invariance.

However, as already explained in refs. [10-12], the harmonics  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a$ , whose dynamics is described by the action  $S_{harmonic}$  (2.39), are pure-gauge degrees of freedom and, therefore, their independence on the world-sheet parameter  $\xi$  does not spoil the reparametrization invariance of the physical superstring dynamics described by (2.67). In fact, in the hamiltonian framework (in which we always work) the reparametrization invariance is accounted for by the presence of the first-class constraints  $\hat{T}_A(\xi)$  (2.74), satisfying the correct Virasoro algebra (2.77). Therefore, there is no breaking of reparametrization invariance in the present canonical hamiltonian formalism.

Moreover, as stressed in [15,34] nothing prevents us from taking the harmonic auxiliary variables  $v, u$  to depend also on  $\xi$  by a straightforward generalization of (2.39), (2.68)-(2.71). In the latter case, however, the expressions for the modified superstring constraints  $\hat{T}_A$  (2.74),  $\hat{D}_A^a$  (2.76) and the PB algebra (2.77)-(2.84) become more complicated.

Actually, if we delete the fermionic string variables  $\Psi_A^a$  from the harmonic GS action (2.66), (2.67) and work instead (as in our original ref. [10]) with the covariantly disentangled first class

$$D_A^{+\frac{1}{2}a} \equiv v^{+\frac{1}{2}} \sigma^a \Pi_A D_A$$

and second class

$$G_A^{+\frac{1}{2}a} \equiv \frac{1}{2} v^{-\frac{1}{2}} \sigma^a \sigma^+ D_A$$

constraints, then it is possible to rewrite  $S_{harmonic}$  (2.39) in a manifestly reparametrization invariant form by promoting  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a$  to depend also on  $\xi$  [35].

The set of auxiliary variables used in ref. [35] exactly corresponds to the harmonic variables (2.21) introduced in [10-15] while the constraints in [35] are identical to a subset of the harmonic constraints in [12]\*.

Actually, using the auxiliary variables  $v_{A\alpha}, \tilde{v}_A^\alpha$  introduced in [37], one can construct a simpler manifestly reparametrization invariant harmonic GS action. Here the symbols

$$A \equiv (a, \dot{a}) \quad , \quad B \equiv (b, \dot{b}) \quad , \quad a, b, \dot{a}, \dot{b} = 1, \dots, 8$$

label pairs of Lorentz-invariant internal  $SO(8)$  (s) and (c) -spinor indices. The explicit form of  $S_{auxiliary}$  entering the modified GS action

$$\hat{S}_{GS} = S_{GS} \text{ (eq. (2.3))} + S_{auxiliary} \quad (2.96)$$

\* The harmonic variables used by Kallosh and Rahmanov in [35]  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^k, u_\mu^{\dot{k}}, (k, \dot{k} = 1, \dots, 4)$  correspond to the harmonic variables  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a, w_\alpha^k, w_\alpha^{\dot{k}}, (k, \dot{k} = 1, \dots, 4)$  of ref. [12] through the relation  $u_\mu^k = w_\alpha^k u_\mu^a$ ,  $u_\mu^{\dot{k}} = w_\alpha^{\dot{k}} u_\mu^a$ . The sets of harmonic constraints  $\{H\}, \{F\}, \{K\}$  in [35] correspond to  $\{D^{-(eq.(2.25)above)}, E^{IJ}, E^{+-}\}, \{D^{+a}(eq.(2.26)), E^{+I}\}, \{(\frac{1}{2}v^{+\frac{1}{2}}\sigma^{ab}\frac{\partial}{\partial v^{+\frac{1}{2}}}) \text{ part of } D^{ab}(eq.(2.24))\}$  of [12]. Here  $E^{IJ}, E^{+-}, E^{+I}$  are the "second generation" harmonic constraints involving  $w_\alpha^k, w_\alpha^{\dot{k}}$  which helped us in [12] to reduce covariantly  $SO(8)$  to  $SU(4) \times U(1)$ .

reads:

$$S_{\text{auxiliary}} = \int d\tau d\xi \sqrt{-g} [(\tilde{p}_A^\alpha)_z \partial_z v_\alpha^A + (p_\alpha^A)_z \partial_z \tilde{v}_A^\alpha - \mu_{z\bar{z}}^{AB} \Psi_{AB} - \lambda_{\bar{z}}^{AB} (\mathcal{D}_{AB})_z] \quad (2.97)$$

where  $\Psi_{AB}$  and  $\mathcal{D}_{AB}$  [37]:

$$\Psi_{AB}(\tau, \xi) \equiv v_{A\alpha}(\tau, \xi) \tilde{v}_B^\alpha(\tau, \xi) - \delta_{AB} = 0 \quad (2.98)$$

$$(\mathcal{D}_{AB})_z \equiv v_{A\alpha}(\tilde{p}_B^\alpha)_z - \tilde{v}_A^\alpha (p_{B\alpha})_z \quad (2.99)$$

are  $2 \times 256$  Lorentz-covariant and functionally independent Dirac first-class constraints, responsible for the pure-gauge nature of the  $2 \times 256$  auxiliary variables  $v_{A\alpha}, \tilde{v}_A^\alpha$ .

In (2.97)(2.99) $_z, \bar{z}$  denote (anti)self-dual world-sheet indices defined through the 2 dimensional world-sheet (anti)self-duality projectors ( e.g. [1]):

$$P_+^{nm} \equiv \frac{1}{2}(g^{mn} + i \frac{\epsilon^{nm}}{\sqrt{-g}}) = e_z^n e_{\bar{z}}^m$$

$$P_-^{nm} \equiv \frac{1}{2}(g^{mn} - i \frac{\epsilon^{nm}}{\sqrt{-g}}) = e_{\bar{z}}^n e_z^m$$

where  $e_{z, \bar{z}}^n$  are world-sheet zweibeins corresponding to the world-sheet metric  $g_{mn}$ . Then:

$$\begin{aligned} (p_A^\alpha)_z &= e_z^n (p_A^\alpha)_n, & e_{\bar{z}}^n (p_A^\alpha)_n &= 0 \\ \lambda_{\bar{z}}^{AB} &= e_{\bar{z}}^n \lambda_n^{AB}, & e_z^n \lambda_n^{AB} &= 0, \end{aligned}$$

and similarly for  $\mu_{z, \bar{z}}$ .

With the help of the auxiliary dynamical variables  $(v_\alpha^A, \tilde{v}_A^\alpha)$ , we can now express the fermionic  $\kappa$ -gauge invariance of  $\hat{S}$  (2.96) in a Lorentz-covariant and irreducible way:

$$\delta_\kappa \theta_\alpha = i(\mathbb{I}_{\bar{z}})_{\alpha\beta} \tilde{v}_a^\beta \kappa_z^a \quad (2.100)$$

where the gauge parameter  $\kappa_z^a$  has only 8 (and therefore - independent) Lorentz invariant components\*.

One can continue covariantly the quantization procedure in the Lagrangian formalism by imposing covariant gauge-fixing condition for the irreducible  $\kappa$ -gauge symmetry (2.94) (cf. [45]):

$$\chi_\alpha \equiv \tilde{v}_a^\alpha \theta_\alpha = 0 \quad (2.101)$$

The corresponding gauge fixing in the hamiltonian formalism was used in [15] in the process of constructing the covariant vertices (2.90), (2.91).

By further imposing the gauge conditions [35]:

$$\lambda_{\bar{z}}^{AB} = \mu_{z\bar{z}}^{AB} = 0$$

one may obtain a gauge fixed action of the form:

$$\hat{S}_{GS}|_{gauge\ fixed} = S_{GS}|_{\theta\theta=0} + S_{auxiliary}|_{\mu=\lambda=0} + (\text{ghost terms}) \quad (2.102)$$

Let us emphasize that (2.102) is not manifestly super-Poincare invariant since the gauge fixing condition (2.101) apparently breaks half of the space-time SUSY. Consequently, the supersymmetry algebra becomes nonlinearly realized as in the non-covariant light-cone formalism (see e.g. [1]).

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\* Recall  $a = 1, \dots, 8$  which is the correct number of independent  $\kappa$ -gauge symmetries (cf. [9]).

Constructing systems as (2.96)-(2.99) or [35] in which the number of new constraints equals the number of new auxiliary variables is not difficult [37,34] and it is esthetically appealing\* but it is not a necessary condition for the consistency of the model. This was already shown in [11] for our case and it is well known in general from the harmonic superspace approach [31,46]. Namely, the "missing" gauge symmetries are substituted in the harmonic superspace approach by the requirement for specific dependence of the superfields on the auxiliary variables (2.28)-(2.35). For additional details, see the appendix C.

The Lagrangian formulations of the type (2.96) are useful if one wants to quantize covariantly the GS superstring within the Lagrangian functional-integral approach [47]. However, for our main objective: an explicitly space-time supersymmetric superstring quantum field theory, it is preferable to use (as we do in the present work) the hamiltonian formalism and a set of variables which are **strongly** confined on the harmonic constraint shell (2.21).

Moreover using at this stage the Lagrangian formalism might prove treacherous since the gauge invariances of the GS superstring, even after their covariant disentangling with the help of the auxiliary variables, still do not form a Lie group. The corresponding Lagrangian formalism for constrained systems with an open constraint algebra [48,49] is not guaranteed to be unitary (as explained in detail in [49]), while in the hamiltonian BFV formalism unitarity is guaranteed by well established theorems [22].

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\* Using such systems, in which the auxiliary variables are **not strongly** constrained by equations of the type (2.21),(2.40), one obtained interesting relations between supersymmetric particles, twistors and higher N SYM in 4 dimensions [34]

### 3. Super-Poincare Covariant Quantization *a la* Dirac of the BS superparticle

Before entering in the details of the construction of our gauge covariant and manifestly super-Poincare covariant field theory for D=10 SYM, we discuss the first quantized theory of the zero-mode (point-particle) limit of the GS superstring, i.e. the (N=1) BS superparticle.

In the present section we describe the super-Poincare covariant first quantization of the latter in the Dirac canonical formalism.

The resulting first quantized system will consist in a overdetermined set of linear Dirac constraint equations which are independent and in involution. This is insured by the existence of a super-Poincare covariant nilpotent BRST charge  $Q_0$  of the BS superparticle - the point particle limit of  $Q_{BRST}$  (2.86-2.89).

In section 5, we will establish the equivalence between our Dirac system and the free D=10 SYM superfield equations (the linearized Nilsson constraint equations).

To first quantize the system means to promote the classical variables to quantum operators and endow them with an (anti-)commutation algebra determined by the Poisson algebra of the respective classical variables.

One then finds a linear space which supports a faithful and irreducible representation of this algebra. The elements of this space are interpreted as the states of the quantum system.

The physical states of the D=10 N=1 BS superparticle are the ones which fulfill the Dirac constraint equations (as it will be explained below, they are matrix equations for a vector-valued  $\phi$  in our representation space)

$$p^2 \phi = 0 \tag{3.1}$$



$$\hat{D}^a \phi \equiv [D^a + \frac{1}{\sqrt{p^+}}(\not{p}\sigma^+ \sigma^b v^{-\frac{1}{2}})\Psi_b]\phi = 0 \quad (3.2)$$

$$\begin{aligned} \hat{D}^{ab} \phi \equiv & [u_\mu^a \frac{\partial}{\partial u_{\mu b}} - u_\mu^b \frac{\partial}{\partial u_{\mu a}} \\ & + \frac{1}{2}(v^{+\frac{1}{2}} \sigma^{ab} \frac{\partial}{\partial v^{+\frac{1}{2}}} + v^{-\frac{1}{2}} \sigma^{ab} \frac{\partial}{\partial v^{-\frac{1}{2}}}) + \tilde{R}^{ab}] \phi = 0, \end{aligned} \quad (3.3)$$

$$D^{-+} \phi \equiv \frac{1}{2}(v_\alpha^{+\frac{1}{2}} \frac{\partial}{\partial v_\alpha^{+\frac{1}{2}}} - v_\alpha^{-\frac{1}{2}} \frac{\partial}{\partial v_\alpha^{-\frac{1}{2}}}) \phi = 0 \quad (3.4)$$

$$D^{+a} \phi \equiv (u_\mu^+ \frac{\partial}{\partial u_{\mu a}} + \frac{1}{2} v^{-\frac{1}{2}} \sigma^+ \sigma^a \frac{\partial}{\partial v^{-\frac{1}{2}}}) \phi = 0 \quad (3.5)$$

$$\hat{D}^{-a} \phi \equiv [(u_\mu^- \frac{\partial}{\partial u_{\mu a}} + \frac{1}{2} v^{+\frac{1}{2}} \sigma^- \sigma^a \frac{\partial}{\partial v^{+\frac{1}{2}}}) - (p^+)^{-1} p_b \tilde{R}^{ab}] \phi = 0, \quad (3.6)$$

where:

$$\tilde{R}^{ab} \equiv \frac{1}{4}(\tilde{\gamma}^{ab})_{cd} \Psi^c \Psi^d \quad (3.7)$$

and the linear operators of the left hand side of (3.1)-(3.6) are the quantized first class constraints (2.58).

In passing to second quantization one reinterprets the quantum states as classical fields, and the constraint equation (3.1)-(3.6) as the free field equations (recall that the superparticle hamiltonian is weakly zero).

There are two main ways to represent the quantum algebra associated with a Poisson-bracket algebra:

- 1) through matrices- in this case the states are represented by the vectors on which the matrices act.
- 2) through functions depending on a maximal set of commuting variables on which the operators corresponding to the other variables act as differentiations. Way 2) is universally applicable while way 1) is useful for grassman-odd variables.

In our case we will use a matrix representation with respect to  $\Psi^a$  and a functional representation with respect to the other variables  $z = (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}})$ . It is more convenient to use the matrix representation with respect to  $\Psi^a$  rather than the functional one because a maximal set of commuting combinations of  $\Psi^a$ 's would contain only four of them and it would require a second generation of harmonics [12] to express it in a  $SO(8)$  covariant way. In the following, we will call each vector of functions representing a quantum state a "wave function" for conciseness. Let us explicitate the matrix structure of (3.1)- (3.6) following from the matrix representation of the quantum operators corresponding to the variables  $\Psi^a$ .

The Grassman variables  $\Psi^a$  are defined in (2.64) to fulfill Poisson Brackets relations (2.50) which at the quantum level determine the anticommutation relations of the corresponding operators (which we denote also by  $\Psi^a$ ). According to these anticommutation relations, the operators  $\Psi^a$  form an 8-dimensional Clifford algebra:

$$\{\Psi^a, \Psi^b\} = C^{ab}. \quad (3.8)$$

Therefore the operators  $\Psi^a$  can be faithfully and irreducibly represented by  $16 \times 16$   $SO(8)$  Dirac  $\Gamma$ -matrices:

$$\Psi^a = \frac{1}{\sqrt{2}} \Gamma_8^a. \quad (3.9)$$

The index  $a$  of  $\Psi^a$  transforms under the  $SO(8)$  generators (3.3) according to the relation:

$$[\hat{D}^{ab}, \Psi^c] = -(\tilde{S}^{ab})_{cd} \Psi^d. \quad (3.10)$$

$$(\tilde{S}^{ab})_{cd} \equiv \frac{1}{2}(\tilde{\gamma}^{ab})_{cd} = v^{-\frac{1}{2}} \sigma_c \sigma^{ab} \sigma^+ \sigma_d v^{-\frac{1}{2}} \quad (3.11)$$

Consequently, the  $\Psi^a$  are in the harmonic (c)-spinor representation (B.8).

See the appendix B for the construction and the properties of the remarkable D=10 Lorentz-invariant harmonic  $SO(8)$  algebra. Due to the triality properties (B.16-B.17) of the harmonic  $SO(8)$  representations, the  $\Psi^a$ 's will relate states which are in the harmonic (s) representation to states in the harmonic (v) representation. One should be careful to the fact that the same index  $a$  might be carried by objects transforming under different harmonic  $SO(8)$  representations; in fact  $A^a$  and  $F^a$  below are such examples (see also appendix B). Moreover, since  $\Psi^a$  are grassman-odd, they will relate bosons to fermions. In conclusion, the  $\Psi^a$ 's are represented by the  $16 \times 16$  matrices:

$$\Psi^a = \frac{1}{\sqrt{2}} \Gamma_8^a = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}(\gamma^a)_{bc} \\ \frac{1}{\sqrt{2}}(\tilde{\gamma}^a)_{bc} & 0 \end{bmatrix} \quad (3.12)$$

where

$$\begin{aligned} (\gamma^a)_{bc} &\equiv \sqrt{2} v^{+\frac{1}{2}} \sigma_b \sigma^a \sigma_c v^{-\frac{1}{2}} \\ (\tilde{\gamma}^a)_{bc} &\equiv \sqrt{2} v^{-\frac{1}{2}} \sigma_b \sigma^a \sigma_c v^{+\frac{1}{2}} \end{aligned} \quad (3.13)$$

are the Lorentz-invariant harmonic D=8  $\sigma$ -matrices (B.1)(B.2).  $\Psi^a$  act on states of the form (recall  $z \equiv (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}})$ ):

$$\phi(z) = \begin{bmatrix} F^a(z) \\ B^a(z) \end{bmatrix}, \quad (3.14)$$

where  $F^a$  are fermions and  $B^a$  are bosons. Let us stress that the wave function  $\phi(z)$  (3.14) is real (only in this case it will describe on shell the D=10 SYM multiplet; see section 5).

The internal  $SO(8)$  rotation properties of these objects are obtained by looking of how they are acted upon by the "non-orbital"  $\tilde{R}^{ab}$  part in

$\hat{D}^{ab}$  (recall (2.56-2.57)):

$$\begin{aligned}\tilde{R}^{ab} &= \frac{1}{4}(\tilde{\gamma}^{ab})_{cd}\Psi^{[c}\Psi^{d]} = \begin{bmatrix} (\frac{1}{4}(\tilde{\gamma}^{ab})_{cd}\frac{1}{2}(\gamma^{cd}) & 0 \\ 0 & (\frac{1}{4}(\tilde{\gamma}^{ab})_{cd}\frac{1}{2}(\tilde{\gamma}^{cd}) \end{bmatrix} \\ &= \begin{bmatrix} S'^{ab} & 0 \\ 0 & V^{ab} \end{bmatrix}\end{aligned}\quad (3.15)$$

where

$$(V^{ab})_{cd} = C^{ac}C^{bd} - C^{ad}C^{bc} \quad (3.16)$$

is the harmonic SO(8) (v) representation (B.9) and  $S'^{ab}$  is a representation related to the harmonic SO(8) (s) representation  $S^{ab}$  (see (B.8)):

$$(S^{ab})_{cd} = \frac{1}{2}(\gamma^{ab})_{cd} = \frac{1}{2}v^{+\frac{1}{2}}\sigma_c\sigma^{ab}\sigma^{-}\sigma_d v^{+\frac{1}{2}} \quad (3.17)$$

through a similarity transformation  $U$ :

$$S'^{ab} = US^{ab}U^{-1} \quad (3.18)$$

$$[U]^{ab} \equiv \sqrt{2}r_c(\gamma^c)^{ab} \equiv 2(v^{+\frac{1}{2}}\sigma^a\sigma^c\sigma^b v^{-\frac{1}{2}})(v^{+\frac{1}{2}}\sigma_c v^{-\frac{1}{2}}) = C^{ab} - 4r^a r^b \quad (3.19)$$

where

$$r_c \equiv v^{+\frac{1}{2}}\sigma_c v^{-\frac{1}{2}} \quad (3.20)$$

Note that

$$U^{-1} = U = U^T.$$

Consequently, we can now write the Dirac constraint equations (3.1)-(3.6) defining the physical quantum states in our matrix representation.

The harmonic constraint equations are:

$$(\hat{D}^{ab}\phi)^c \equiv [(D^{ab} + \tilde{R}^{ab})\phi]^c = \begin{bmatrix} D^{ab}F^c + (S'^{ab})_d^c F^d \\ D^{ab}B^c + (V^{ab})_d^c B^d \end{bmatrix} = 0 \quad (3.21)$$

$$([D^{-+}\phi]^c = \begin{bmatrix} D^{-+}F^c \\ D^{-+}B^c \end{bmatrix} = 0 \quad (3.22)$$

$$([D^{+a}\phi]^c = \begin{bmatrix} D^{+a}F^c \\ D^{+a}B^c \end{bmatrix} = 0 \quad (3.23)$$

$$(\hat{D}^{-a}\phi)^b \equiv [(D^{-a} - \tilde{R}^{ac}\frac{p_c}{p^+})\phi]^b = \begin{bmatrix} D^{-a}F^b - \frac{p_c}{p^+}(S'^{ac})^b_d F^d \\ D^{-a}B^b - \frac{p_c}{p^+}(V^{ac})^b_d B^d \end{bmatrix} = 0 \quad (3.24)$$

In view of (3.18), in order to have  $F^a$  transform in the standard (s) representation and also, in order to absorb the factor  $\frac{1}{\sqrt{p^+}}$  in (3.2) it is natural to work with new superfield wave functions  $\phi'(z)$  which are obtained from  $\phi(z)$  through the following linear transformation:

$$\phi(z) \rightarrow \phi'(z) = \begin{bmatrix} \sqrt{p^+}U^{ab} & 0 \\ 0 & C^{ab} \end{bmatrix} \begin{bmatrix} F_b(z) \\ B_b(z) \end{bmatrix} = \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^a(z) \end{bmatrix} \quad (3.25)$$

In terms of  $\phi'(z)$  (3.25), the Dirac constraint equations (3.1)-(3.6) acquire the form:

$$(-\partial^2)\phi' \equiv \begin{bmatrix} (-\partial^2)Y^{+\frac{1}{2}a} \\ (-\partial^2)B^a \end{bmatrix} = 0 \quad (3.26)$$

$$\hat{D}^\alpha\phi' \equiv \begin{bmatrix} D^\alpha Y^{+\frac{1}{2}a} - i(\beta\sigma^b\sigma^a v^{+\frac{1}{2}})^\alpha B_b \\ D^\alpha B^a - \frac{1}{\beta^+}(\beta\sigma^a\sigma^b v^{+\frac{1}{2}})^\alpha Y_b^{+\frac{1}{2}} \end{bmatrix} \quad (3.27)$$

$$D^{+a}\phi \equiv \begin{bmatrix} D^{+a}Y^{+\frac{1}{2}a} \\ D^{+a}B^a \end{bmatrix} = 0 \quad (3.28)$$

$$\hat{D}^{-a}\phi \equiv \begin{bmatrix} (D^{-a} - \frac{1}{2}\frac{\partial^a}{\partial\tau})Y^{+\frac{1}{2}b} - \frac{\partial}{\partial\tau}(S^{ac})^b{}_d Y^{+\frac{1}{2}d} \\ D^{-a}B^b - \frac{\partial}{\partial\tau}(V^{ac})^b{}_d B^d \end{bmatrix} = 0 \quad (3.29)$$

$$\hat{D}^{ab}\phi \equiv \begin{bmatrix} D^{ab}Y^{+\frac{1}{2}c} + (S^{ab})^c{}_d Y^{+\frac{1}{2}d} \\ D^{ab}B^c + (V^{ab})^c{}_d B^d \end{bmatrix} = 0 \quad (3.30)$$

$$\hat{D}^{-+}\phi \equiv \begin{bmatrix} (D^{-+} - \frac{1}{2})Y^{+\frac{1}{2}a} \\ D^{-+}B^a \end{bmatrix} = 0 \quad (3.31)$$

where  $D^{ab}, D^{-+}, D^{\pm a}$  are as in (2.24)-(2.26) and  $\partial^+ \equiv u_\mu^+ \partial^\mu$ ,  $\partial^a \equiv u_\mu^a \partial^\mu$ . Henceforth, the prime on  $\phi$  will be omitted.

The constraint equations (3.30), (3.31) express the fact that the wave function  $\phi(z)$  (3.25) is a harmonic  $SO(8) \times SO(1, 1)$  invariant. This is natural generalization of the properties of the harmonic superfields belonging to the space  $\mathcal{H}_0$  defined by (2.28), (2.29).

The harmonic superfields (2.28) identically satisfied the harmonic equations (2.35) where the harmonic "spin" part  $\tilde{R}^{ab}$  (3.15) is absent, since (2.28) do not carry external overall  $SO(8) \times SO(1, 1)$  indices unlike the case of  $\phi(z)$  (3.25). Therefore, it is natural to call  $Y^{+\frac{1}{2}a}(z), B^a(z)$  **harmonic superfields with external  $SO(8) \times SO(1, 1)$  indices** (see also appendix C). In order to see their structure, one has to actually solve (3.30), (3.31) explicitly.

To achieve this, one expresses first the functions  $Y^{+\frac{1}{2}a}(z)$  and  $B^a(z)$  in terms of new functions  $Y_\alpha(z)$  and  $B^\mu(z)$  which carry external Lorentz

indices but **not** external  $SO(8) \times SO(1, 1)$  harmonic indices:

$$Y^{+\frac{1}{2}a} = (v^{+\frac{1}{2}}\sigma^a)^\alpha Y_\alpha \quad (3.32)$$

$$B^a = u_\mu^a B^\mu \quad (3.33)$$

In terms of the new functions, the equations (3.30) and (3.31) reduce to the requirement that the fields  $Y_\alpha, B^\mu$  are invariant under the orbital  $SO(8) \times SO(1, 1)$  rotations of  $v_\alpha^{\pm\frac{1}{2}}, u_\mu^a$

$$D^{-+} \begin{bmatrix} Y_\alpha \\ B^\mu \end{bmatrix} = 0 \quad (3.34)$$

$$D^{ab} \begin{bmatrix} Y_\alpha \\ B_\mu \end{bmatrix} = 0 \quad (3.35)$$

i.e.  $Y_\alpha(z), B^\mu(z)$  are general harmonic superfields (without external  $SO(8) \times SO(1, 1)$  indices) belonging to the space  $\mathcal{H}_0$  specified by (2.28),(2.29).

The representation (3.32),(3.33) is unique because the harmonic objects  $u_\mu^a$  and  $(v^{+\frac{1}{2}}\sigma^a)^\alpha$  have exactly the same "internal"  $SO(8) \times SO(1, 1)$  properties as  $B^a(z)$  and  $Y^{+\frac{1}{2}a}(z)$  (recall eqs. (3.30) (3.31)):

$$D^{ab}(v^{+\frac{1}{2}}\sigma^c)^\alpha = -(S^{ab})_d^\alpha (v^{+\frac{1}{2}}\sigma^d)^\alpha \quad (3.36)$$

$$D^{ab}u_\mu^a = -(V^{ab})_d^\mu u_\mu^d$$

and, moreover, the objects  $u_\mu^a$  and  $(v^{+\frac{1}{2}}\sigma^a)^\alpha$  are the only harmonic objects to have the property (3.36)(and the correct  $SO(1,1)$  charges).

From now on we shall work only on the space  $\mathcal{H}$  of superfield wave functions of the form given by (3.32)(3.33)

$$\phi(z) = \begin{bmatrix} (v^{+\frac{1}{2}}\sigma^a)^{\alpha} Y_{\alpha}(z) \\ u_{\mu}^a B^{\mu}(z) \end{bmatrix} \quad (3.37)$$

Since on  $\mathcal{H}$  (3.37) the Dirac constraint equations (3.30), (3.31) are fulfilled identically, i.e. realized operatorially,  $\hat{D}^{ab}$  and  $\hat{D}^{-+}$  can be dropped from among the set of Dirac constraints to be imposed on the physical states.

The remaining constraints  $(-\partial^2)$ ,  $\hat{D}^{\alpha}$ ,  $D^{+a}$ ,  $\hat{D}^{-a}$  will be imposed only "weakly" as conditions on the physical states (3.26)-(3.29). In order to analyze their implications it is useful to perform the following transformation on  $Y_{\alpha}(z)$  and  $B^{\mu}(z)$  in (3.37):

$$\begin{bmatrix} Y_{\alpha} \\ B^{\mu} \end{bmatrix} \rightarrow \begin{bmatrix} A^{\alpha} \\ A^{\mu} \end{bmatrix}$$

$$Y_{\alpha}(z) = \frac{i}{2} \partial^{+} (\sigma^{-})_{\alpha\beta} [A^{\beta}(z) + i D^{\beta} \lambda(z)] \quad (3.38)$$

$$B^{\mu}(z) = A^{\mu}(z) + \partial^{\mu} \lambda(z) \quad (3.39)$$

where

$$\lambda(z) \equiv - \int_{x^{-}}^{z^{-}} u_{\mu}^{+} A^{\mu}(x(y^{-}; u, v), \theta, u, v) dy^{-} \quad (3.40)$$

$x^{-} \equiv u_{\mu}^{-} x^{\mu}$ ,  $x^{\mu}(y^{-}; u, v) \equiv (\eta^{\mu\nu} + u^{+\mu} u^{-\nu}) x_{\nu} - u^{+\mu} y^{-}$ ,  $\partial^{+} \equiv u_{\mu}^{+} \partial^{\mu}$ .

Inserting (3.38)-(3.40) into (3.37) one can easily show that the Dirac constraint equations (3.26)-(3.29) for the covariantly quantized N=1 BS superparticle result in the linearized Nilsson constraint equations of the free D=10 N=1 SYM for  $A^{\alpha}$ ,  $A^{\mu}$  which become independent on  $(u, v)$ .



This statement, instead of being directly proved here, will arise as a simple consequence of the more general considerations in the section 5.

## 4.

#### 4. Covariant BFV-BRST First- and Second- Quantization of the BS superparticle

In this section we perform the super-Poincare covariant first-quantization of the  $N=1$   $D=10$  BS superparticle in the BFV-BRST formalism and indicate its equivalence with the canonical Dirac quantization of the preceding question. We also write down a superspace free-field action for the linearized  $D=10$  SYM in terms of unconstrained superfields yielding as equations of motion the Dirac constraint equations (3.26)-(3.29) for the superfield wave function  $\phi(z)$  of the  $N=1$  BS superparticle.

From the mathematical point of view the Dirac system (3.26)-(3.29) is an overdetermined system of 33 matrix equations ( $33 =$  number  $\mathcal{N}$  of Dirac constraints  $(-\partial^2), \hat{D}^\alpha, D^{+a}, \hat{D}^{-a}$ ) for only one vector-valued function  $\phi(z)$ . This overdetermined system is however consistent (integrable) since the linear operators  $(-\partial^2), \hat{D}^\alpha, D^{+a}, \hat{D}^{-a}$  acting on  $\phi(z)$  (3.37) form a closed algebra under (anti-)commutation (cf. (2.59)-(2.62) in the classical theory):

$$\{\hat{D}^\alpha, \hat{D}^\beta\} = -i \frac{(\sigma^+)^{\alpha\beta}}{\partial^+} (-\partial^2) \quad (4.1)$$

$$[\hat{D}^{-a}, \hat{D}^{-b}] = \frac{1}{(\partial^+)^2} \begin{bmatrix} S^{ab} & 0 \\ 0 & V^{ab} \end{bmatrix} (-\partial^2) \quad (4.2)$$

$$[\hat{D}^{-a}, \hat{D}^\alpha] = \begin{bmatrix} 0 & U\gamma_b \\ \frac{i}{2\sqrt{2}} \tilde{\gamma}_b U & 0 \end{bmatrix} \frac{i}{2\sqrt{2}} (\sigma^+ \sigma^{ab} v^{-\frac{1}{2}})^\alpha \frac{1}{\partial^+} (-\partial^2) \quad (4.3)$$

$$[\hat{D}^{+a}, \hat{D}^{-b}] = C^{ab} \hat{D}^{-+} + \hat{D}^{ab} = 0 \quad (\text{on the space } \mathcal{H} \text{ (3.37)}) \quad (4.4)$$

the rest of the commutators being identically zero. Here, once again the notations (3.13), (3.16), (3.17), (3.19) were used.

The BRST charge  $Q_0$  corresponding to the operator algebra (4.1)-(4.4) precisely coincides with the zero mode (point particle) limit of  $Q_{BRST}$  (2.86)-(2.89) of the harmonic GS superstring, where the contributions of  $\hat{D}^{ab}$  and  $D^{-+}$  are deleted (because we are working on the space  $\mathcal{H}$  (3.37) of harmonic superfields). We write  $Q_0$  in matrix form:

$$Q_0 = \begin{bmatrix} Q_0^{(YY)} C^{ab} & [Q_0^{(YB)}]^{ab} \\ [Q_0^{(BY)}]_{ab} & Q_0^{(BB)} C^{ab} \end{bmatrix} \quad (4.5)$$

$$Q_0^{(YY)} = c(-\partial^2) + \chi_\alpha D^\alpha - (2i\partial^+)^{-1}(\chi\sigma^+\chi)\frac{\partial}{\partial c} + i\eta_a^- D^{+a} + i\eta_a^+[D^{-a} - \frac{1}{2}\frac{\partial^a}{\partial^+} - \frac{\partial_b}{\partial^+} S^{ab} - \frac{1}{2}(\partial^+)^{-2}\eta_b^+ S^{ab}\frac{\partial}{\partial c}] \quad (4.6)$$

$$Q_0^{(BB)} = c(-\partial^2) + \chi_\alpha D^\alpha - (2i\partial^+)^{-1}(\chi\sigma^+\chi)\frac{\partial}{\partial c} + i\eta_a^- D^{+a} + i\eta_a^+[D^{-a} - \frac{\partial_b}{\partial^+} V^{ab} - \frac{1}{2}(\partial^+)^{-2}\eta_b^+ V^{ab}\frac{\partial}{\partial c}] \quad (4.7)$$

$$[Q_0^{(YB)}]^{ab} = -i(\chi \rho \sigma^b \sigma^a v^{+\frac{1}{2}}) + \frac{i\eta_c^+}{\sqrt{2(2\partial^+)}}(\chi\sigma^+\sigma^{cd}v^{-\frac{1}{2}})(U\gamma_d)^{ab}\frac{\partial}{\partial c} \quad (4.8)$$

$$[Q_0^{(BY)}]_{ab} = -\frac{1}{\partial^+}(\chi \rho \sigma^a \sigma^b v^{+\frac{1}{2}}) - \frac{\eta_c^+ \sqrt{2}}{(2\partial^+)^2}(\chi\sigma^+\sigma^{cd}v^{-\frac{1}{2}})(\tilde{\gamma}_d U)^{ab}\frac{\partial}{\partial c} \quad (4.9)$$

The whole information about the algebra (4.1)-(4.4) is simply encoded in the nilpotency property of  $Q_0$  (4.5)-(4.9):

$$Q_0^2 = 0 \quad (4.10)$$

In the BFV-BRST formalism  $Q_0$  (4.5)-(4.9) is a linear operator acting on the space  $\tilde{\mathcal{H}}$  of ghost-haunted harmonic superfields.  $\tilde{\mathcal{H}}$  consists of fields of

the following form (cf. (3.37)):

$$\Phi(z, \eta) \equiv \begin{bmatrix} \mathcal{Y}^{+\frac{1}{2}a}(z, \eta) \\ \mathcal{B}^a(z, \eta) \end{bmatrix} \quad (4.11)$$

with the short-hand notations:

$$\begin{aligned} z &\equiv (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}}) \\ \eta &\equiv (\eta^A) \equiv (c, \chi_\alpha, \eta^{\pm a}) \end{aligned} \quad (4.12)$$

The property that  $\Phi$  (4.11) is a  $SO(8) \times SO(1, 1)$  harmonic invariant is now expressed by the requirement that the ghost-haunted generators of  $SO(8) \times SO(1, 1)$  annihilate  $\Phi(z, \eta)$  ( these equations replace (3.30),(3.31) which were fulfilled in the space  $\mathcal{H}$  (3.37)):

$$(\hat{D}^{-+} + \eta^{+a} \frac{\partial}{\partial \eta^{+a}} - \eta^{-a} \frac{\partial}{\partial \eta^{-a}}) \begin{bmatrix} \mathcal{Y} \\ \mathcal{B} \end{bmatrix} = 0 \quad (4.13)$$

$$(\hat{D}^{ab} + \eta^{+a} \frac{\partial}{\partial \eta_b^+} - \eta^{+b} \frac{\partial}{\partial \eta_a^+} + \eta^{-a} \frac{\partial}{\partial \eta_b^-} - \eta^{-b} \frac{\partial}{\partial \eta_a^-}) \begin{bmatrix} \mathcal{Y} \\ \mathcal{B} \end{bmatrix} = 0 \quad (4.14)$$

The explicit form of  $\Phi(z, \eta)$  satisfying (4.13),(4.14) is given (in complete analogy with (3.32),(3.33)) as:

$$\mathcal{Y}^{+\frac{1}{2}a}(z, \eta) = (v^{+\frac{1}{2}\sigma^a})^\alpha \mathcal{Y}_\alpha(z, \eta) \quad (4.15)$$

$$\mathcal{B}^a(z, \eta) = u_\mu^a \mathcal{B}^\mu(z, \eta) \quad (4.16)$$

$$\mathcal{Y}_\alpha(z, \eta) = \sum_{\{\lambda\}\{\nu\}} \left( \frac{u_{\lambda_1}^+}{p^+} \right) \dots \left( \frac{u_{\lambda_n}^+}{p^+} \right) \left( \frac{u_{\nu_1}^-}{p^-} \right) \dots \left( \frac{u_{\nu_m}^-}{p^-} \right) \mathcal{Y}_{(n,m)\alpha}^{\{\lambda\}\{\nu\}}(z, \eta) \quad (4.17)$$

$$\mathcal{B}^\mu(z, \eta) = \sum_{\{\lambda\}\{\nu\}} \left( \frac{u_{\lambda_1}^+}{p^+} \right) \dots \left( \frac{u_{\lambda_n}^+}{p^+} \right) \left( \frac{u_{\nu_1}^-}{p^-} \right) \dots \left( \frac{u_{\nu_m}^-}{p^-} \right) \mathcal{B}^{\mu\{\lambda\}\{\nu\}}(z, \eta) \quad (4.18)$$

Each coefficient field in the expansions (4.17),(4.18) is an arbitrary analytic harmonic ghost-haunted superfield whose formal expansion in terms of  $v_\alpha^{\pm\frac{1}{2}}$ ,  $u_\mu^a$  now reads (cf. (2.29)):

$$\begin{aligned} \mathcal{X}(z, \eta) = & \sum_{\{\mu\}\{\nu\}\{\lambda\}} [u_{\mu_1}^{a_1} \dots u_{\mu_n}^{a_n} \eta^{+b_1} \dots \eta^{+b_m} \eta^{-c_1} \dots \eta^{-c_l}]_{SO(8) \text{ singlet}} \\ & \times u_{\nu_1}^+ \dots u_{\nu_k}^+ u_{\lambda_1}^- \dots u_{\lambda_{k+m-l}}^- \mathcal{X}^{\{\mu\}\{\nu\}\{\lambda\}}(x, \theta, c, \chi_\alpha). \end{aligned} \quad (4.19)$$

where  $\mathcal{X}$  stands for any  $\mathcal{B}^{\mu\{\lambda\}\{\nu\}}$  or  $\mathcal{Y}^{\{\lambda\}\{\nu\}}_{(n,m)\alpha}$  which appear in the right-hand-side of the expansions (4.18),(4.17).

One can now perform a transformation of  $\mathcal{Y}^{+\frac{1}{2}a}$ ,  $\mathcal{B}^a$  in complete analogy with (3.38)-(3.40) and rewrite (4.11),(4.15),(4.16) in the form:

$$\begin{aligned} \Phi(z, \eta) & \equiv \begin{bmatrix} \mathcal{Y}^{+\frac{1}{2}a}(z, \eta) \\ \mathcal{B}^a(z, \eta) \end{bmatrix} \\ & = \begin{bmatrix} \frac{i}{2} \partial^+ (v^{+\frac{1}{2}} \sigma^a \sigma^-)_{\rho} [\mathcal{A}^\beta(z, \eta) + i D^\beta \lambda(z, \eta)] \\ u_\mu^a [\mathcal{A}^\mu(z, \eta) + \partial^\mu \lambda(z, \eta)] \end{bmatrix} \end{aligned} \quad (4.20)$$

where  $\lambda(z, \eta)$  is a functional of  $\mathcal{A}^\mu(z, \eta)$  defined in complete analogy with eq. (3.40):

$$\lambda(z) \equiv - \int_{x^-} u_\mu^+ \mathcal{A}^\mu(x(y^-; u, v), \theta, u, v; \eta) dy^- \quad (4.21)$$

The original harmonic superfield  $\phi(z)$  (3.37) enters in the ghost-haunted harmonic superfield  $\Phi(z, \eta)$  (4.11) as the zeroth order term in

the ghost expansion:

$$\Phi(z, \eta) = \phi(z) + \sum_{n \geq 1} \frac{1}{n!} \eta^{A_1} \dots \eta^{A_n} \phi_{A_1 \dots A_n}(z) \quad (4.22)$$

Eq. (4.20) together with (4.22) implies:

$$\begin{aligned} \mathcal{A}^\alpha(z, \eta) &= A^\alpha(z) + \sum_{n \geq 1} \frac{1}{n!} \eta^{B_1} \dots \eta^{B_n} A_{B_1 \dots B_n}^\alpha(z) \\ \mathcal{A}^\mu(z, \eta) &= A^\mu(z) + \sum_{n \geq 1} \frac{1}{n!} \eta^{B_1} \dots \eta^{B_n} A_{B_1 \dots B_n}^\mu(z) \end{aligned} \quad (4.23)$$

where  $A^\alpha(z), A^\mu(z)$  are the D=10 SYM supergauge potentials.

In what follows it will be very useful to employ the following condensed notations for the linear generators (3.26)-(3.29) and their respective ghosts:

$$\left[ \begin{array}{c|cccc} L_A & -\partial^2 & \hat{D}^\alpha & D^{+a} & \hat{D}^{-a} \\ \hline \eta^A & c & \chi_\alpha & \eta^{-a} & \eta^{+a} \end{array} \right] \quad (4.24)$$

In terms of (4.24), the algebra (4.1)-(4.4) and the BRST charge (4.5)-(4.9) are written short-hand as:

$$[L_A, L_B] \equiv L_A L_B + (-1)^{\epsilon_A \epsilon_B + 1} L_B L_A = f_{AB}^C L_C \quad (4.25)$$

$$Q_0 = \eta^A L_A + \frac{1}{2} (-1)^{\epsilon_B} \eta^B \eta^C f_{CB}^A \frac{\partial}{\partial \eta^A} \quad (4.26)$$

In (4.25)-(4.26)  $\epsilon_A$  denote the Grassmann parity of  $L_A$ . The corresponding ghosts  $\eta^A$  have accordingly the opposite parity  $\epsilon(\eta^A) = \epsilon_A + 1$ .

The key ingredient of the canonical BFV-BRST formalism [22] is that one can rewrite the consistent overdetermined system of (matrix) Dirac constraint equations (3.26)-(3.29) for  $\phi(z)$  (3.37) as a single linear matrix equation for  $\tilde{\Phi}(z, \eta)$  (4.13):

$$Q_0 \tilde{\Phi}(z, \eta) = 0 \quad (4.27)$$

An important property of (3.60) is that it possesses a ghost-haunted gauge invariance as a consequence of the nilpotency of  $Q_0$  (4.10)\*:

$$\delta_\Lambda \tilde{\Phi}(z, \eta) = Q_0 \Lambda(z, \eta) \quad (4.28)$$

A fundamental result of the BFV-BRST quantization is the general theorem [22] about the equivalence of the BFV-BRST physical state conditions (4.27),(4.28) with the Dirac constraint equations for the physical wave function (using notations (4.22),(4.25),(4.26)):

$$L_A \phi(z) = 0, \quad A = 1, \dots, \mathcal{N} \quad (4.29)$$

Here is a brief illustration of the above general theorem. Indeed, inserting the ghost expansion of  $\tilde{\Phi}(z, \eta)$  (4.22) and the similar expansion for the gauge parameter  $\Lambda(z, \eta)$  in (4.28):

$$\Lambda(z, \eta) = \Lambda_0(z) + \sum_{n \geq 1} \frac{1}{n!} \eta^{A_1} \dots \eta^{A_n} \Lambda_{A_1 \dots A_n}(z) \quad (4.30)$$

into (4.27) and (4.28) and employing the condensed notations (4.24)-

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\* Due to nilpotency of  $Q_0$  (4.10)  $\Lambda(z, \eta)$  is defined itself only modulo transformation of the type (4.28):  $\Lambda \sim \Lambda + Q_0 \Lambda'$  for arbitrary  $\Lambda'(z, \eta)$

(4.26), one obtains:

$$L_A \phi(z) = 0, \text{ (short hand for the Dirac system (3.26) - (3.29))} \quad (4.31)$$

$$\delta_\Lambda \phi(z) = 0; \quad (4.32)$$

$$L_A \phi_B(z) + (-1)^{\epsilon_A \epsilon_B + 1} L_B \phi_A(z) - f_{AB}^C \phi_C(z) = 0 \quad (4.33)$$

$$\delta_\Lambda \phi_A(z) = L_A \Lambda_0(z); \quad (4.34)$$

$$\begin{aligned} & [(-1)^{\epsilon_A} \sum_{i=1}^n (\epsilon_{B_i} + 1) (L_A \phi_{B_1 \dots B_n}(z) \\ & - \frac{n}{2} (-1)^{(\epsilon_C + \epsilon_{B_1})} \sum_{j=2}^n (\epsilon_{B_j} + 1) f_{AB_1}^C \phi_{CB_2 \dots B_n}(z))]_{\text{antisymm}(A, B_1, \dots, B_n)} = 0 \end{aligned} \quad (4.35)$$

$$\begin{aligned} \delta_\Lambda \phi_{B_1 B_2 \dots B_n}(z) = & [(-1)^{\epsilon_{B_1}} \sum_{j=2}^n (\epsilon_{B_j} + 1) (L_{B_1} \Lambda_{B_2 \dots B_n}(z) \\ & - \frac{(n-1)}{2} (-1)^{(\epsilon_C + \epsilon_{B_2})} \sum_{k=3}^n (\epsilon_{B_k} + 1) f_{B_1 B_2}^C \Lambda_{CB_3 \dots B_n}(z))]_{\text{antisymm}(B_1, \dots, B_n)} \end{aligned} \quad (4.36)$$

for general  $n$ . Antisymmetrization in (4.35), (4.36) is defined as:

$$\mathcal{M}_{\dots AB \dots} = (-1)^{(\epsilon_A + 1)(\epsilon_B + 1)} \mathcal{M}_{\dots BA \dots} \quad (4.37)$$

Now, using (4.25) in the equivalent form:

$$[(-1)^{\epsilon_A(\epsilon_B + 1)} (L_A L_B - \frac{1}{2} f_{AB}^C L_C)]_{\text{antisymm}(AB)} = 0 \quad (4.38)$$

one can easily check that the general solutions of (4.33), (4.35) are pure-gauges ones (cf. [22]):

$$\phi_A(z) = \delta_\Lambda \phi_A(z) \text{ (eq. (4.31)) for arbitrary } \Lambda_0(z),$$

$$\phi_{B_1 \dots B_n}(z) = \delta_\Lambda \phi_{B_1 \dots B_n}(z) \text{ (eq. (4.33)) for arbitrary } \Lambda_{B_1 \dots B_{n-1}}(z), \quad (4.39)$$

whereas the zeroth order term  $\phi(z)$  in the ghost expansion (4.22) is



gauge-invariant (4.32) and satisfies the canonical system of Dirac constraint equations (4.31).

Now, after establishing the equivalence between the BFV-BRST quantization scheme (eqs. (4.27),(4.28)) and the canonical Dirac formalism (eq. (4.31)), we can write down a field theory action principle yielding the whole overdetermined set of Dirac constraint equations (4.31) as equations of motion. To this end it is sufficient to construct an action which to generate (4.27) as variation equation and to possess ghost-haunted gauge invariance under the transformation (4.28). The action, we are looking for, reads:

$$S_0 = \frac{1}{2} \int dz d\eta \hat{H} \Phi(z, \eta) Q_0 \Phi(z, \eta) \quad (4.40)$$

Here  $\hat{H}$  is a linear operator fulfilling the properties (" $T$ " denotes operator transposition)

$$\hat{H}^T = \hat{H} \quad , \quad Q_0^T \hat{H} = \hat{H} Q_0. \quad (4.41)$$

Now, (4.41) together with the nilpotency of  $Q_0$  (4.10) assure the invariance of  $S_0$  (4.40) under the gauge transformation (4.28). Taking into account the explicit expression of  $Q_0$  (4.5)-(4.9) we find the following form of  $\hat{H}$  for the case of interest - second quantized N=1 BS superparticle or, equivalently, free D=10 N=1 SYM:

$$\hat{H} = \begin{bmatrix} -\frac{1}{2}(K_1 + K_1^T) \frac{1}{\theta^+} & 0 \\ 0 & \frac{1}{2}(K_2 + K_2^T) \end{bmatrix} \quad (4.42)$$

where  $K_{1,2}$  act on the arguments of the corresponding functions  $\mathcal{Y}^{+\frac{1}{2}a}(z, \eta)$  and  $\mathcal{B}^a(z, \eta)$  from (4.11) as follows:

$$\begin{aligned} K_1 : v_\alpha^{\pm\frac{1}{2}} &\rightarrow \pm i v_\alpha^{\pm\frac{1}{2}} \\ c &\rightarrow -c \\ \eta^{\pm a} &\rightarrow -\eta^{\pm a}, \end{aligned}$$

$$\begin{aligned}
K_2 : v_\alpha^{\pm\frac{1}{2}} &\rightarrow \pm i v_\alpha^{\pm\frac{1}{2}} \\
\chi_\alpha &\rightarrow -\chi_\alpha.
\end{aligned}
\tag{4.43}$$

Thus, formula (4.40) is the superspace action for the linearized D=10 SYM in terms of unconstrained (off-shell) superfields which possesses a Witten's type [21] BFV gauge invariance (4.28).

The off-shell superspace action for the linearized D=10 SYM previously proposed in ref. [12] can be regarded as a **gauge-fixed** action with respect to (4.28).

In the next section we shall derive an appropriate nonlinear generalization of the Dirac constraint equations (3.26)-(3.29) which will be shown to be equivalent to the Nilsson constraints for D=10 SYM and thus, will provide complete on-shell description of the interacting D=10 SYM theory. In section 6, a general action principle for arbitrary consistent overdetermined systems of nonlinear field equations will be developed, which will enable us to derive the full nonlinear generalization of the superspace action (4.40) and the BFV gauge invariance (4.28) of the linearized D=10 SYM theory.

## 5. Harmonic Superfield Representation for the Nilsson SYM Constraints

As we have already discussed in section 1, the complete on-shell description of D=10 N=1 SYM theory is given by the Nilsson constraint equations [23,6,24,25]:

$$F^{\alpha\beta} \equiv \frac{1}{g} (\{\nabla^\alpha, \nabla^\beta\} - 2i \mathcal{N}^{\alpha\beta}) = 0 \quad (5.1)$$

We use the standard notations:

$$\nabla^\alpha \equiv D^\alpha + g[A^\alpha, \cdot]$$

$$\nabla^\mu \equiv \partial^\mu + ig[A^\mu, \cdot]$$

$$D^\alpha \equiv \frac{\partial}{\partial \theta_\alpha} + i \rho^{\alpha\beta} \theta_\beta$$

$$\mathcal{N}^{\alpha\beta} \equiv \nabla_\mu (\sigma^\mu)^{\alpha\beta}$$

$$F^{\alpha\mu} \equiv D^\alpha A^\mu + i\partial^\mu A^\alpha + g[A^\alpha, A^\mu],$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]. \quad (5.2)$$

The fundamental fields in the above equations are  $A^\mu(x, \theta)$  - the vector superfield gauge potential and  $A^\alpha(x, \theta)$  - the superfield Majorana-Weyl spinor gauge potential.  $g$  denotes the coupling constant.

The Bianchi identities for  $\nabla^\alpha$ ,  $\nabla^\mu$  are in fact the consistency conditions for the overdetermined nonlinear system (5.1). Multiple application of these identities yields as a consequence of (5.1) the following additional equations for  $A^\alpha, A^\mu$  [24,25]:

$$F^{\alpha\mu} - (\sigma^\mu W)^\alpha = 0 \quad (5.3)$$

$$\nabla^\alpha F^{\mu\nu} = ((\sigma^\mu \nabla^\nu - \sigma^\nu \nabla^\mu) W)^\alpha \quad (5.4)$$

$$\nabla^\alpha W_\beta = -\frac{i}{2} (\sigma^{\mu\nu})_\beta{}^\alpha F_{\mu\nu} \quad (5.5)$$

$$\nabla^\mu F_{\mu\nu} = g W_{\sigma\nu} W \quad (5.6)$$

$$\mathcal{N}W = 0 \quad (5.7)$$

where  $W_\alpha$  is a Majorana-Weyl spinor defined by (5.3).

Our aim now is to transform the nonlinear system (5.1),(5.3)-(5.7) into an **equivalent** system of nonlinear equations in terms of harmonic superfields such that the linearized form of the latter to coincide exactly with the system of Dirac constraint equations (3.26)-(3.29) for the wave function of the covariantly quantized D=10 N=1 BS superparticle. This will provide the complete proof that the covariantly quantized D=10 N=1 harmonic BS superparticle (2.63)-(2.65) describes on-shell the (linearized) D=10 SYM multiplet.

To this end we regard  $A^\alpha, A^\mu$  in (5.1),(5.3)-(5.7) as **harmonic superfields** (2.28)-(2.29), i.e. as functions on the extended superspace  $z = (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}})$  [10-15] identically satisfying

$$(D^{ab}, D^{-+}) \begin{bmatrix} A^\alpha(z) \\ A^\mu(z) \end{bmatrix} = 0 \quad (5.8)$$

In order to insure the on-shell independence of  $A^\alpha, A^\mu$  on the auxiliary

harmonic variables  $(u, v)$  we add the harmonic differential equations:

$$D^{\pm a} \begin{bmatrix} A^a(z) \\ A^\mu(z) \end{bmatrix} = 0 \quad (5.9)$$

(cf. the discussion in section 2 leading to eqs. (2.36),(2.37); the harmonic differential operators  $D^{ab}, D^{-+}, D^{\pm a}$  appearing in (5.8),(5.9) are the same as in (2.24)-(2.26)).

Now, let us consider the following nonlinear field transformation:

$$\begin{bmatrix} A^a(z) \\ A^\mu(z) \end{bmatrix} \rightarrow \Phi(z) = \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^a(z) \end{bmatrix} \quad (5.10)$$

$$Y^{+\frac{1}{2}a}(z) = \frac{i}{2}(v^{+\frac{1}{2}}\sigma^a\sigma^-)_{\alpha}\partial^{+}[\Omega^{-1}(z)A^{\alpha}(z)\Omega(z) + \frac{1}{g}\Omega^{-1}(z)D^{\alpha}\Omega(z)] \quad (5.11)$$

$$B^a(z) = u_{\mu}^a[\Omega^{-1}(z)A^{\mu}(z)\Omega(z) - \frac{i}{g}\Omega^{-1}(z)\partial^{\mu}\Omega(z)] \quad (5.12)$$

(here  $\partial^{+} \equiv u_{\mu}^{+}\partial^{\mu}$ ). The superfield  $\Omega(z)$  in (5.11),(5.12) takes values in the YM gauge group and it is a functional of  $A^{\mu}(z)$ , solving the equation  $(u_{\mu}^{+}\nabla^{\mu})\Omega = 0$ :

$$\Omega(z) = Pexp\{-ig \int_{x^{-}}^{x^{-}} u_{\mu}^{+} A^{\mu}(x(y^{-}; u, v), \theta, u, v) dy^{-}\} \quad (5.13)$$

$$x^{-} \equiv u_{\mu}^{-} x^{\mu} \quad , \quad x^{\mu}(y^{-}; u, v) \equiv (\eta^{\mu\nu} + u^{+\mu}u^{-\nu})x_{\nu} - u^{+\mu}y^{-}.$$

Now, eqs. (5.10)-(5.13) are easily recognized as the nonlinear (non-abelian) analogue of eqs. (3.37)-(3.40) related with the superfield wave function of the  $D=10$   $N=1$  BS superparticle.

Let us now derive the nonlinear equations satisfied by  $Y^{+\frac{1}{2}a}(z)$  (5.11),  $B^a(z)$  (5.12), which are implied by the (nonlinear) system (5.1), (5.3)-(5.9). First of all we get:

$$(\hat{D}^{ab}, \hat{D}^{-+}) \begin{bmatrix} Y^{+\frac{1}{2}c}(z) \\ B^c(z) \end{bmatrix} = 0 \quad (5.14)$$

where

$$\hat{D}^{ab} = D^{ab} + \begin{bmatrix} S^{ab} & 0 \\ 0 & V^{ab} \end{bmatrix}$$

$$\hat{D}^{-+} = D^{-+} + \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \quad (5.15)$$

with  $S^{ab}, V^{ab}$  the same as in (3.16),(3.17). Therefore,  $\phi(z)$  (5.10) is itself harmonic superfield with external overall  $SO(8) \times SO(1,1)$  indices belonging to the space  $\mathcal{H}$  (3.37)

Further, acting with  $D^{\pm a}$  on both sides of eqs. (5.11),(5.12) and using eqs. (5.9),(5.13),(5.3),(5.6) together with the formulae from appendix B, we obtain the following equations for  $\phi(z)$  (5.10) with explicitly separated linear and nonlinear parts:

$$D^{+a}\phi(z) = 0 \quad (5.16)$$

$$\hat{D}^{-a}\phi(z) + \begin{bmatrix} [V_2^{-a}(\phi|z)]^{(Y)} \\ [V_2^{-a}(\phi|z)]^{(B)} \end{bmatrix} = 0 \quad (5.17)$$

In (5.17), the linear operator  $\hat{D}^{-a}$  is the same as in (3.29) (i.e.  $\hat{D}^{-a}$  is the modified  $D^{-a}$  operator due to the harmonic "spin" part of  $\hat{D}^{ab}$  and the

non-zero  $SO(1,1)$  charge matrix of  $\hat{D}^{-+}$  in (5.15). The nonlinear parts in (5.17) read:

$$[V_2^{-a}(\phi|z)]^{(Y)b} \equiv ig[Y^{+\frac{1}{2}b}, \frac{1}{\partial^+} B^a] + i\frac{g}{2}\frac{1}{\partial^+}[Y^{+\frac{1}{2}b}, B^a] + ig(S^{ac})^{bd}\frac{1}{\partial^+}[Y_d^{+\frac{1}{2}}, B_c] \quad (5.18)$$

$$[V_2^{-a}(\phi|z)]^{(B)b} \equiv gC^{ab}\frac{1}{(\partial^+)^2}(\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\} - i[B_c, \partial^+ B^c]) + ig[B^b, \frac{1}{\partial^+} B^a] \quad (5.19)$$

In the course of derivation of eqs. (5.17) and below the following useful relation is used:

$$Y^{+\frac{1}{2}a}(z) = (v^{+\frac{1}{2}}\sigma^a)^\alpha \Omega^{-1}(z) W_\alpha(z) \Omega(z) \quad (5.20)$$

which is a consequence from (5.3) and (5.11).

The next step is to operate with  $D^\alpha$  on both sides of eqs. (5.10),(5.12) and use eqs. (5.3)-(5.5) and (5.13),(5.20) to obtain (disentangling again linear and nonlinear parts):

$$\hat{D}^\alpha \phi(z) + \begin{bmatrix} [V_1^\alpha(\phi|z)]^{(Y)} \\ [V_1^\alpha(\phi|z)]^{(B)} \end{bmatrix} = 0 \quad (5.21)$$

where  $\hat{D}^\alpha$  is the linear operator defined in (3.27) and

$$[V_1^\alpha(\phi|z)]^{(Y)a} \equiv 2ig(v^{+\frac{1}{2}}\sigma^b)^\alpha \left\{ \frac{1}{\partial^+} Y_b^{+\frac{1}{2}}, Y^{+\frac{1}{2}a} \right\} - \frac{g}{2}(v^{+\frac{1}{2}}\sigma^a \sigma^{bc})^\alpha [B_b, B_c] - ig(v^{+\frac{1}{2}}\sigma^a)^\alpha \frac{1}{\partial^+} (\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\} - i[B_c, \partial^+ B^c]) \quad (5.22)$$

$$[V_1^\alpha(\phi|z)]^{(B)a} \equiv 2ig(v^{+\frac{1}{2}}\sigma^b)^\alpha \frac{1}{\partial^+} \left[ \frac{1}{\partial^+} Y_b^{+\frac{1}{2}}, \partial^+ B^a \right] - ig(v^{+\frac{1}{2}}\sigma^b \sigma^a \sigma^c)^\alpha \frac{1}{\partial^+} [B_c, Y_b^{+\frac{1}{2}}] \quad (5.23)$$

Finally, from eqs. (5.7) and (5.6) and substituting eqs. (5.11),(5.12),

(5.20) we get:

$$\begin{aligned} 0 &= (v^{+\frac{1}{2}}\sigma^a)\alpha\Omega^{-1}\mathcal{N}_{\alpha\beta}(\mathcal{N}W)^\beta\Omega \\ &= (-\partial^2)Y^{+\frac{1}{2}a}(z) + [V_0(\phi|z)]^{(Y)a}; \end{aligned} \quad (5.24)$$

$$\begin{aligned} 0 &= -u_\nu^a\Omega^{-1}(\nabla_\mu F^{\mu\nu} - gW\sigma^\nu W)\Omega \\ &= (-\partial^2)B^a(z) + [V_0(\phi|z)]^{(B)a}; \end{aligned} \quad (5.25)$$

where the nonlinear parts read:

$$\begin{aligned} [V_0(\phi|z)]^{(Y)a} &\equiv -ig(\partial^b[B_b, Y^{+\frac{1}{2}a}] + [B_b, \nabla'^b Y^{+\frac{1}{2}a}] + [\partial^+ B_b, \frac{1}{\partial^+} \nabla'^b Y^{+\frac{1}{2}a}]) \\ &\quad + 2ig\partial^+ [\frac{1}{(\partial^+)^2} (\nabla'_c \partial^+ B^c - g\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\}), Y^{+\frac{1}{2}a}] \\ &\quad - 2ig[\partial^+ B_c, (S^{cd})^{ab} \frac{1}{\partial^+} \nabla'_d Y_b^{+\frac{1}{2}}] - ig[F'_{cd}, (S^{cd})^{ab} Y_b^{+\frac{1}{2}}]; \end{aligned} \quad (5.26)$$

$$\begin{aligned} [V_0(\phi|z)]^{(B)a} &\equiv \partial_b(\nabla'^a B^b) - \partial^+ \nabla'^a \frac{1}{(\partial^+)^2} (\nabla'_c \partial^+ B^c - g\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\}) \\ &\quad + ig[\frac{1}{(\partial^+)^2} (\nabla'_c \partial^+ B^c - g\{Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}}\}), \partial^+ B^a] \\ &\quad + ig[B_b, F'^{ab}] + 2g(S^{ac})_{bd} \{ \frac{1}{\partial^+} \nabla'_c Y^{+\frac{1}{2}b}, Y^{+\frac{1}{2}d} \} - g\{ \frac{1}{\partial^+} \nabla'^a Y^{+\frac{1}{2}c}, Y_c^{+\frac{1}{2}} \} \end{aligned} \quad (5.27)$$

with the following notations:

$$\nabla'^a \equiv \partial^a + ig[B^a, \cdot]$$

$$F'^{ab} \equiv \partial^a B^b - \partial^b B^a + ig[B^a, B^b]$$

and  $(S^{ab})_{cd}$  as in (3.17).

Thus, the nonlinear system (5.1), (5.3)-(5.7) of the Nilsson constraint equations and their consequences from the Bianchi identities together



with (5.8),(5.9) implying the on-shell independence of  $A^\alpha(x, \theta, u, v)$ ,  $A^\mu(x, \theta, u, v)$  on the auxiliary harmonic variables  $(u, v)$  is reduced via the nonlinear field transformation (5.10)-(5.13) to the nonlinear system (5.16),(5.17),(5.21),(5.24),(5.25) for the harmonic superfields  $\phi(z)$  (5.10):

$$D^{+a}\phi(z) = 0 \quad (5.28)$$

$$\hat{D}^{-a}\phi(z) = 0 \quad (5.29)$$

$$\hat{D}^\alpha\phi(z) + V_1^\alpha(\phi|z) = 0 \quad (5.30)$$

$$(-\partial^2)\phi(z) + V_0(\phi|z) = 0 \quad (3.31)$$

with the nonlinear parts defined in (5.18),(5.19),(5.22), (5.23),(5.26), (5.27). (Since  $\phi(z)$  (5.10) are harmonic superfields, (5.14) are identically satisfied).

Now we shall establish the inverse statement, namely, starting from the nonlinear system (5.28)-(5.31) for the harmonic superfields  $\phi(z)$  (5.10), we can exactly recover the original system (5.1),(5.3)-(5.7) in terms of the ordinary superfields  $A^\alpha(x, \theta)$ ,  $A^\mu(x, \theta)$ . To this end we consider the following nonlinear field transformation

$$\phi(z) \equiv \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^a(z) \end{bmatrix} \rightarrow \begin{bmatrix} A^\alpha(z) \\ A^\mu(z) \end{bmatrix} \quad (5.32)$$

$$A^\alpha(z) = 2i(v^{+\frac{1}{2}}\sigma^a)^\alpha\hat{\Omega}(z)(\frac{1}{\partial^+}Y_a^{+\frac{1}{2}}(z))\hat{\Omega}^{-1}(z) - \frac{1}{g}D^\alpha\hat{\Omega}(z).\hat{\Omega}^{-1}(z) \quad (5.33)$$

$$A^\mu(z) = \hat{\Omega}(z)u_\mu^a B^a(z)\hat{\Omega}^{-1}(z) - u_\mu^+ \hat{\Omega} \frac{1}{(\partial^+)^2} [\nabla_c^+ \partial^+ B^c(z) - g\{Y^{+\frac{1}{2}c}(z), Y_c^{+\frac{1}{2}}(z)\}]\hat{\Omega}^{-1}(z) + \frac{i}{g}\partial^\mu\hat{\Omega}(z).\hat{\Omega}^{-1}(z) \quad (5.34)$$

where the harmonic superfield  $\hat{\Omega}(z)$  takes values in the YM gauge group

and it is functional of  $B^a(z)$  defined by the equations:

$$D^{+a}\hat{\Omega} = 0 \quad (5.35)$$

$$\hat{\Omega}^{-1}D^{-a}\hat{\Omega} = -ig\frac{1}{\partial^+}B^a \quad (5.36)$$

Note, that eqs. (5.17) (together with (5.19)) are the integrability conditions for the overdetermined system (5.36).

From the explicit form of (5.33),(5.34) it is seen that the new fields  $A^\alpha(z)$ ,  $A^\mu(z)$  are harmonic superfields (cf. (2.28),(2.29)), i.e. the equations:

$$(D^{ab}, D^{-+}) \begin{bmatrix} A^\alpha(z) \\ A^\mu(z) \end{bmatrix} = 0 \quad (5.37)$$

are identically fulfilled.

First, applying the harmonic operators  $D^{\pm a}$  (2.26) on  $A^\alpha(z)$ ,  $A^\mu(z)$  as defined by (5.33)-(5.36) and using (5.28),(5.29) and (5.19) we easily obtain:

$$D^{\pm a} \begin{bmatrix} A^\alpha(z) \\ A^\mu(z) \end{bmatrix} = 0 \quad (5.38)$$

which together with the identically fulfilled (5.37) yields the on-shell independence of  $A^\alpha(z)$ ,  $A^\mu(z)$  on  $(u, v)$ :

$$A^\alpha = A^\alpha(x, \theta), \quad A^\mu = A^\mu(x, \theta) \quad (5.39)$$

As a second step we consider the following expression:

$$\frac{1}{g}(\{\nabla^\alpha, \nabla^\beta\} - 2i\mathcal{N}^{ab}) \equiv -\frac{1}{16}(\sigma_\mu)^{\alpha\beta}F^\mu - \frac{1}{32(5!)}(\sigma_{\mu_1\dots\mu_5})^{\alpha\beta}F^{\mu_1\dots\mu_5} \quad (5.40)$$

where the covariant derivatives  $\nabla^\alpha = D^\alpha + g[A^\alpha, \cdot]$ ,  $\nabla^\mu = \partial^\mu + ig[A^\mu, \cdot]$ ,

are defined with the supergauge potentials from (5.33),(5.34). The coefficients of the  $\sigma$ -matrix expansion in (5.40) are (cf. [11] and appendix A):

$$F^\mu \equiv (\sigma^\mu)_{\alpha\beta}(D^\alpha A^\beta + D^\beta A^\alpha + g\{A^\alpha, A^\beta\}) - 32A^\mu \quad (5.41)$$

$$F^{\mu_1 \dots \mu_5} \equiv (\sigma^{\mu_1 \dots \mu_5})_{\alpha\beta}(D^\alpha A^\beta + D^\beta A^\alpha + g\{A^\alpha, A^\beta\}) \quad (5.42)$$

with  $A^\alpha, A^\mu$  from (5.33),(5.34).

According to (5.39)  $F^\mu$  (5.41) and  $F^{\mu_1 \dots \mu_5}$  (5.42) do not depend on the auxiliary harmonic variables  $(u, v)$ , i.e.:

$$F^\mu = F^\mu(x, \theta), \quad F^{\mu_1 \dots \mu_5} = F^{\mu_1 \dots \mu_5}(x, \theta) \quad (5.43)$$

Now, using the nonlinear definitions (5.33),(5.34) for  $A^\alpha, A^\mu$  and the obvious relation

$$\{D^\alpha, D^\beta\} = 2i \beta^{ab}$$

one can easily show that

$$\begin{aligned} \hat{\Omega}^{-1}(z)(u_\mu^+ F^\mu(x, \theta))\hat{\Omega}(z) &= 0, \\ \hat{\Omega}^{-1}(z)(u_{\mu_1}^{a_1} \dots u_{\mu_4}^{a_4} u_{\mu_5}^+ F^{\mu_1 \dots \mu_5}(x, \theta))\hat{\Omega}(z) &= 0 \end{aligned} \quad (5.44)$$

Since the harmonic coefficients  $u_\mu^+, u_{\mu_1}^{a_1} \dots u_{\mu_4}^{a_4} u_{\mu_5}^+$  in (5.44) are arbitrary and since  $F^\mu, F^{\mu_1 \dots \mu_5}$  do not depend on  $(u, v)$ , (5.44) actually imply:

$$F^\mu = 0, \quad F^{\mu_1 \dots \mu_5} = 0 \quad (5.45)$$

and, therefore, inserting (5.45) into (5.40):

$$F^{\alpha\beta} \equiv \frac{1}{g}(\{\nabla^\alpha, \nabla^\beta\} - 2i \mathcal{N}^{\alpha\beta}) = 0$$

which are exactly the original Nilsson constraint equations (5.1).

As a third step we introduce a harmonic superfield  $W_\alpha(z)$  in the following way:

$$Y^{+\frac{1}{2}\alpha}(z) = (v^{+\frac{1}{2}}\sigma^\alpha)^\alpha \hat{\Omega}^{-1}(z) W_\alpha(z) \hat{\Omega}(z) \quad (5.46)$$

(i.e.  $W_\alpha(z) = \hat{\Omega}(z) Y_\alpha(z) \hat{\Omega}^{-1}(z)$  in the notations (3.32)).

Now using eqs. (5.16)-(5.18) for  $Y^{+\frac{1}{2}\alpha}(z)$ , we easily get:

$$D^{\pm\alpha} W_\alpha(z) = 0 \quad (5.47)$$

which together with the identically fulfilled  $(D^{ab}, D^{+-})W_\alpha(z) = 0$  implies, that  $W_\alpha$  does not depend on  $(u, v)$ :

$$W_\alpha = W_\alpha(x, \theta) \quad (5.48)$$

Inserting (5.46) into (5.33) we get a relation between  $A^\alpha$  and  $W_\alpha$ :

$$\hat{\Omega}^{-1}(\sigma^+ W)^\alpha \hat{\Omega} = i\partial^+(\hat{\Omega}^{-1} A^\alpha \hat{\Omega} + \frac{1}{g} \hat{\Omega}^{-1} D^\alpha \hat{\Omega}) \quad (5.49)$$

Using  $u_\mu^+ A^\mu = ig\partial^+ \hat{\Omega} \hat{\Omega}^{-1}$  (following from (5.34) by multiplying both sides with  $u_\mu^+$ ) we can rewrite (5.49) in the following form:

$$\hat{\Omega}^{-1}(z) u_\mu^+ [F^{\alpha\mu}(x, \theta) - (\sigma^\mu)^{\alpha\beta} W_\beta(x, \theta)] \hat{\Omega}(z) = 0 \quad (5.50)$$

where

$$F^{\alpha\mu} \equiv D^\alpha A^\mu + i\partial^\mu A^\alpha + g[A^\alpha, A^\mu] \text{ (cf. notations (5.2))}$$

Thus (5.50) actually imply:

$$F^{\alpha\mu}(x, \theta) - (\sigma^\mu W)^\alpha(x, \theta) = 0$$

i.e. the original ordinary superfield eq. (5.3).

Now, having established properties (5.39),(5.48), i.e. the on-shell independence on  $(u, v)$  of  $A^\alpha, A^\mu, W_\alpha$  defined in terms of  $Y^{+\frac{1}{2}a}(z), B^a(z)$  through (5.33),(5.34) and (5.46), it is straightforward to derive the following consequences of the nonlinear system (5.28)-(5.31) for  $Y^{+\frac{1}{2}a}(z), B^a(z)$ :

$$\hat{\Omega}^{-1}(z)u_\mu^+u_\nu^+[\nabla^\alpha F^{\mu\nu}(x, \theta) - ((\sigma^\mu\nabla^\nu - \sigma^\nu\nabla^\mu)W)^\alpha(x, \theta)]\hat{\Omega}(z) = 0 \quad (5.51)$$

$$\hat{\Omega}^{-1}(z)(v^{+\frac{1}{2}}\sigma^a)^\beta[\nabla^\alpha W_\beta(x, \theta) + \frac{i}{2}(\sigma^{\mu\nu})_\beta{}^\alpha F_{\mu\nu}(x, \theta)]\hat{\Omega}(z) = 0 \quad (5.52)$$

$$\hat{\Omega}^{-1}(z)u_\nu^+[\nabla_\mu F^{\mu\nu}(x, \theta) - gW\sigma^\nu W(x, \theta)]\hat{\Omega}(z) = 0 \quad (5.53)$$

$$\begin{aligned} & \hat{\Omega}^{-1}(z)\frac{1}{2}(v^{+\frac{1}{2}}\sigma^b\sigma^a)_\alpha[(\nabla W)^\alpha(x, \theta)]\hat{\Omega}(z) \\ &= \partial^+[(D^{-a} - \frac{1}{2}\frac{\partial^a}{\partial^+})Y^{+\frac{1}{2}b}(z) - \frac{\partial^c}{\partial^+}(S^{ac})^b{}_d Y^{+\frac{1}{2}d}(z)] + \partial^+[V_2^{-a}(\phi|z)]^{(Y)b} \\ &= 0(\text{eq. (5.17)}) \end{aligned} \quad (5.54)$$

Once again, since the terms in the square brackets on the left-hand-sides of (5.51)-(5.54) do not depend on the harmonic variables  $(u, v)$ , these equations imply the rest (5.4)-(5.7) of the nonlinear system for the ordinary superfields  $A^\alpha(x, \theta), A^\mu(x, \theta)$ .

This finishes the proof of the equivalence between the Nilsson constraint equations (5.1) together with their consequences from the Bianchi identities (5.3)-(5.7) in terms of ordinary superfields  $A^\alpha(x, \theta), A^\mu(x, \theta)$  and the nonlinear system (5.28)-(5.31) in terms of harmonic superfields  $Y^{+\frac{1}{2}a}(z), B^a(z)$ , where both sets of superfields are related through the nonlinear field transformation (5.10)-(5.13). Thus the system (5.28)-(5.31) provides alternatively the complete on-shell superspace description of D=10 N=1 SYM.

In particular, one immediately notices that in the linearized case ( $g = 0$ , i.e.  $V_0(\phi|z), V_1^\alpha(\phi|z), V_2^{-\alpha}(\phi|z) = 0$ ) the harmonic superfield system (5.28)-(5.31) precisely reduces to the system of Dirac constraint equations (3.26)-(3.29) for the wave function  $\phi(z)$  of the super-Poincare covariantly quantized D=10 N=1 BS superparticle.

In the next section we shall derive a superspace action in terms of unconstrained off-shell (ghost-haunted harmonic) superfields (a nonlinear generalization of the superspace action (4.40)) which will produce (5.28)-(5.31) and, therefore, (5.1) and (5.3)-(5.7) as equations of motion.

## 6. Off-Shell Superspace Action for D=10 SYM

In this section we shall review our general construction of action principle for arbitrary consistent overdetermined systems of nonlinear field equations [20] and, subsequently, shall apply it to derive a superspace action for D=10 SYM in terms of unconstrained (off-shell) superfields (cf. also [20]).

Let us consider the following general overdetermined system of  $\mathcal{N} > 1$  nonlinear equations:

$$\mathcal{L}_A(\phi|z) \equiv L_A \phi(z) + V_A(\phi|z) = 0, \quad A = 1, \dots, \mathcal{N} \quad (6.1)$$

$$V_A(\phi|z) \equiv \sum_{n \geq 0} \int dz_1 \dots dz_{n+2} V_A^{(n+2)}(z; z_1, \dots, z_{n+2}) \phi(z_1) \dots \phi(z_{n+2}) \quad (6.2)$$

In (6.1) the function  $\phi(z)$  is defined on a (graded) linear space  $\mathcal{R}$  and it takes values in another (graded) vector space  $\mathcal{U}$ , i.e. has a vector index  $\phi = (\phi^a(z))$ . Also,  $\phi(z)$  is taken to be **real**.  $L_A$  are (graded) linear operators with at most second order derivatives and are, correspondingly, matrices ( $L_A \equiv (L_A^{ab})$ ) in the vector space  $\mathcal{U}$ . Clearly,  $V_A(\phi|z) = ([V_A(\phi|z)]^a)$  are also vectors in  $\mathcal{U}$ . In the general discussion of this section the vector indices  $a, b$  will be suppressed for brevity.

Comparing (6.1) with (4.29) we see that the system (6.1) may be considered, from the point of view of second quantization as nonlinear generalization of the Dirac constraint equations for a first-quantized system with first-class Dirac constraints  $\{L_A\}$ ,  $A = 1, \dots, \mathcal{N}$ . Therefore the system (6.1) represents the nonlinear field equations of motion to be derived from an underlying field theory action which has to be a nonlinear generalization of (4.29)

The necessary conditions for consistency of the overdetermined system (6.1) are obtained by multiple application of antisymmetrized products of the linear operators  $L_B$  on  $\mathcal{L}_A(\phi|z)$  (6.1) and by requiring the result to vanish when eqs. (6.1) are fulfilled. The first consistency condition

$$L_A \mathcal{L}_B(\phi|z) + (-1)^{\epsilon_A \epsilon_B + 1} L_B \mathcal{L}_A(\phi|z) = 0$$

yields for the linear and nonlinear parts respectively:

$$[L_A, L_B] \equiv L_A L_B + (-1)^{\epsilon_A \epsilon_B + 1} L_B L_A = f_{AB}^C L_C \quad (6.3)$$

(cf. (4.25));

$$\begin{aligned} & L_A V_B(\phi|z) + (-1)^{\epsilon_A \epsilon_B + 1} L_B V_A(\phi|z) - f_{AB}^C V_C(\phi|z) \\ &= \int dz' \left[ \frac{\delta V_B(\phi|z)}{\delta \phi(z')} \mathcal{L}_A(\phi|z') + (-1)^{\epsilon_A \epsilon_B + 1} \frac{\delta V_A(\phi|z)}{\delta \phi(z')} \mathcal{L}_B(\phi|z') \right] \quad (6.4) \\ & (= 0 \text{ on the surface of equations (6.1)}) \end{aligned}$$

In (6.3), (6.4)  $f_{AB}^C$  are in general linear operators and  $\epsilon_A, \epsilon_B$  are the Grassmann parities of  $L_A, L_B$  correspondingly. In Eq. (6.4) the operators  $L_A$  act on  $V_B(\phi|z)$  defined in eq. (6.2) as on functions of  $z$ .

The next consistency condition

$$\begin{aligned} & [L_C (-1)^{\epsilon_B + \epsilon_C} L_A V_B(\phi|z)]_{\text{antisymm}(A,B,C)} \\ &= 0 \text{ on the surface of eqs. (6.1)} \end{aligned}$$

gives using (6.3), (6.4):

$$[f_{ABC}^{(2)} \overset{DE}{(-1)^{\epsilon_D} f_{AD}^G}]_{\text{antisymm}(A,B,C)} V_G(\phi|z) = 0$$



where the operator  $f_{ABC}^{(2) DE}$  is defined by:

$$f_{ABC}^{(2) DE} L_E = ((-1)^{\epsilon_D + \epsilon_B + 1} \{(-1)^{\epsilon_D \epsilon_C} [f_{AB}^D, L_C] + f_{AG}^D f_{GC}^D\})_{\text{antisymm}(ABC)} \quad (6.5)$$

and antisymmetrization means the same as in (4.37):

$$\mathcal{M}_{...BA...} = (-1)^{(\epsilon_A + 1)(\epsilon_B + 1)} \mathcal{M}_{...AB...}$$

For most interesting systems it turns out that:

$$f_{ABC}^{(2) DE} = 0 \quad (6.6)$$

Let us immediately note, that if the set of operators  $L_A$  is viewed as a first-quantized system of Dirac first class hamiltonian constraints (cf. (6.3) and (4.25)), then  $f_{ABC}^{(2) DE}$  defined by (6.5) is precisely the so called second order BVFV structure function [22]. Its vanishing (6.6) means that the corresponding hamiltonian system is first-rank, i.e. the corresponding BRST charge does not possess higher order ghost terms, as in (4.26).

Our general construction of an action principle for the system (6.1) works under the following general assumptions:

- (i) The number  $N_b$  of bosonic operators  $L_A$  in (6.1) (i.e. with  $\epsilon_A = 0$ ) has to be **odd**;
- (ii) The linear operators  $L_A$  must be functionally independent;
- (iii) Condition (6.6) holds.

Condition (iii) means that the only nontrivial consistency conditions for the system (6.1) are given by (6.3), (6.4).

From the point of view of second quantization, conditions (ii) and (iii) mean that the underlying first-quantized system of Dirac first-class constraints  $\{L_A\}$  is BVFV-irreducible and first-rank.

Since the system (6.1) comprises  $\mathcal{N} = N_b + N_f > 1$  matrix equations it is of course impossible to find an action functional  $S = S[\phi]$ , depending on  $\phi(z)$  alone, such that (6.1) would arise as equations of motion

$$\frac{\delta S}{\delta \phi(z)} = 0.$$

Our general construction of an action principle for the overdetermined system (6.1) proceeds in the following series of steps.

The **first step** is to rewrite the overdetermined set (6.1) of  $\mathcal{N}$  (matrix) equations as a single (matrix) equation in terms of a (vector valued) field  $\tilde{\Phi}(z, \eta)$  depending on auxiliary variables collectively denoted by  $\eta$ . The original field  $\phi(z)$  from (6.1) enters as:

$$\tilde{\Phi}(z, \eta) = \phi(z) + \bar{\tilde{\Phi}}(z, \eta) \quad (6.7)$$

$$\bar{\tilde{\Phi}}(z, \eta) = \sum_{n \geq 1} \frac{1}{n!} \eta^{A_1} \dots \eta^{A_n} \phi_{A_1 \dots A_n}(z)$$

To this end we take:

$$\eta = (\eta^A) = (c^i, \chi^\alpha) \quad i = 1, \dots, N_b, \quad \alpha = 1, \dots, N_f, \quad A = 1, \dots, \mathcal{N} = N_f + N_b \quad (6.8)$$

to be the ghost variables associated with  $L_A$ , i.e. having opposite Grassmann parity  $\epsilon(\eta^A) = \epsilon_A + 1$ . Since  $\phi(z)$  was taken to be real, the ghost-haunted field  $\tilde{\Phi}(z, \eta)$  is likewise real.

Then (6.7) is exactly the ghost-haunted wave function (4.22) entering the BFV-BRST quantization (4.27)-(4.28).

The new single (matrix) equation for  $\tilde{\Phi}(z, \eta)$  replacing the system (6.1) is of the following general form:

$$Q(\tilde{\Phi}|z, \eta) \equiv Q_0 \tilde{\Phi}(z, \eta) + \mathcal{V}(\tilde{\Phi}|z, \eta) = 0 \quad (6.9)$$

$$\mathcal{V}(\Phi|z, \eta) \equiv \sum_{n \geq 0} \int dz_1 d\eta_1 \dots dz_{n+2} d\eta_{n+2} \times \\ \mathcal{V}^{(n+2)}(z, \eta; z_1, \eta_1, \dots, z_{n+2}, \eta_{n+2}) \Phi(z_1, \eta_1) \dots \Phi(z_{n+2}, \eta_{n+2}) \quad (6.10)$$

The linear operator  $Q_0$  entering (6.9) is the BRST charge [22] corresponding to the algebra (6.3):

$$Q_0 = \eta^A L_A + \frac{1}{2} (-1)^{\epsilon_B} \eta^B \eta^C f_{CB}^A \frac{\partial}{\partial \eta^A} \quad (6.11)$$

and  $\mathcal{V}(\Phi, z, \eta)$  possesses the properties ( $\delta(\eta) \equiv \prod_{A=1}^N \delta(\eta^A)$ ):

$$\int d\eta \delta(\eta) \mathcal{V}(\Phi|z, \eta) = 0 \quad (6.12a)$$

$$\int d\eta \delta(\eta) \frac{\partial}{\partial \eta^A} \mathcal{V}(\Phi|z, \eta) = V_A(\phi|z) \quad (6.12b)$$

Eqs. (6.11), (6.12b) ensure that the single equation (6.9) for  $\Phi(z, \eta)$  contains the original nonlinear system (6.1):

$$0 = \int d\eta \delta(\eta) \frac{\partial}{\partial \eta^A} Q(\Phi|z, \eta) = L_A \phi(z) + V_A(\phi|z)$$

Let us point out that in each ghost integral **first** the integration over the fermionic ghosts  $c^i$  (6.8) is performed:

$$\int d\eta \mathcal{F}(\eta) = \int d\chi \left[ \int dc \mathcal{F}(c, \eta) \right] \quad (6.13)$$

$$\int d c c^{i_1} \dots c^{i_M} = \delta_{MN} \epsilon^{i_1 \dots i_N}$$

Clearly, (6.11) tells us that (6.9) is precisely the appropriate nonlinear generalization of the BFV-equation (4.27)- i.e. the BFV physical state condition.

The **second step** is to find the gauge invariance exhibited by the new single equation (6.9) such that the equations of motion implied by (6.9) for the "non-physical" part  $\tilde{\Phi}(z, \eta)$  of the ghost-haunted field  $\Phi(z, \eta)$  (6.7) should have pure-gauge solutions, whereas eqs (6.1) for the original field  $\phi(z)$  should be gauge-invariant. This gauge symmetry must yield the appropriate nonlinear generalizations of (4.28), (4.32), (4.34), (4.36).

The required gauge invariance has the form:

$$\begin{aligned} \delta_{\Lambda} \Phi(z, \eta) &= \int dz' d\eta' \Lambda(z', \eta') \frac{\delta Q(\Phi|z, \eta)}{\delta \Phi(z', \eta')} \\ &= Q_0 \Lambda(z, \eta) + \int dz' d\eta' \Lambda(z', \eta') \frac{\delta \mathcal{V}(\Phi|z, \eta)}{\delta \Phi(z', \eta')} \end{aligned} \quad (6.14)$$

and the gauge invariance of (6.9) under (6.14) implies:\*

$$\int dz' d\eta' Q(\Phi|z', \eta') \frac{\delta Q(\Phi|z, \eta)}{\delta \Phi(z', \eta')} = 0 \quad (6.15)$$

Inserting in (6.15) the expansion (6.9) for  $Q(\Phi|z', \eta')$  one gets:

$$Q_0^2 = 0$$

(i.e.  $Q_0$  is a nilpotent operator which is true by construction, see eqs. (6.11), (6.6)),

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\* In fact, due to (6.15), the gauge parameter  $\Lambda(z, \eta)$  in (6.14) is defined itself only modulo nonlinear transformations

$$\Lambda(z, \eta) \sim \Lambda(z, \eta) + \int dz' d\eta' \Lambda'(z', \eta') \frac{\delta Q(\Phi|z, \eta)}{\delta \Phi(z', \eta')}$$

and

$$Q_0 \mathcal{V}(\Phi|z, \eta) + \int dz' d\eta' [Q_0 \Phi(z', \eta') + \mathcal{V}(\Phi|z', \eta')] \frac{\delta \mathcal{V}(\Phi|z, \eta)}{\delta \Phi(z', \eta')} = 0. \quad (6.16)$$

Therefore, it is natural to call eq. (6.15) the nonlinear nilpotency condition.

Also note, that due to (6.12a), the original field  $\phi(z)$  is inert under the gauge transformation (6.14):

$$\begin{aligned} \delta_\Lambda \phi(z) &= \int d\eta \delta(\eta) \delta_\Lambda \Phi(z, \eta) \\ &= \int dz' d\eta' \Lambda(z', \eta') \frac{\delta}{\delta \Phi(z', \eta')} \left[ \int d\eta \delta(\eta) Q(\Phi|z, \eta) \right] \equiv 0 \end{aligned}$$

exactly as in the linear case (4.32).

The **third step** is to derive the the action, invariant under (6.14) and producing (6.9) as equation of motion. It is easily found to be:

$$\begin{aligned} S &= \int dz d\eta \hat{H} \Phi(z, \eta) \bar{Q}(\Phi|z, \eta) \\ &= \frac{1}{2} \int dz d\eta \hat{H} \Phi(z, \eta) Q_0(\Phi|z, \eta) + \int dz d\eta \hat{H} \Phi(z, \eta) \bar{\mathcal{V}}(\Phi|z, \eta) \end{aligned} \quad (6.17)$$

with notations explained as follows. The linear operator  $\hat{H}$  is defined to fulfill ("T"-denotes operator transposition):

$$\begin{aligned} \hat{H}^T &= \hat{H} \\ Q_0^T \hat{H} &= \hat{H} Q_0. \end{aligned} \quad (6.18)$$

A typical form of  $\hat{H}$  is

$$\hat{H} \Phi(z, \eta) = R \Phi(\rho_1 z, \rho_2 z)$$

where  $R$  is a matrix acting on the vector-valued field,  $\rho_{1,2}$  are numbers taking the values  $\pm 1, \pm i$  (cf. (4.42), (4.43)). Let us recall that, since

$\Phi(z, \eta)$  is real, the free part of the action (6.17) is bilinear (instead of hermitean) form in  $\Phi$ . The functional  $\bar{Q}(\Phi|z, \eta)$  is defined through the relation:

$$[1 + \int dz' d\eta' \Phi(z', \eta') \frac{\delta}{\delta \Phi(z', \eta')}] \bar{Q}(\Phi|z, \eta) = Q(\Phi|z, \eta) \quad (6.19)$$

which simply means:

$$\bar{Q}(\Phi|z, \eta) = \frac{1}{2} Q_0 \Phi(z, \eta) + \bar{\mathcal{V}}(\Phi|z, \eta) \quad (6.20)$$

where  $\bar{\mathcal{V}}(\Phi|z, \eta)$  is given by a series of the same form as for  $\mathcal{V}(\Phi|z, \eta)$  (6.10) with additional multiplication of each  $\mathcal{V}^{(n+2)}$  by the factor  $(n+3)^{-1}$ :

$$\bar{\mathcal{V}}(\Phi|z, \eta) = \sum_{n \geq 0} \frac{1}{n+3} \int dz_1 d\eta_1 \dots dz_{n+2} d\eta_{n+2} \times \mathcal{V}^{(n+2)}(z, \eta; z_1, \eta_1, \dots, z_{n+2}, \eta_{n+2}) \Phi(z_1, \eta_1) \dots \Phi(z_{n+2}, \eta_{n+2}) \quad (6.21)$$

Since  $\bar{Q}(\Phi|z, \eta)$  (6.20)-(6.21) enters the action functional (6.17) where one can freely symmetrize the fields  $\Phi(z, \eta)$  entering in the various terms, we immediately find that  $\bar{Q}$  (6.20) or, equivalently,  $Q$  (6.9) should satisfy the antisymmetry condition:

$$\frac{\delta \hat{H} Q(\Phi|z, \eta)}{\delta \Phi(z', \eta')} = - \frac{\delta \hat{H} Q(\Phi|z', \eta')}{\delta \Phi(z, \eta)} \quad (6.22)$$

The minus sign in (6.22) is due to the anticommutativity of the ghost measures (recall  $N_b \equiv$  number of  $c^i = \text{odd}$ )

$$\int d\chi dc \int d\chi' dc' = - \int d\chi' dc' \int d\chi dc$$

Now, it is straightforward to show that the action (6.17) is indeed invariant under the gauge transformation (6.14) provided the nonlinear nilpotency (6.15) and the antisymmetry condition (6.22) hold.

Clearly the action (6.17) is precisely the nonlinear generalization of the free BFV-BRST action (4.40).

The final step is to derive the explicit expression of  $\mathcal{V}(\Phi|z, \eta)$  (6.10) such that (6.15), (6.22) and (6.12) are satisfied. Using (6.11) and the consistency conditions (6.4) and inserting them into eq. (6.16) we find:

$$\begin{aligned} \mathcal{V}(\Phi|z, \eta) &= \eta^A V_A(\Phi(\cdot, \eta)|z) \\ &= \sum_{n \geq 0} \int dz_1 \dots dz_{n+2} \eta^A V_A^{(n+2)}(z; z_1, \dots, z_{n+2}) \Phi(z_1, \eta) \dots \Phi(z_{n+2}, \eta) \end{aligned} \quad (6.23)$$

and similarly:

$$\begin{aligned} \bar{\mathcal{V}}(\Phi|z, \eta) &= \sum_{n \geq 0} \frac{1}{n+3} \int dz_1 \dots dz_{n+2} \eta^A V_A^{(n+2)}(z; z_1, \dots, z_{n+2}) \Phi(z_1, \eta) \dots \Phi(z_{n+2}, \eta) \end{aligned} \quad (6.24)$$

where the kernels  $V_A^{(n+2)}$  are exactly the same as in (6.2).

Eqs. (6.23)(6.24) are the principal result in the present general construction since now each object  $Q_0$ (6.11),  $\bar{\mathcal{V}}$ (6.21), (and similarly  $\mathcal{V}$ (6.10)) entering the action (6.17) is explicitly expressed in terms of objects  $\{L_A\}$ ,  $\{V_A^{(n+2)}\}$  entering the original nonlinear system (6.1)(6.2).

Let us now apply our general action principle to construct an off-shell superspace action for D=10 SYM.

From the mathematical point of view, the system (5.1),(5.3)-(5.7) (the Nilsson constraints plus their consequences from the Bianchi identities for  $\nabla^\alpha$ ,  $\nabla^\mu$  (5.2)), which provides the complete on-shell description of D=10 SYM [6,24,25], is a consistent overdetermined system of nonlinear equations for the supergauge potentials  $A^\alpha(x, \theta)$ ,  $A^\mu(x, \theta)$ . However, one can easily show that it cannot be written in the form (6.1) with Lorentz-covariant and independent linear operators  $L_A$ , and, moreover, the condition (6.6) is not satisfied (i.e. the system is of higher rank).

On the other hand, it was shown in detail in section 5, that the nonlinear system (5.1),(5.3)-(5.7) is equivalent to the nonlinear system (5.28)-(5.31) in terms of the harmonic superfield

$$\Phi(z) \equiv \begin{bmatrix} Y^{+\frac{1}{2}a}(z) \\ B^a(z) \end{bmatrix}$$

which is related to  $A^\alpha, A^\mu$  from (5.1),(5.3)-(5.7) through the nonlinear field transformation (5.10)-(5.13). Therefore, the harmonic superfield representation (5.28)-(5.31) of the D=10 SYM on-shell equations (5.1),(5.3)-(5.7) is a consistent overdetermined system of nonlinear field equations fulfilling all conditions (i),(ii),(iii) above for our action principle to work. Indeed:

(i) The number of bosonic operators  $L_A: (-\partial)^2, D^{+a}, \hat{D}^{-a}$  from (5.28)-(5.31) is odd (=17);

(ii) All linear operators  $\{L_A\} \equiv \{(-\partial)^2, \hat{D}^\alpha, D^{+a}, \hat{D}^{-a}\}$  in (5.28)-(5.31) are BFV-irreducible, i.e. functionally independent;

(iii) The set of  $\{L_A\}$  is first-rank, i.e. the second order BFV structure function vanishes (6.6).

Thus our action principle (eq. (6.17)) yields the following superspace action in terms of off-shell unconstrained superfields for D=10 SYM:

$$\begin{aligned} S_{SYM} = & \frac{1}{2} \int dz d\eta \hat{H}\Phi(z, \eta) Q_0 \Phi(z, \eta) \\ & + \int dz d\eta \hat{H}\Phi(z, \eta) [cV_0(\Phi(\cdot, \eta))|z] + \chi_\alpha V_1^\alpha(\Phi(\cdot, \eta))|z + \eta_a^+ V_2^{-a}(\Phi(\cdot, \eta))|z] \end{aligned} \quad (6.25)$$

with the notations:

$$z \equiv (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}}),$$



$$dz \equiv (d^{10}x^\mu)(d^{16}\theta_\alpha)(d^{80}u_\mu^a)(d^{32}v_\alpha^{\pm\frac{1}{2}})$$

$$\prod_{a,b} \delta(u_\mu^a u^{b\mu} - C^{ab}) \prod_{a,\pm} \delta(u_\mu^a v^{\pm\frac{1}{2}} \sigma^\mu v^{\pm\frac{1}{2}}) \delta((v^{+\frac{1}{2}} \sigma_\mu v^{+\frac{1}{2}})(v^{-\frac{1}{2}} \sigma^\mu v^{-\frac{1}{2}}) + 1);$$
(6.26)

$$\eta \equiv (\eta^A) \equiv (c, \chi_\alpha, \eta^{\pm a})$$

$$d\eta \equiv dc(d^{16}\chi_\alpha)(d^8\eta^{+a})(d^8\eta^{-a});$$
(6.27)

$$\Phi(z, \eta) \equiv \begin{bmatrix} \mathcal{Y}^{+\frac{1}{2}a}(z, \eta) \\ \mathcal{B}^a(z, \eta) \end{bmatrix} \equiv \begin{bmatrix} (v^{+\frac{1}{2}} \sigma^a)^\alpha \mathcal{Y}_\alpha(z, \eta) \\ u_\mu^a \mathcal{B}^\mu(z, \eta) \end{bmatrix};$$
(6.28)

where  $\mathcal{Y}_\alpha(z, \eta)$ ,  $\mathcal{B}^\mu(z, \eta)$  are ghost haunted superfields without external  $SO(8) \times SO(1, 1)$  indices (cf. (4.11),(4.15),(4.16)) and the functionals  $V_0(\Phi(\cdot, \eta)|z)$ ,  $V_1^\alpha(\Phi(\cdot, \eta)|z)$ ,  $V_2^{-a}(\Phi(\cdot, \eta)|z)$  in the interacting part of  $S_{SYM}$  (6.25) are exactly the same as (5.26),(5.27),(5.22),(5.23),(5.18),(5.19), where the usual real harmonic superfields  $Y^{+\frac{1}{2}a}(z)$ ,  $B^a(z)$  are substituted with the corresponding real ghost-haunted harmonic superfields  $\mathcal{Y}^{+\frac{1}{2}a}(z, \eta)$ ,  $\mathcal{B}^a(z, \eta)$  (6.28).

The way the supergauge potentials  $A^\alpha(x, \theta)$ ,  $A^\mu(x, \theta)$  of  $D=10$  SYM enter in the action (6.25) is given by the following nonlinear ghost-haunted superfield transformation:

$$\begin{bmatrix} \mathcal{Y}^{+\frac{1}{2}a}(z, \eta) \\ \mathcal{B}^a(z, \eta) \end{bmatrix} \text{ (eq. (6.28))} \rightarrow \begin{bmatrix} \mathcal{A}^\alpha(z, \eta) \\ \mathcal{A}^\mu(z, \eta) \end{bmatrix}$$
(6.29)

$$\Phi(z, \eta) \equiv \begin{bmatrix} \frac{i}{2}(v^{+\frac{1}{2}} \sigma^a \sigma^-)_\alpha \partial^+ [ \Omega^{-1}(z, \eta) \mathcal{A}^\alpha(z, \eta) \Omega(z, \eta) + \frac{1}{g} \Omega^{-1}(z, \eta) D^\alpha \Omega(z, \eta) ] \\ u_\mu^a [ \Omega^{-1}(z, \eta) \mathcal{A}^\mu(z, \eta) \Omega(z, \eta) - \frac{i}{g} \Omega^{-1}(z, \eta) \partial^\mu \Omega(z, \eta) ] \end{bmatrix}$$
(6.30)

where  $\Omega(z, \eta)$  is a functional of  $\mathcal{A}^\mu(z, \eta)$  taking values in the YM gauge

group and it is defined in complete analogy with (5.13):

$$\Omega(z) = P \exp\left\{-ig \int_{y^-}^{x^-} u_{\mu}^+ \mathcal{A}^{\mu}(x(y^-; u, v), \theta, u, v; \eta) dy^-\right\} \quad (6.31)$$

$x^- \equiv u_{\mu}^- x^{\mu}$ ,  $x^{\mu}(y^-; u, v) \equiv (\eta^{\mu\nu} + u^{+\mu} u^{-\nu}) x_{\nu} - u^{+\mu} y^-$ . Thus, the zeroth order term in the ghost expansion of  $\Phi(z, \eta)$  (6.29)-(6.31) exactly coincides with the harmonic SYM superfield  $\phi(z)$  (5.10)-(5.13) and, therefore, the usual SYM supergauge potentials  $A^{\alpha}(x, \theta)$ ,  $A^{\mu}(x, \theta)$  are identified as the **harmonic**  $(u, v)$  **independent** parts of the zeroth order terms in the ghost expansions (4.23) of  $\mathcal{A}^{\alpha}(z, \eta)$ ,  $\mathcal{A}^{\mu}(z, \eta)$  from (6.30) exactly as in the linearized case (section 4).

As a final remark, let us stress that the superspace action (6.25) is also manifestly invariant under the superspace YM gauge transformation of the ghost-haunted superfields  $\mathcal{A}^{\mu}(z, \eta)$ ,  $\mathcal{A}^{\alpha}(z, \eta)$ :

$$\mathcal{A}^{\mu}(z, \eta) \rightarrow (\mathcal{A}^{\omega})^{\mu}(z, \eta) = \omega^{-1}(z, \eta) (\mathcal{A}^{\mu}(z, \eta) - \frac{i}{g} \partial^{\mu} \omega(z, \eta))$$

$$\mathcal{A}^{\alpha}(z, \eta) \rightarrow (\mathcal{A}^{\omega})^{\alpha}(z, \eta) = \omega^{-1}(z, \eta) (\mathcal{A}^{\alpha}(z, \eta) + \frac{1}{g} D^{\alpha} \omega(z, \eta)) \quad (6.32)$$

This is because the action (6.25) depends on  $\mathcal{A}^{\alpha}(z, \eta)$ ,  $\mathcal{A}^{\mu}(z, \eta)$  only through the ghost-haunted superfield expression  $\Phi(z, \eta)$  (6.30) which is itself invariant under (6.32).

Let us recapitulate the results of this section. We described here a general construction [20] of an off-shell action principle for arbitrary consistent overdetermined systems of nonlinear field equations.

The main tool is the BFV-BRST ghost formalism [22]. The action (6.17) resembles the Siegel-Zwiebach-Witten-Neveu-West [21] construction of (super)string field actions but does not involve the peculiarities

(star products, Chern-Simons forms etc.) specific to the field theory of the Ramond-Neveu-Schwarz (RNS) (super)string.

The main application presented here is the construction of a superspace action (6.25) for D=10 N=1 SYM in terms of unconstrained (off-shell) superfields. This action contains both cubic and quartic interaction terms. The corresponding superfields (6.28) depend besides on the ordinary superspace coordinates  $(x^\mu, \theta_\alpha)$  also on the auxiliary (harmonic-like) bosonic variables  $(u, v)$  (eq. (2.21)) [10-15] and on a number of BFV-BRST ghost variables  $\eta^A$  (6.27). Thus, these generalized superfields contain an infinite number of pure-gauge and auxiliary fields which are eliminated through the Witten-type nonlinear BFV gauge invariance (6.14) and through the usual superspace YM gauge invariance (6.33) of our superspace action.

Let us particularly stress that, in our formalism, the YM gauge invariance (6.33) is **not** a part of the Witten-type gauge invariance (6.14) but it is an independent symmetry of our action (6.25). This phenomenon is most easily understood in the context of the heterotic GS superstring. Already its zero-mode (point-particle) limit contains the gauge invariant SYM whereas in the RNS formalism the YM gauge invariance arises **from** the Witten's gauge invariance at the first excited string level in the NS sector.

## 7. Conclusions and outlook

The main objectives of the present paper may be summarized as follows.

(1)

We describe in a pedagogical way the main ideas and concepts in the harmonic superstring program aimed at a consistent manifestly super-Poincare covariant quantization of space-time supersymmetric strings (the GS superstrings).

The first crucial step is introduction of auxiliary harmonic variables allowing covariant disentangling of local fermionic gauge-invariances of the superstring. The next crucial step is the introduction of additional fermionic string coordinates enabling us to convert the set of mixed first- and second- class Dirac hamiltonian constraints of the GS superstring into a set of super-Poincare covariant, functionally independent (BFV-irreducible) first-class constraints only.

This is inevitable in order to preserve manifest supersymmetry (Dirac brackets due to second-class constraints would ruin the superspace geometry by causing the superstring coordinates  $x^\mu, \theta_\alpha$  not to commute among themselves).

The introduction of the auxiliary harmonic and fermionic string variables is accompanied by introduction of appropriate additional gauge invariances beyond those of the GS superstring such that the new system (called harmonic GS superstring) is physically equivalent to the original GS model. We also made contact with a more recent formulation [35] extending the harmonic superstring program from the canonical Hamiltonian formalism to the Lagrangian functional-integral quantization formalism.

(2)

The effectiveness of the harmonic superstring program was further explicitly demonstrated by providing the full first-quantization analysis of the zero-mode (point particle) limit of the GS-superstring - the  $D=10$  ( $N=1$ ) BS superparticle. The main result here is the derivation of the linearized Nilsson curvature constraints for  $D=10$  SYM **from** and establishing their equivalence to the manifestly super-Poincare covariant Dirac constraint equations for the  $D=10$   $N=1$  BS superparticle.

(3)

The preceding result was further generalized to the full nonlinear case by deriving a harmonic superfield representation of the nonlinear Nilsson constraints of  $D=10$  SYM reducing in the the linearized case to the system of Dirac constraint equations for the  $D=10$   $N=1$  BS superparticle.

(4)

We described the main steps of our construction of a covariant action principle for a very broad class of consistent overdetermined systems of nonlinear field equations. The only conditions for their structure are the following:

The linear parts of the equations are identified as a system of quantized Dirac first-class constraints belonging to an underlying particle-like (or string-like) system which are BFV-irreducible and first rank (i.e. the second and higher BFV structure functions vanish and the corresponding BRST charge does not exhibit neither higher ghost terms nor ghosts for ghosts).

In particular, a system of consistency equations on the interacting parts of the above nonlinear equations was formulated (eq. (6.4)) which allows in principle to find interacting (nonlinear) modifications of Dirac constraint equations for particle-like and string-like systems, i.e. to find the corresponding interacting field theoretic equations of motion

(5)

Our general action principle for overdetermined systems of nonlinear field equations was applied to derive a superspace action for D=10 SYM in terms of unconstrained off-shell superfields, starting from the harmonic superfield representation of the Nilsson curvature constraints for D=10 SYM. Thus, a solution was found to the long standing problem of an off-shell superspace formulation of D=10 SYM. The same formalism can be applied to D=4 N=4 SYM and similar supersymmetric gauge theories which are formulated in terms of geometrical constraints on some of the relevant curvatures.

Although the D=10 SYM action (6.25) is manifestly off-shell supersymmetric, this is at the price of having covariant nonlocal factors  $(\partial^+)^{-1}$  (recall  $\partial^+ \equiv v^{+\frac{1}{2}} \rho v^{+\frac{1}{2}}$ ). One may hope that by combining the present approach of section 6 with the formalism developed in [51] one will be able by further appropriate nonlinear field transformations of  $\Phi(z, \eta)$  (6.28) to eliminate the nonlocality  $(\partial^+)^{-1}$  factors.

The next most ambitious task is to apply the formalism presented in this paper to attack the issue of a manifestly super-Poincare covariant field theory of the GS superstrings. The main problem here will be to find solutions for the field-theoretic superstring vertices coming from the string generalization of the consistency eqs. (6.4).

## Appendix A. D=10 and D=8 Spinor Conventions

The D=10  $\gamma$  - matrices and D=10 charge conjugation matrix are taken in the following representation:

$$\Gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\dot{\alpha}}^{\beta} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}}_{\beta} & 0 \end{pmatrix}$$

$$C_{10} = \begin{pmatrix} 0 & C^{\alpha\dot{\beta}} \\ (-C)^{\dot{\alpha}\beta} & 0 \end{pmatrix}$$

$$\Gamma^{11} \equiv \Gamma^0 \Gamma^1 \dots \Gamma^9 = \begin{pmatrix} \delta_{\dot{\alpha}}^{\beta} & 0 \\ 0 & -\delta_{\dot{\alpha}}^{\beta} \end{pmatrix}$$

Indices of D=10 left- (right-) handed MW spinors  $\phi_{\alpha}, \psi_{\dot{\alpha}}$  are raised by means of  $C_{10}$  :

$$\phi^{\dot{\alpha}} = (-C)^{\dot{\alpha}\beta} \phi_{\beta}$$

$$\psi^{\alpha} = C^{\alpha\dot{\beta}} \psi_{\dot{\beta}}$$

Throughout the paper we use D=10  $\sigma$  - matrices with undotted indices only :

$$(\sigma^\mu)^{\alpha\beta} = C^{\alpha\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}}_{\beta}$$

$$(\sigma^\mu)_{\alpha\beta} = (-C)^{-1}_{\dot{\beta}\dot{\alpha}} (\sigma^\mu)^{\dot{\alpha}}_{\beta}$$

$$(\sigma^\mu)_{\alpha\gamma} (\sigma^\nu)^{\gamma\beta} + (\sigma^\nu)_{\alpha\gamma} (\sigma^\mu)^{\gamma\beta} = -2\delta_{\alpha}^{\beta} \eta^{\mu\nu}$$

$$\eta_{\mu\nu} = \text{diag}(-, +, \dots, +)$$

The standard basis in the space of the D=10  $\gamma$  - matrices is

$$\Gamma^{\mu_1 \dots \mu_n} \equiv \Gamma^{[\mu_1 \Gamma^{\mu_2} \dots \Gamma^{\mu_n}]}, \quad n = 0, 1, \dots, 10,$$

where the square brackets denote antisymmetrization with respect to the enclosed indices . These matrices have the following properties:

$$(\Gamma^{\mu_1 \dots \mu_{2r+1}} C_{10}^{-1})^T = (-1)^r \Gamma^{\mu_1 \dots \mu_{2r+1}} C_{10}^{-1} \quad (A.1)$$

$$(C_{10}^{-1} \Gamma^{\mu_1 \dots \mu_{2r}})^T = (-1)^{r+1} C_{10}^{-1} \Gamma^{\mu_1 \dots \mu_{2r}}$$

$$\Gamma^{\mu_1 \dots \mu_n} \Gamma^{11} = (-1)^{\frac{n(n-1)}{2}} \frac{1}{(10-n)!} \epsilon^{\mu_1 \dots \mu_n \nu_1 \dots \nu_{10-n}} \Gamma_{\nu_1 \dots \nu_{10-n}}$$

$$\text{tr}(\Gamma^{\mu_1 \dots \mu_n} \Gamma^{\nu_1 \dots \nu_n}) = 32 \delta^{nm} (-1)^{\frac{1}{2}[n]} \times$$

$$\times \det \begin{pmatrix} \eta^{\mu_1 \nu_1} & \dots & \eta^{\mu_1 \nu_n} \\ \dots & \dots & \dots \\ \eta^{\mu_n \nu_1} & \dots & \eta^{\mu_n \nu_n} \end{pmatrix}$$

where  $[n] = n$  for  $n = \text{even}$  and  $[n] = n - 1$  for  $n = \text{odd}$  .

Equations (A.1) imply for the  $\sigma$  matrices  $\sigma^{\mu_1 \dots \mu_n} \equiv \sigma^{[\mu_1 \dots \mu_n]}$  :

$$(\sigma_{\mu_1 \dots \mu_{2r+1}})^{\alpha\beta} = (-1)^r (\sigma_{\mu_1 \dots \mu_{2r+1}})^{\beta\alpha} ; \quad (A.2)$$

$$(\sigma^{\mu_1 \dots \mu_{2r+1}})_{\alpha\beta} = (-1)^{r+1} \frac{1}{(9-2r)!} \epsilon^{\mu_1 \dots \mu_{2r+1} \nu_1 \dots \nu_{9-2r}} (\sigma_{\nu_1 \dots \nu_{9-2r}})_{\alpha\beta} , \quad (A.3)$$

$$(\sigma^{\mu_1 \dots \mu_{2r+1}})^{\alpha\beta} = (-1)^r \frac{1}{(9-2r)!} \epsilon^{\mu_1 \dots \mu_{2r+1} \nu_1 \dots \nu_{9-2r}} (\sigma_{\nu_1 \dots \nu_{9-2r}})^{\alpha\beta} .$$

Accounting for (A.2),(A.3) and (A.1) any (anti)-symmetric matrix



$A^{sym}(A^{asym})$  on the D=10 spinor space can be decomposed as follows:

$$A_{\alpha\beta}^{sym} = \sum_{[\mu]} A_{[\mu]} \sigma_{\alpha\beta}^{[\mu]} \quad (A.4)$$

where  $[\mu] = \mu, [\mu_1 \dots \mu_5]$ ,

$$A_{\alpha\beta}^{asym} = A_{[\mu_1 \mu_2 \mu_3]} \sigma_{\alpha\beta}^{\mu_1 \mu_2 \mu_3} ,$$

$$A_{\mu} = -\frac{1}{16} A_{\alpha\beta}^{sym} \sigma_{\mu}^{\alpha\beta} ,$$

$$A_{[\mu_1 \mu_2 \mu_3]} = -\frac{1}{16(3!)} A_{\alpha\beta}^{asym} \sigma_{\mu_1 \mu_2 \mu_3}^{\alpha\beta} ,$$

$$A_{[\mu_1 \dots \mu_5]} = -\frac{1}{32(5!)} A_{\alpha\beta}^{asym} \sigma_{\mu_1 \dots \mu_5}^{\alpha\beta} .$$

Note that the coefficient  $A_{[\mu_1 \dots \mu_5]}$  is self-dual due to (A.3).

Let us also list the following useful properties of the D=10  $\sigma$  - matrices :

$$(\sigma_{\mu})^{\alpha\beta} (\sigma^{\mu})^{\gamma\delta} + (\sigma_{\mu})^{\beta\gamma} (\sigma^{\mu})^{\alpha\delta} + (\sigma_{\mu})^{\gamma\alpha} (\sigma^{\mu})^{\beta\delta} = 0 \quad (A.5)$$

$$\sigma^{\mu} \sigma^{\nu_1 \dots \nu_n} = \sigma^{\mu \nu_1 \dots \nu_n} + \sum_{k=1}^n (-1)^k \eta^{\mu \nu_k} \sigma^{\nu_1 \dots \hat{k} \dots \nu_n}$$

where  $\hat{k}$  means that the index  $\nu_k$  is missing .

For the D=8  $\gamma$ -matrices and D=8 charge conjugation matrix we use the following representation :

$$\Gamma_8^i = \begin{pmatrix} 0 & (\gamma^i)_a^b \\ (\tilde{\gamma}^i)_a^b & 0 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} C^{ab} & 0 \\ 0 & (-C)^{\dot{a}\dot{b}} \end{pmatrix}$$

$$C^{ab} = C^{ba}$$

Indices of SO(8) (s) and (c) spinors  $\phi_a, \psi_{\dot{a}}$  are raised as :

$$\phi^a = C^{ab}\phi_b, \quad \psi^{\dot{a}} = (-C)^{\dot{a}\dot{b}}\psi_{\dot{b}}$$

## Appendix B. D=10 Lorentz-Invariant Harmonic SO(8) Algebra

It is a remarkable result that with the help of the harmonic variables  $v_\alpha^{\pm\frac{1}{2}}$ ,  $u_\mu^a$  (2.21) carrying Lorentz-spinor and Lorentz-vector indices, one can imbed the  $SO(8)$  Clifford algebra into the  $SO(1,9)$  in a completely  $SO(1,9)$  invariant way. The construction (2.22), which we use throughout this paper, crucially depends on the existence of the D=10 Fierz identity (1.1)(or (A.5)).

Define the following  $8 \times 8$  matrices in terms of harmonics and D=10  $\sigma$ -matrices:

$$(\gamma^a)_{bc} \equiv \sqrt{2}v^{+\frac{1}{2}}\sigma_b\sigma^a\sigma_c v^{-\frac{1}{2}}, \quad (B.1)$$

$$(\tilde{\gamma}^a)_{bc} \equiv \sqrt{2}v^{-\frac{1}{2}}\sigma_b\sigma^a\sigma_c v^{+\frac{1}{2}}, \quad (B.2)$$

$$\sigma^a \equiv u_\mu^a \sigma^\mu$$

Here the index  $a$  labels the matrices  $\gamma^a$ ,  $\tilde{\gamma}^a$ , while  $b, c$  are  $8 \times 8$  matrix indices. Since  $(\sigma^\mu)_{\alpha\beta} = (\sigma^\mu)_{\beta\alpha}$  it follows that

$$(\gamma^a)_{bc} = (\tilde{\gamma}^a)_{cb}$$

Using the Fierz identity (1.1) it is straightforward to compute that (B.1),(B.2) obey the anticommutation relations (written in a matrix form):

$$\gamma^a \tilde{\gamma}^b + \gamma^b \tilde{\gamma}^a = 2C^{ab} \quad (B.3)$$

$$\tilde{\gamma}^a \gamma^b + \tilde{\gamma}^b \gamma^a = 2C^{ab}$$

and therefore, define a representation of the  $SO(8)$  Clifford algebra which

we call harmonic representation:

$$\Gamma_8^a \equiv \begin{bmatrix} 0 & (\gamma^a)_{bc} \\ (\tilde{\gamma}^a)_{bc} & 0 \end{bmatrix} \quad (B.4)$$

Having the harmonic  $SO(8)$  Clifford algebra (B.3)(B.4) one can now construct the generators of the  $SO(8)$  Lie algebra corresponding to the three inequivalent 8 dimensional representations (s),(c) and (v). It is natural to call these representations harmonic (s),(c),(v) representations. Introducing the two possible antisymmetrized products of (B.1) and (B.2):

$$(\gamma^{ab})_{cd} \equiv \frac{1}{2}(\gamma^a \tilde{\gamma}^b - \gamma^b \tilde{\gamma}^a)_{cd} = v^{+\frac{1}{2}} \sigma_c \sigma^a \sigma^b \sigma^- \sigma_d v^{+\frac{1}{2}}, \quad (B.5)$$

$$(\tilde{\gamma}^{ab})_{cd} \equiv \frac{1}{2}(\tilde{\gamma}^a \gamma^b - \tilde{\gamma}^b \gamma^a)_{cd} = v^{-\frac{1}{2}} \sigma_c \sigma^a \sigma^b \sigma^+ \sigma_d v^{-\frac{1}{2}}, \quad (B.6)$$

where the second equalities are a consequence of the Fierz identity (1.1), one gets the following expressions for the desired  $SO(8)$  generators:

i) harmonic (s) spinor representation

$$S^{ab} = \frac{1}{2} \gamma^{ab} \quad (B.7)$$

ii) harmonic (c) spinor representation

$$\tilde{S}^{ab} = \frac{1}{2} \tilde{\gamma}^{ab} \quad (B.8)$$

iii) harmonic vector (v) representation

$$(V^{ab})^{cd} = C^{ac} C^{bd} - C^{ad} C^{bc} \quad (B.9)$$

Let us stress the fact that all Lorentz invariant harmonic  $SO(8)$  indices  $a, b, c, d, \dots$  appearing in the present formalism, belong to one and the same

fixed 8-dimensional representation of  $SO(8)$ . This is to be contrasted with the noncovariant light-cone formalism where all types of  $SO(8)$  indices ((v),(s) and (c)) do appear [1].

The triality property of the harmonic (s),(c),(v) representations (B.7)-(B.9) are encoded in the relations:

$$\gamma^c \tilde{S}^{ab} - S^{ab} \gamma^c = C^{ac} \gamma^b - C^{bc} \gamma^a \quad (B.10)$$

$$\tilde{\gamma}^c S^{ab} - \tilde{S}^{ab} \gamma^c = C^{ac} \tilde{\gamma}^b - C^{bc} \tilde{\gamma}^a \quad (B.11)$$

$$(\gamma^c V^{ab})_e{}^d = (\gamma^c)_e{}^a C^{bd} - (\gamma^c)_e{}^b C^{ad} \quad (B.12)$$

$$(\tilde{\gamma}^c V^{ab})_e{}^d = (\tilde{\gamma}^c)_e{}^a C^{bd} - (\tilde{\gamma}^c)_e{}^b C^{ad} \quad (B.13)$$

$$(V^{ab} \gamma^c)_e{}^d = C^{ad} (\gamma^c)_e{}^b - C^{bd} (\gamma^c)_e{}^a \quad (B.14)$$

$$(V^{ab} \tilde{\gamma}^c)_e{}^d = C^{ad} (\tilde{\gamma}^c)_e{}^b - C^{bd} (\tilde{\gamma}^c)_e{}^a \quad (B.15)$$

Using (B.12-B.15) one can show that given two objects  $A^a$ ,  $B^a$  transforming under two different harmonic representations out of (B.7-B.9), then the objects

$$C^a = (\gamma^a)_{bc} A^b B^c \quad (B.16)$$

$$\tilde{C}^a = (\tilde{\gamma}^a)_{bc} A^b B^c \quad (B.17)$$

transform under the third one.

Here is a list of useful formulas, frequently used in the text:

$$D^{ab}(v^{+\frac{1}{2}}\sigma^c)^\alpha = -(S^{ab})_d^c(v^{+\frac{1}{2}}\sigma^d)^\alpha \quad (B.18)$$

$$D^{ab}(v^{-\frac{1}{2}}\sigma^c)^\alpha = -(\tilde{S}^{ab})_d^c(v^{-\frac{1}{2}}\sigma^d)^\alpha \quad (B.19)$$

$$D^{ab}u_\mu^c = -(V^{ab})_d^c u_\mu^d \quad (B.20)$$

$$D^{-a}(v^{+\frac{1}{2}}\sigma^b)^\alpha = \frac{1}{\sqrt{2}}(\gamma^a)_c^b(v^{-\frac{1}{2}}\sigma^c)^\alpha \quad (B.21)$$

$$D^{+a}(v^{-\frac{1}{2}}\sigma^b)^\alpha = \frac{1}{\sqrt{2}}(\tilde{\gamma}^a)_c^b(v^{+\frac{1}{2}}\sigma^c)^\alpha \quad (B.22)$$

where  $S^{ab}$ ,  $\tilde{S}^{ab}$ ,  $V^{ab}$  are the same as in (B.7)-(B.9). As a consequence of (B.18-B.22) we obtain the remarkable result that the harmonic  $SO(8)$  Clifford algebra (B.1)-(B.4) and consequently, the harmonic  $SO(8)$  algebra (B.5)-(B.9) are invariant under all harmonic differential operators which span the  $SO(1, 9)$  algebra (2.27):

$$(D^{ab}, D^{-+}, D^{+a}, D^{-a}) \begin{bmatrix} \gamma^c \\ \gamma^{cd} \\ \tilde{\gamma}^c \\ \tilde{\gamma}^{cd} \end{bmatrix} = 0. \quad (B.23)$$

### Appendix C. General Harmonic Superfields and Pure-Gauge nature of the harmonic variables

In the  $D=4$  harmonic superspace approach [31] harmonic superfields are defined as functions on the extended  $N$ -superspace  $z = (x^\mu, \theta_\alpha^i; u)$ . ( $i = 1, \dots, N$ ) where the variables  $u$  belong to a compact homogenous space  $\frac{G}{H}$ .

For  $N = 2, 3$ ,  $G$  is the group of automorphisms of the extended super-Poincare algebra  $G = SU(N)$ , whereas  $H = [U(1)]^{N-1}$  [31].

In the present  $D=10$  case the appropriate homogenous space  $\frac{\mathcal{L}}{SO(8) \times SO(1,1)}$  is noncompact, since the analog of the group-space  $G$  is here the space  $\mathcal{L}$  defined by the kinematical constraints (2.21) on  $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$ .  $H = SO(8) \times SO(1,1)$  is the internal group of local rotations of  $u_\mu^a, v_\alpha^{\pm\frac{1}{2}}$  (2.21). The fact that our harmonic superfields  $\phi(z)$ ,  $z \equiv (x^\mu, \theta_\alpha, u_\mu^a, v_\alpha^{\pm\frac{1}{2}})$  are actually functions on  $\frac{\mathcal{L}}{SO(8) \times SO(1,1)}$  is expressed by the property that they identically satisfy:

$$\hat{D}^{ab} \phi(z) \equiv (D^{ab} + \Sigma^{ab}) \phi(z) = 0 \quad (C.1)$$

$$\hat{D}^{-+} \phi(z) \equiv (D^{-+} - \hat{q}) \phi(z) = 0 \quad (C.2)$$

In (C.1),(C.2)  $D^{ab}, D^{+-}$  are the same as in (2.24),(2.25), i.e. they are "orbital" parts of the  $SO(8)$  and  $SO(1,1)$  rotations, whereas  $\Sigma^{ab}$  denotes the "spin" part of  $SO(8)$  and  $\hat{q}$  denotes the  $SO(1,1)$  charge matrix.

In general,  $\phi(z)$  may be a direct sum of components transforming under different inequivalent representations of the "spin"-part  $\Sigma^{ab}$  in (C.1) and possessing different half-integer or integer  $SO(1,1)$  charges in (C.2).

This is precisely the case in the present formalism - see eqs. (3.25) (3.30),(3.31). Therefore, it is sufficient to analyze (C.1),(C.2) for har-

monic superfields of the form:

$$\begin{aligned}\phi(z) &\equiv \phi^{(q)}(c)\{+\frac{1}{2}d\}\{-\frac{1}{2}e\}(z) \\ (c) &\equiv (c_1, \dots, c_l), \{+\frac{1}{2}d\} \equiv (+\frac{1}{2}d_1, \dots, +\frac{1}{2}d_m), \{-\frac{1}{2}e\} \equiv (-\frac{1}{2}e_1, \dots, -\frac{1}{2}e_n),\end{aligned}\quad (C.3)$$

with an overall  $SO(1, 1)$  charge  $q + \frac{1}{2}(m-n)$  ( $q$  is integer) and whose external  $SO(8)$  indices  $(c_1, \dots, c_l)$ ,  $(d_1, \dots, d_m)$ ,  $(e_1, \dots, e_n)$  transform respectively under the harmonic (v),(s) and (c) representations (B.7)-(B.9):

$$[D^{ab} + \sum_{i=1}^l V^{ab}(i) + \sum_{j=1}^m S^{ab}(j) + \sum_{k=1}^n \bar{S}^{ab}(k)]\phi(z) = 0 \quad (C.4)$$

$$[D^{-+} - (q + \frac{m}{2} - \frac{n}{2})]\phi(z) = 0 \quad (C.5)$$

In (C.4)  $V^{ab}(i)$  denotes the action of  $V^{ab}$  (B.7) on the  $i$ -th index  $c_i$ :

$$V^{ab}(i)\phi(z) \equiv (V^{ab})^{c_i}_{c'_i} \phi^{(q)}(c_1, \dots, c'_i, \dots, c_l)\{+\frac{1}{2}d\}\{-\frac{1}{2}e\}$$

and similarly for  $S^{ab}(j)$ ,  $\bar{S}^{ab}(k)$ .

Now, recalling formulae (B.18)-(B.20) we find that each external  $SO(8) \times SO(1, 1)$  index  $c_i$ ,  $(+\frac{1}{2}d_j)$ ,  $(-\frac{1}{2}e_k)$  of  $\phi(z)$  (C.3) can be unambiguously saturated by  $u_{\mu_i}^{c_i}$ ,  $(v + \frac{1}{2}\sigma^{d_j})^{\alpha_j}$ ,  $(v - \frac{1}{2}\sigma^{e_k})^{\beta_k}$ . On the other hand the integer charge  $q$  can be unambiguously saturated by  $q$  vectors  $u_{\mu}^{\pm} \equiv v^{\pm\frac{1}{2}}\sigma_{\mu}v^{\pm\frac{1}{2}}$  (depending on the sign of  $q$ ). Therefore:

$$\begin{aligned}\phi(z) \text{ (eq. (C.3))} &= u_{\mu_1}^{\pm} \dots u_{\mu_q}^{\pm} u_{\nu_1}^{c_1} \dots u_{\nu_l}^{c_l} \times \\ &(v + \frac{1}{2}\sigma^{d_1})^{\alpha_1} \dots (v + \frac{1}{2}\sigma^{d_m})^{\alpha_m} (v - \frac{1}{2}\sigma^{e_1})^{\beta_1} \dots (v - \frac{1}{2}\sigma^{e_n})^{\beta_n} \phi_{\{\alpha\}\{\beta\}}^{\{\mu\}\{\nu\}}(z) \\ &\{\mu\} \equiv (\mu_1, \dots, \mu_q), \{\nu\} \equiv (\nu_1, \dots, \nu_l), \\ &\{\alpha\} \equiv (\alpha_1, \dots, \alpha_m), \{\beta\} \equiv (\beta_1, \dots, \beta_n),\end{aligned}\quad (C.6)$$



where the coefficient superfields **identically** satisfy:

$$(D^{ab}, D^{-+})\phi_{\{\alpha\}\{\beta\}}^{\{\mu\}\{\nu\}}(z) = 0 \quad (C.7)$$

i.e. they belong to the space  $\mathcal{H}_0$  (2.28),(2.29) of harmonic superfields without any external  $SO(8) \times SO(1,1)$  indices.

Next, we observe that harmonic superfields belonging to  $\mathcal{H}_0$  in fact depend on the Lorentz-spinor harmonics  $v_\alpha^{\pm\frac{1}{2}}$  not in an arbitrary way but only through the light-like composites  $u_\mu^\pm = v^{\pm\frac{1}{2}}\sigma_\mu v^{\pm\frac{1}{2}}$  (see (2.28),(2.29)). Therefore the field from  $\mathcal{H}_0$  depend only on 45 independent combinations of the harmonic variables  $u_\mu^a, u_\mu^\pm$  (accounting for their kinematical constraints), which implies that the general solution of the 45 Dirac constraint equations

$$(D^{ab}, D^{-+}, D^{+a}, D^{-a})\phi(z) = 0, \text{ (on } \mathcal{H}_0 \text{ (2.28), (2.29))} \quad (C.8)$$

is  $\phi(z) = \phi(x, \theta)$ , i.e. it is constant with respect to  $(u, v)$  (cf. (2.37)).

The analog of the Dirac system (C.8) for the more general harmonic superfields (C.6) with external  $SO(8) \times SO(1,1)$  indices reads:

$$(\hat{D}^{ab}, \hat{D}^{-+}, D^{+a}, \hat{D}^{-a})\phi(z) = 0 \quad (C.9)$$

where  $\hat{D}^{ab}, \hat{D}^{-+}$  have "spin" parts  $\Sigma^{ab}, \hat{q}$  as in (C.4),(C.5) and  $\hat{D}^{-a}$  accordingly reads (3.29):

$$\hat{D}^{-a} \equiv D^{-a} - \frac{\partial^a}{\partial^+} \hat{q} - \frac{\partial_b}{\partial^+} \Sigma^{ab} \quad (C.10)$$

Let us point out that the system (C.9) is consistent only on-shell, i.e. when  $(-\partial^2)\phi(z) = 0$ , since (recall eq. (4.2)):

$$[\hat{D}^{-a}, \hat{D}^{-b}] = (\partial^+)^{-2} \Sigma^{ab} (-\partial^2)$$

Now, inserting the general expression (C.6) into the system (C.9) and using the formulae (B.18)-(B.22) we get precisely eqs. (C.8) for the coeffi-

cient harmonic superfield  $\phi_{\{\alpha\}\{\beta\}}^{\{\mu\}\{\nu\}}(z)$ . Therefore, the general solution of eqs (C.9) read:

$$\begin{aligned} \phi(z)|_{on-shell} = & u_{\mu_1}^{\pm} \dots u_{\mu_q}^{\pm} u_{\nu_1}^{c_1} \dots u_{\nu_l}^{c_l} \times \\ & (v^{+\frac{1}{2}} \sigma^{d_1})^{\alpha_1} \dots (v^{+\frac{1}{2}} \sigma^{d_m})^{\alpha_m} (v^{-\frac{1}{2}} \sigma^{e_1})^{\beta_1} \dots (v^{-\frac{1}{2}} \sigma^{e_n})^{\beta_n} \phi_{\{\alpha\}\{\beta\}}^{\{\mu\}\{\nu\}}(x, \theta) \end{aligned} \quad (C.11)$$

Eqs. (C.11) is the precise statement of the on-shell pure-gauge nature of the auxiliary harmonic variables  $(u, v)$  (2.21) for arbitrary harmonic superfields carrying external  $SO(8) \times SO(1, 1)$  indices. Namely, on-shell, the whole dependence of  $\phi(z)$  on  $(u, v)$  is only through a fixed monomial in  $(u, v)$  carrying the external  $SO(8) \times SO(1, 1)$  indices of  $\phi(z)$  whereas the physical fields are contained in the ordinary superfield  $\phi_{\{\alpha\}\{\beta\}}^{\{\mu\}\{\nu\}}(x, \theta)$ .

Property (C.11) exactly parallels analogous properties of D=4 harmonic superfields in [31].

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