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LATTICE MODELS AND CONFORMAL FIELD THEORIES

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1. INTRODUCTION

The recent developments of conformal field theory have considerably increased our understanding of two dimensional critical phenomena[1,2]. They however do not refer in general to any particular lattice or continuum model familiar to statistical mechanics. From a practical point of view it is necessary to understand better the connection between critical physical systems and the conformal theories to which they correspond. In two dimensions, the general arguments of Landau type that usually allow to classify universality classes are not sufficient, and more specific methods are required. Also it must be noticed that constraints which are natural in string theory are often not satisfied in statistical mechanics, and the study of physical questions can require the introduction of rather unusual conformal theories. For instance, geometrical problems like percolation or polymers are described by non minimal, non unitary $c = 0$ theories. From a more conceptual point of view, the conformal invariance formalism bears striking similarities with the theory of integrable models. The precise understanding of the connections between these two fields should be an important progress.

In this paper, we review several works that aim at answering the above questions. They are all partly based on free field (or Coulomb gas) representations that seem to play a rather unifying role in the subject.

In the first part we discuss the general scheme that allows one to derive the conformal theory associated to a critical (integrable) lattice model. We show in particular how the central charge, critical exponents, and torus partition function can be obtained using renormalisation group arguments.

In the second part we discuss, in the case of $c < 1$ models, the common quantum group structure that appears in the integrable lattice models and in the theory of Virasoro algebra representations.

In the third part we finally discuss relations between off-critical integrable models, and conformal theories in finite geometries.

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2. CENTRAL CHARGE, CRITICAL EXPONENTS AND PARTITION FUNCTION OF LATTICE MODELS

There are several methods to obtain the exponents of lattice models. The calculation of two point functions at criticality is a difficult task, and has not been achieved so far, except for the energy and spin operators in the Ising case[3]. The calculation of order parameters can been performed more easily with the corner transfer matrix[4,5] technique; it requires however the knowledge of integrable weights *off* criticality, and these are not yet obtained in all cases (E_6 - E_7 - E_8 models of [6] for instance). The method we present here, although less rigorous, is more general. As we shall see in the next section, it allows as well the derivation of modular invariant partition functions. The basic step consists in transforming the model to study, or a model supposed to be in the same universality class, into a restricted solid on solid model which is assumed (and this can usually be justified) to flow under renormalization group onto a free Gaussian model[7-9]. Various observables are then transformed into vortex or spin wave operators[7], and their exponents readily calculated as functions of the coupling constant. The latter can finally be obtained by reference to some other, exactly known, quantity. On a torus, the study of topological effects due to boundary conditions allow as well the calculation of the modular invariant partition function[10].

a) Potts model - 6 vertex model

To illustrate these concepts we consider in some details the Potts model, which is defined by associating a variable $\sigma = 1,\dots,Q$ to each site of the square lattice $\mathcal{L}$, with an action

$$\mathcal{A} = -\frac{1}{T}\sum_{\langle jk\rangle}\delta_{\sigma_j\sigma_k} \tag{2.1}$$

This definition can be extended to $Q \in \mathbb{R}$ by considering the high temperature expansion of the partition function

$$\mathcal{Z}_Q = \sum_{\text{graphs}} W(\mathcal{G}) = \sum_{\text{graphs}} (e^{1/T}-1)^{\mathcal{N}_B} Q^{\mathcal{N}_C} \tag{2.2}$$

where the graphs are obtained by putting $\mathcal{N}_B$ bonds on the edges of the lattice which form $\mathcal{N}_c$ clusters i.e. connected components, including isolated points. Eq.(2.2) is more easily handled using a polygon decomposition[11] of the surrounding lattice $\mathcal{S}$, here another square lattice (Fig.1). If $\mathcal{N}_L$ is the number of loops in a given graph of (2.2) and $\mathcal{N}_S$ the total number of sites in $\mathcal{L}$, then by Euler's relation $\mathcal{N}_L = \mathcal{N}_B + \mathcal{N}_C - \mathcal{N}_S$. In a plane, the number $\mathcal{N}_p$ of polygons reads

$$\mathcal{N}_p = \mathcal{N}_L + \mathcal{N}_c \tag{2.3}$$

Hence (2.2) can be rewritten as

$$\mathcal{Z}_Q = Q^{\mathcal{N}_S/2} \sum_{\text{graphs}} [(e^{1/T}-1)\; Q^{-1/2}]^{\mathcal{N}_B} Q^{\mathcal{N}_P/2} \tag{2.4}$$

Model (2.4) is known to have a second order phase transition for $Q \in [0,4]$, the critical temperature being such that $(e^{1/T}-1)\; Q^{-1/2} = 1$ (self dual line[12]). The weight $Q^{1/2}$ per polygon can now be reformulated locally[11] by first giving an arbitrary orientation to each polygon, and then associating to each left (right) turn a term exp $+(-)\; i\alpha/4$. Since on a plane

$$|n_\ell - n_r| = 4 \tag{2.5}$$

for the square lattice, summing over all orientations gives the desired factor provided

$$Q^{1/2} = 2 \cos \alpha, \quad \alpha \in [0,\; \pi/2] \tag{2.6}$$

It is now convenient to stick again the oriented contours at each node of S; one recovers in this way the 6-vertex model[12,13] (Fig.2) with weights[11]

$$\begin{aligned} a &= b = 1 \\ c &= 2 \cos \alpha/2 \end{aligned} \tag{2.7}$$

The Potts model (2.4) on its self dual line is thus equivalent to the 6-vertex model, up to boundary terms. The latter do not modify the thermodynamic quantities, but as we explain latter, they affect the conformal ones.

The program of reformulating the original model as a solid on solid (SOS) model is now easily carried out[14]. Given a configuration of vertices one introduces height variables φ that live on the faces of S, and are defined recursively by the constraint that two neighboring heights differ by $\pm\pi$, the highest one being on the left of the arrow that separates them. This makes sense because there is no vertex with a non zero divergence allowed. It is now assumed[8] that this solid on solid model flows, under renormalization, to a gaussian one with action

$$\mathcal{A} = \frac{g}{4\pi} \int |\vec{\nabla}\varphi|^2 \; d^2x \tag{2.8}$$

The coupling constant can be determined as follows. We look at the dimension of operators O_M that create locally a new vertex of the kind shown in Fig.3. These vertices induce defects in the above definition of heights, since φ is no more uniquely defined. Indeed going around C (Fig.3) gives $\oint_C \vec{\nabla}\varphi \cdot d\vec{\ell} = -4\pi$. The operator O_M is a vortex operator[7], with charge $M = \frac{1}{2\pi} \oint \vec{\nabla}\varphi \cdot d\vec{\ell} = -2$. Its dimension is easily obtained in the continuum limit (2.7) as $x_M = g\, M^2/2 = 2g$. On the other hand it is possible to calculate the singularity of the free energy of the 6-vertex model as one introduces a small amount of type d vertices[12].

[40] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter, G.R.W. Quis el

Matching x_M with Baxter's formula gives, introducing

$$\Delta = \frac{a^2+b^2-c^2}{2ab} = -\cos\alpha \qquad \alpha \in [0,\pi] \tag{2.9}$$

the result

$$g = 1 - \frac{\alpha}{\pi} \tag{2.10}$$

(the 6-vertex model ceases[12] to be critical for $|\Delta| > 1$). In the Potts case we have

$$\Delta = -\frac{1}{2} Q^{1/2} \tag{2.11}$$

so only the region $\Delta \in [-1,0]$ is covered. As Q varies from 0 to 4, g goes from $\frac{1}{2}$ to 1.

Once g is known (in the scale where topological defects are not renormalized), the dimensions of observables that are equivalent to combinations of spin wave ($O_E = e^{iE\varphi}$) and vortex operators are easily determined

$$x_{EM} = \frac{E^2}{2g} + g\,\frac{M^2}{2}, \qquad S_{EM} = EM$$

or

$$h(\bar{h}) = \frac{1}{4}\left(\frac{E}{\sqrt{g}} \pm M\sqrt{g}\right)^2 \tag{2.12}$$

The 6-vertex model is thus described by a free Gaussian theory with central charge 1 and operators (2.12). It has a simple variant, the Ashkin Teller[12] model that contains[15-17] also twist fields (operators creating a branch point singularity where φ changes of sign) with dimension $h(\bar{h}) = 1/16$.

b) Central charge

The transcription of the $\sqrt{Q}$ per loop term into a local weight depending on left and right turns (2.5) is sensitive to curvature. If we think for instance of a lattice built on a cube with a loop encircling a corner[18] as in Fig.4 then $|n_\ell - n_r| = \pm 3$ instead of ± 4. To correct this, an electric charge has to be added in 0, which corresponds in the lagrangian version (2.8) to a new term $\sim ie_0 \int R\varphi\, d^2x$ where R is the curvature. Even on a plane where R = 0, this term is known[19] to modify the central charge due to boundary conditions. The simplest way[20] of getting c is to consider a cylinder (Fig.5) where non contractible loops have $|n_\ell - n_r| = 0$ and hence a weight 2 instead of $\sqrt{Q}$. This weight can be corrected by adding charges $\pm e_0$ at $\pm\infty$ in such a way that $2\cos \pi e_0 = \sqrt{Q}$ i.e. $e_0 = \frac{\alpha}{\pi}$. The finite size behaviour of the free energy is then

FIGURE CAPTIONS

modified, and the central charge, which was one for the 6-vertex model, reads now $c = 1 - 12\, x_{e_0,0}$, i.e.

$$c = 1 - 6\, e_0^2/g \tag{2.13}$$

Another consequence of the form of the weights is the existence of a non trivial operator with mean value one[8]. Indeed the insertion of $\exp - 2ie_0\, \varphi$ inside a clockwise oriented loop changes the angular weight of the latter from $\exp -i\alpha$ to $\exp[-i\alpha + 2ie_0\pi] = \exp i\alpha$, and similarly for an anticlockwise loop the weight $\exp i\alpha$ is changed into $\exp -i\alpha$. Summing ones both orientation gives thus the same factor $\sqrt{Q} = 2\cos\alpha$ as before, hence

$$\langle \exp - 2ie_0\varphi \rangle = 1 \tag{2.14}$$

It is of course assumed here that the height at infinity takes some fixed value: (2.14) does not hold on a torus.

If we define $\alpha = \pi/\mu+1$, (2.13) gives

$$c = 1 - \frac{6}{\mu(\mu+1)} \tag{2.14}$$

We recover the values $c = 1$ for $Q = 4$, $c = 4/5$ for $Q = 3$, $c = \frac{1}{2}$ for $Q = 2$. An important point for physical applications is $Q = 1$ which corresponds to the percolation problem[21]. From (2.14), $c = 0$ in this case, a result which was expected since in the limit $Q \to 1$ the partition function $\mathfrak{Z}_Q \to 1$. (2.14) reproduces, for μ integer = m, the central charges of the unitary series. The associated values of Q are the Behara numbers[12] $Q = 4\cos^2\pi/m+1$.

c) Critical exponents

We discuss the example of percolation[20] ($Q = 1$). An important quantity here is the fractal dimension D_H of the "hull" (i.e. the boundary[21] of the infinite cluster) which is measured[22] $D_H \simeq 1.75$. To calculate it with the above formalism, we introduce[22] the correlator

$$G(\vec{r}_1 - \vec{r}_2) = \frac{1}{\mathfrak{Z}_Q} \sum_{\text{graphs}}{}' W(\mathcal{G}) \tag{2.15}$$

where the sum is taken only over graphs such that $\vec{r}_1$ and $\vec{r}_2$ are two points on the surrounding lattice at the corners of the same polygon $\mathcal{P}$. (2.15) is translated in the SOS language by modifying first the orientation of one line of $\mathcal{P}$ so that both sides of $\mathcal{P}$ now go from $\vec{r}_1$ to $\vec{r}_2$. The resulting configuration has now a vortex of charge $M = 1(-1)$ in $\vec{r}_1\,(\vec{r}_2)$. The Boltzmann weight in the SOS model does not, however, exactly correspond to (2.15). There is an additional[8,9] curvature factor $\exp +(-)2i\alpha$ for each left (right) turn of the polygon around one of the extremities (Fig.6). It can be compensated by adding in $\vec{r}_1$ and $\vec{r}_2$ the same electric charge $E = -e_0$. Since $E_1 + E_2 = -2e_0$ is precisely

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the charge associated to (2.14), this does not modify, as is necessary, the weight of surrounding polygons.

The breaking of electrical neutrality is repaired by adding a charge $+2e_0$ at infinity, and one gets

$$x = -\frac{e_0^2}{2g} + \frac{g}{2} \tag{2.16}$$

For percolation $g = \frac{2}{3}$, $e_0 = \frac{1}{3}$, so $x = \frac{1}{4}$. Scaling arguments[22] give now $D_H = 2-x = 7/4$.

Using this method, various exponents for percolation or polymers have been calculated[23] that have an important physical meaning. It is important to notice however that all the corresponding operators are outside the minimal unitary[24] set, reduced here to the identity with $h_{11} = 0$ since $c = 0$; for instance $x = 1/4 = 2h_{22}$.

d) Partition functions

We consider a torus with complex periods 1 and τ. The universal part of the 6-vertex model partition function is evaluated using the mapping onto (2.8). Because the arrow-heights correspondence is purely local, φ cannot in general be uniquely defined on the torus, but presents some winding numbers[25]

$$\begin{cases} \varphi(z+1,\bar{z}+1) = \varphi(z,\bar{z}) + 2\pi M \\ \varphi(z+\tau,\bar{z}+\bar{\tau}) = \varphi(z,\bar{z}) + 2\pi M' \end{cases} \tag{2.17}$$

In a given sector characterized by M, M', the functional integral

$$Z_{MM'} = \int_{\varphi \text{ satisfying } (2.17)} [D\varphi] \quad e^{-\mathcal{A}} \tag{2.18}$$

is easily evaluated by decomposing φ into a classical (soliton) and a quantum part, and using the known result

$$Z_{00} = \sqrt{\frac{g}{\text{Im}\tau}} \, \frac{1}{\eta\bar{\eta}} \tag{2.19}$$

where η is Dedekind's function $\eta = q^{1/24} \prod_1^\infty (1-q^n)$, $q = e^{2i\pi\tau}$. One finds

$$Z_{MM'} = Z_{00} \exp - \pi g \frac{|M-M'\tau|^2}{\text{Im}\tau} \tag{2.20}$$

$Z_{MM'}$ has the modular transformation properties from its definition

$$Z_{MM'}(\tau) = Z_{aM'+bM,cM'+dM}\left(\frac{a\tau+b}{c\tau+d}\right) \qquad (2.21)$$

The total 6-vertex' model partition function is obtained by summing over M and M' and is then modular invariant. Restricting to an even × even lattice, M and M' $\in \mathbb{Z}$ one gets

$$\mathfrak{Z}_{6\text{-vertex}} \longrightarrow Z_c(g) = \sum_{MM'\in\mathbb{Z}} Z_{MM'}(g) = \frac{1}{\eta\bar{\eta}} \sum_{E,M\in\mathbb{Z}} q^{h_{EM}} \bar{q}^{\bar{h}_{EM}} \qquad (2.22)$$

where last equality is obtained by Poisson formula. (2.22) exhibits the operator content (2.12). It contains also operators of dimensions (1,0) (resp.(01)) associated to the currents $\partial\varphi(\bar{\partial}\varphi)$. If g = 1, (2.22) has additional currents O_{11} ($O_{1,-1}$) and can be shown[25] to coïncide with the SU(2) level 1 Wess Zumino model partition function[26]. Accordingly, the lattice model present a (global) SU(2) invariance. (2.22) satisfies also the duality transformation formula

$$Z_c(g) = Z_c(1/g) \qquad (2.23)$$

Although the 6-vertex model has $g \in [0,1]$, (2.22) is defined for any g. With $g \in [n^2,(n+1)^2]$, it describes[27] a n-critical 6-vertex model.

To obtain the partition function of the Potts model is a more difficult task. First, all polygons that wind around the torus have a weight 2 in the 6-vertex model formulation that must be corrected into $\sqrt{Q}$. The extra term equivalent to the charges $\pm e_0$ at infinity in the cylinder case can be built[10] by noticing that

- if two non contractible loops coexist on the torus, then they are homotopic

- if a non contratible loops intersects the periods $1(\tau)$ $n(n')$ times, then n and n' are coprimes ($n\wedge n'=1$). One is then led to introduce the interaction between winding numbers $\cos \pi e_0\, M\wedge M'$. The other point is that on the torus, although Eulers' relation remains valid, (2.3) can be violated for clusters that have a "cross topology"[10] (Fig.7) for which $\mathcal{N}_L + \mathcal{N}_c - \mathcal{N}_p = 2$. Equation (2.4) gives to such graphs the relative weight 1 instead of Q. One thus must odd to (2.4), (Q-1) times the partition function restricted to clusters with cross topology; the latter are selected by giving a weight 0 to non contractible contours. One finds finally

$$\mathfrak{Z}_Q \longrightarrow Z_Q = \sum_{M,M'\in\mathbb{Z}} Z_{MM'}(g)\left[\cos \pi e_0\, M\wedge M' + \frac{Q-1}{2} \cos\frac{\pi}{2} M\wedge M'\right] \qquad (2.24)$$

For simple values of e_0, the sum (2.24) can be recast into linear combinations of Z_c[10]. One finds

$$Z_{Q=4} = [3Z_c(4) - Z_c(1)]/2$$
$$Z_{Q=3} = \left[Z_c\left(\frac{10}{3}\right) - Z_c\left(\frac{5}{6}\right) - Z_c\left(\frac{15}{2}\right) + Z_c(30)\right]\Big/2 \qquad (2.25)$$
$$Z_{Q=2} = [Z_c(12) - Z_c(4/3)]/2$$
$$Z_{Q=1} = [Z_c(4) - Z_c(3/2)]/2 = 1$$

The results for $Q = 2,3$ agree with modular invariant combinations of characters derived in [28]. The result for $Q = 1$ is also expected since $\mathfrak{Z}_{Q=1} = 1$.

For other values of Q, it can be shown[10] that (2.24) decomposes onto infinite sums of characters with real (negative) coefficients. In particular at the Behara numbers values, (2.24) doest not reproduce the modular invariant combinations obtained in [26]. In the percolation case, geometrical quantities of interest can be obtained by considering[23] derivatives of Z_Q.

The ADE integrable lattice models[6] can also be transformed[29] into the 6-vertex model. They then look similar to the Potts model with $Q = 4\cos^2\pi/H$ (where H is the Coexeter number of the algebra) up to boundary terms. One finds in all cases[30]

$$Z_{A,D,E} = \sum_{M,M'\in\mathbb{Z}} Z_{MM'}\left(g = \frac{H-1}{H}\right) \sum_{j\,\in\,\text{exponent}} \cos\left[\frac{\pi j}{H} M\wedge M'\right] \qquad (2.26)$$

where in the last sum j runs over the exponents of the classifying algebra.

e) Generalization to other conformal theories

The preceding picture generalizes[30,31,32] to most of the known theories. At the basis of a given conformal series, there is a vertex model that has an integrable curve in its parameter space. This curve has a critical part C along which a coupling constant varies, and which ends at a self dual point described by a Wess Zumino theory. Minimal models renormalize then onto a set of discrete points on C, and their partition functions can be written similarly to (2.26).

An interesting physical example is provided by the "SU(2) level 2" series. Here the vertex model is the 19-vertex model[33] which is described[30,34] in the continuum limit by a free superfield

$$\mathcal{A} = \frac{g}{\pi}\int \partial\varphi\,\bar\partial\varphi\, d^2x + \int (\psi\bar\partial\psi - \bar\psi\partial\bar\psi)\, d^2x \qquad (2.27)$$

with partition function[30]

$$Z_{SC}(g) = \sum_{r,s=0,1} \mathfrak{Z}_2(r,s) \sum_{\substack{M=r \bmod 2\\ M'=s \bmod 2}} Z_{MM'}(g) \qquad (2.28)$$

where $\mathfrak{Z}_2(r,s)$ is the Ising model partition function with the spin σ twisted by

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$(-1)^r((-1)^s)$ along 1 (τ), and $g \in [0,1/2]$. In this case $Z_{SC}(g) = Z_{SC}(1/4g)$.

At the self dual point $g = 1/2$ (2.28) coincides with the SU(2) level 2 Wess Zumino model partition function. For superminimal ADE models[35] one has then[30]

$$Z_{ADE} = \sum_{r,s=0,1} \mathfrak{Z}_2(r,s) \sum_{\substack{M=r \bmod 2 \\ M'=s \bmod 2}} Z_{MM'}\left(\frac{H-2}{2H}\right) \sum_{j \in \text{exponents}} \cos\frac{\pi j}{H} M \wedge M' \qquad (2.29)$$

3. QUANTUM GROUP STRUCTURES IN INTEGRABLE MODELS AND CONFORMAL THEORIES

It has become clear that integrable lattice models and conformal theories have similar properties, and the developments of both fields are now quite correlated. The precise connection between integrability and conformal invariance is not however fully understood. We discuss here how the concept of quantum group can shed some light on this problem.

a) $SU(2)_t$ representations and spectrum of the XXZ chain

We consider again the Potts model, and turn to a hamiltonian formalism which is more suited for our purpose. In the very anisotropic limit the transfer matrix defines the quantum XXZ chain[12]

$$\mathcal{H} = \frac{\alpha}{2\pi \sin\alpha} \sum_{i=1}^{L-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos\alpha\, \sigma_i^z \sigma_{i+1}^z\right) + i \sin\alpha \left(\sigma_1^z - \sigma_L^z\right) \qquad (3.1)$$

where σ's are Pauli matrices, and where the surface term accounts for the special weight of external vertices[11]. (We have taken an open chain to deal later on with a single Virasoro algebra). If $\alpha = 0$, (3.1) has a global SU(2) invariance, which turns into a local one in the continuum limit described by the Wess Zumino model. If $\alpha \neq 0$, one can show[36] that (3.1) still commutes with generalized objects

$$\begin{cases} S^\pm = \sum_i S_i^\pm; \quad S_i^\pm = t^{\sigma_1^z/2} \cdots t^{\sigma_{i-1}^z/2}\, \sigma_i^\pm/2\, t^{-\sigma_{i+1}^z/2} \cdots t^{-\sigma_L^z/2} \\[2ex] S^3 = \sum_i \sigma_i^z/2 \end{cases} \qquad (3.2)$$

where

$$t = e^{i\alpha} \qquad (3.3)$$

The associated commutation relations are

$$\begin{cases} [S^3,S^{\pm}] = \pm S^{\pm} \\ [S^+,S^-] = \dfrac{t^{2S^3}-t^{-2S^3}}{t-t^{-1}} \end{cases} \tag{3.4}$$

They define[38] the quantum deformation of SU(2): $SU(2)_t$ which can be considered as the "symmetry group" of the chain.

It is now interesting to have a look at the theory of representations[36] of $SU(2)_t$. For this we need the Casimir operator

$$S^2 = S^{x^2} + S^{y^2} + (t+t^{-1}) \left(\frac{t^{S^3}-t^{-S^3}}{t-t^{-1}}\right)^2 \tag{3.5}$$

If $\frac{\alpha}{\pi} = \frac{1}{\mu+1}$ is irrational, the representations $\tilde{\rho}_j$ of $SU(2)_t$ are characterized by the value of

$$S^2 = \left[j + \frac{1}{2}\right]_t^2 = \left(\frac{t^{1/2}t^j - t^{-1/2}t^{-j}}{t-t^{-1}}\right)^2 \tag{3.6}$$

and are isomorphic to SU(2) ones ρ_j. If $\frac{\alpha}{\pi}$ becomes rational, things get different due to the possible periodicity of (3.6) when j varies. We discuss here the case μ integer. Then, (3.6) is invariant under the transformations

$$\begin{aligned} j' &= j + n(n+1) \\ j' &= \mu - j + n(\mu+1) \end{aligned} \tag{3.7}$$

Accordingly, there are states $|\gamma\rangle$ that become highest weights ($S^+|\gamma\rangle = 0$) even if they are the S^- of something, and several sets of vectors $\{\tilde{\rho}_j\}$ which would have formed irreducible representations in the irrational case mix to form an indecomposable representation. We give the example of the final structure of the Hilbert space for $t = e^{i\pi/3}$, L=3

$$\begin{array}{ccc} |\uparrow\uparrow\uparrow\rangle & & \\ \downarrow & & \\ |\gamma\rangle = q|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + q^{-1}|\downarrow\uparrow\uparrow\rangle & |\beta\rangle & |\alpha\rangle \\ \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow \\ q|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + q^{-1}|\downarrow\downarrow\uparrow\rangle & |\beta'\rangle & |\alpha'\rangle \\ \uparrow & & \\ |\downarrow\downarrow\downarrow\rangle & & \end{array} \tag{3.8}$$

where arrows connect states under the action of $S^{\pm}$. Associated to (3.7) is a property of nilpotency[36]

$$(S^{\pm})^{\mu+1} = 0 \tag{3.9}$$

It is now interesting to restrict attention to representations that remain isolated (like $\{|\alpha\rangle, |\alpha'\rangle\}$ in (3.8) and thus are still of $\tilde{\rho}_j$ type. Excluding moreover[36] the case $j = \mu/2$, one finds that the corresponding highest weights are completely characterized by

$$|\alpha_j\rangle \in \frac{\text{Ker } S^+}{\text{Im}(S^+)^{\mu}} \quad ; \; S^3|\alpha_j\rangle = j|\alpha_j\rangle \tag{3.10}$$

and, if $\Gamma_j^{(L)} = \begin{pmatrix} L/2-j \\ L \end{pmatrix} - \begin{pmatrix} L/2-j-1 \\ L \end{pmatrix}$ is the number of spin j representaions in the SU(2) case, their number reads

$$\Omega_j^{(L)} = \Gamma_j^{(L)} - \Gamma_{\mu-j}^{(L)} + \Gamma_{j+\mu+1}^{(L)} - \Gamma_{\mu-j+\mu+1}^{(L)} + \cdots \tag{3.11}$$

The representations are unitary i.e. $(S^+)^+ = S^-$ there.

Going back to the hamiltonian (3.1), Bethe ansatz and numerical calculations give the generating function of scaled gaps associated to all $\tilde{\rho}_j$ as[36,40,41]

$$K_{1,1+2j} = \frac{q^{h_{1,1+2j}} - q^{h_{1,-1-2j}}}{P(q)} \tag{3.12}$$

The generating function restricted to isolated ones as above reads thus, similarly to (3.11)

$$\chi_{1,1+2j} = K_{1,1+2j} - K_{1,1+2(\mu-j)} + \cdots \tag{3.13}$$

and is exactly the Virasoro character[42] associated to an irreducible representation of highest (L_0) weight $h_{1,1+2j}$. This thus point out a relation between the commutant of $SU(2)_t$ and the Virasoro algebra.

It is also interesting to notice that the hamiltonian (3.1) is not completely equivalent to the Potts model, and the partition function $\mathcal{Z}_Q$ (2.4) with free (periodic) boundary conditions in the space (time) direction can be deduced from the spectrum of H by adding[41] boundary terms as in (2.24). One finds

$$\mathcal{Z}_Q \longrightarrow \sum_{j=1}^{[\mu/2]-1} \chi_{1,1+2j} \tag{3.14}$$

Comparison of (3.14) and (3.10) indicates a more formal way of formulating the correspondence between 6-vertex and Potts models by restricting[43] the former to proper $SU(2)_t$ representations.

b) Feigin Fuchs construction

The preceding study bears striking similarities with the theory of Virasoro representations via the bosonic Feigin Fuchs construction (we restrict here again to μ integer)[44,45]. There, one considers a chiral bosonic field with propagator

$$\langle \varphi(z)\varphi(w) \rangle = -\frac{\mu+1}{2\mu} \ell n(z-w) \qquad (3.15)$$

and charge $e_0 = \frac{1}{\mu+1}$ at infinity. The Virasoro generators are obtained as moments of the stress energy tensor

$$T(z) = -\frac{\mu}{\mu+1} :(\partial\varphi)^2: + \frac{i}{\mu+1} \partial^2\varphi \qquad (3.16)$$

and null states are built using screening operators. These are vertex operators $V_{e_\pm}$ with dimension $h = \left(e^2 - 2e_0 e\right)/4g$ equal to one, hence

$$\begin{cases} e_+ = 2 \\ e_- = -2\frac{\mu}{\mu+1} \end{cases} \qquad (3.17)$$

Because h=1, $(Q_\pm, L_n) = 0$ where

$$Q_\pm \sim \frac{1}{2i\pi} \oint V_{e_\pm}(z)\, dz \qquad (3.18)$$

On the other hand the free field model having a U(1) symmetry, $\left[Q^3, L_n\right] = 0$ where

$$Q^3 \sim \frac{1}{2i\pi} \oint \partial\varphi\, dz \qquad (3.19)$$

This gives thus a structure very similar to (3.1), (3.2). Commutation relations can be defined in this continuum limit and one obtains, with the appropriate normalizations

$$\begin{cases} \left[Q^3, Q_+\right] = \frac{\mu+1}{\mu} Q_+ \\ \left[Q^3, Q_-\right] = -Q_- \\ [Q_+, Q_-] = \frac{t^{2Q^3} - t^{-2Q^3}}{t - t^{-1}} \end{cases} \qquad (3.20)$$

with $t = e^{i\pi e_0}$, the last term being essentially due to $e_+ + e_- = 2e_0$. (3.20) reproduces (3.4), up to a mismatch of the first relation. Accordingly one finds (see also [46])

$$\begin{cases} (Q^+)^{\mu} = 0 \\ (Q^-)^{\mu+1} = 0 \end{cases} \tag{3.21}$$

instead of (3.9). The irreductible Virasoro representation is then obtained by considering, in a given fock space $\mathcal{F}_e$ = [polynomials in $\partial\varphi, \partial^2\varphi \ldots$] V_e, cohomology groups similar to (3.10), and

$$\chi_{r,s} = \mathrm{Tr}\, q^{L_0} \text{ (restricted to states with charge } \frac{1-r}{2} e_+ + \frac{1-s}{2} e_- \text{ and belonging to } \frac{\mathrm{Ker}\, S^{+r}}{\mathrm{Im}\, S^{+\mu-r}}) \tag{3.22}$$

A similar Feigin Fuchs construction can be carried out for SU(2) models of level k. One finds then, for a given $t = e^{i\pi/H}$, commutation relations similar to (3.20), with

$$[Q^3, Q_+] = \frac{H}{H-k} Q_+ \tag{3.23}$$

The lattice quantum group is thus obtained formally by taking the k=0 limit of this structure.

4. OFF CRITICAL INTEGRABLE MODELS AND CONFORMAL THEORIES IN FINITE GEOMETRIES

Another interesting relation with conformal theories can be observed in the study of *non critical* integrable systems. We discuss here the case of the 6-vertex model. The latter is critical[12] for $|\Delta| \leq 1$. The region $\Delta > 1$ is completely frozen with all arrows in the same direction and a spontaneous polarization 1. The region $\Delta < -1$ presents a more interesting order; in terms of the solid on solid variables φ, the surface that was rough for $|\Delta| \leq 1$ becomes localized, with a finite correlation length ξ. In this regime the ground states are antiferroelectrically ordered, and the variables φ take two different values B and C on each sublattice. On can then, fixing heights B and C at the boundary, consider the probability of finding the height A at the center of the system P(A/B,C). Parametrizing the weights by

$$a = b = 1$$

$$c = 2\cosh\frac{\tilde{\alpha}}{2}, \quad \tilde{\alpha} \geq 0 \tag{4.1}$$

($\Delta = -\cosh\tilde{\alpha}$) one finds[47], using corner transfer matrix technique

$$P(A/B,C) = \frac{p^{\left(A - \frac{B+C}{2}\right)^2/4}}{\sum_{A'} p^{\left(A' - \frac{B+C}{2}\right)^2/4}} \tag{4.2}$$

where $p = \exp - 4\tilde{\alpha}$.

Now we can consider instead the 6-vertex model at $\Delta = -1$ on a strip of width L and length T, with periodic boundary conditions in the time direction. Suppose first we fix heights to be equal to A (resp.D) in $j = 1$ ($j=L$). Then Z can be directly evaluated via the free field mapping. Separating φ into its quantum and classical parts one gets

$$Z(A/D) = \frac{q^{(A-D)^2/4}}{\eta(q)} \tag{4.3}$$

where $q = \exp - \frac{\pi T}{L}$. The case where heights are fixed to B (resp.C) in $j = L-1$ (resp. $j=L$) can also be treated with Bethe ansatz calculations[41,48] to give

$$Z(A/B,C) = \frac{q^{\left(A - \frac{B+C}{2}\right)^2/4}}{\eta(q)} \tag{4.4}$$

It is remarkable that (4.2) and (4.4) have the same structure; P(A/BC) can indeed be formally calculated by forming the ratio $Z(A/BC)\Big/\sum Z(A'/BC)$ once p is identified with q. This points out a relation between an off critical integrable model and its conformal theory in a finite geometry, once the distance to criticality p in the former case has been identified with the finite size parameter q in the latter. The correspondence $p = q$ is non universal but, in the limit where $\tilde{\alpha} \longrightarrow 0$, it must satisfy, due to finite size scaling

$$q \simeq e^{-2\pi^2/\ln\,\xi} \tag{4.5}$$

This is the case here since[12] $\xi = \exp \pi^2/2\tilde{\alpha}$.

This observation can be (partly) extended[40] to minimal models. In the Ising case for instance, we can consider the partition[49] functions with spin $\sigma = 1$, and $\sigma = 1$ (resp -1) for $j = L$

$$\begin{aligned} Z_{++} &= \chi_{1,1} \\ Z_{+-} &= \chi_{1,3} \end{aligned} \tag{4.6}$$

Forming now the ratio

$$\frac{Z_{++} - Z_{+-}}{Z_{++} + Z_{+-}} = \frac{\prod_0^\infty (1 - q^{n+1/2})}{\prod_0^\infty (1 + q^{n+1/2})} \tag{4.7}$$

gives an expression that agrees exactly with the spontaneous magnetization in the low temperature phase, as calculated by Yang[12], where q has now to be related to T_c-T via the elliptic parametrization of Boltzmann weights.

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REFERENCES

[1] J. Cardy, in "Phase Transitions and Critical Phenomena" Vol.11, (1987).

[2] C. Itzykson, H. Saleur, J.B. Zuber, "Conformal Invariance and Applications to Statistical Mechanics", World Scientific (1988).

[3] T.T. Wu, B. Mc Coy, C.A. Tracy, E. Barouch, Phys. Rev. B13, 316 (1976).

[4] G.E. Andrews, R.J. Baxter, P.J. Forrester, J. Stat. Phys. 35, 193 (1984).

[5] E. Date, M. Jimbo, A. Kuniba, T. Miwa, M. Okado, Nucl. Phys. B290, 231 (1987).

[6] V. Pasquier, Nucl. Phys. B285, 162 (1987).

[7] L.P. Kadanoff, A.C. Brown, Ann. Phys. 121, 318 (1979).

[8] M. Den Nijs, Phys. Rev. B23, 6111 (1981).

[9] B. Nienhuis, in "Phase Transitions and Critical Phenomena" Vol.11 (1987).

[10] B. di Francesco, H. Saleur, J.B. Zuber, J. Stat. Phys. 49, 57 (1987).

[11] R.J. Baxter, S.B. Kelland, F.Y. Wu, J. Phys. A9, 397 (1976).

[12] R.J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press (1982).

[13] E. Lieb, in "Phase Transitions and Critical Phenomena" Vol.1 (1972).

[14] H. Van Beijeren, Phys. Rev. Lett. 38, 993 (1977).

[15] D. Friedan, S. Schenker, in [2].

[16] S.K. Yang, Nucl. Phys. B285, 183 (1987).

[17] H. Saleur, J. Phys. A20, L1127 (1987).

[18] O. Foda, unpublished.

[19] D. Friedan, in " Recent Advances in Field Theory and Statistical Mechanics", Les Houches 1982.

[20] H. Blöte, J. Cardy, M. Nightingale, Phys. Rev. Lett. 56, 742 (1986).

[21] D. Stauffer, "Introduction to Percolation Theory"

[22] B. Duplantier, H. Saleur, Phys. Rev. Lett. 57, 3179 (1986).

[23] H. Saleur, J. Phys. A19, L807 (1986).
B. Duplantier, H. Saleur, Nucl. Phys. B290, 291 (1987)

[24] H. Saleur, J. Phys. A20, 455 (1987).

[25] P. Di Francesco, H. Saleur, J.B. Zuber, Nucl. Phys. B285, 454 (1987).

[26] A. Cappelli, C. Itzykson, J.B. Zuber, Nucl. Phys. B280, 445 (1987).

[27] B. Nienhuis, E.K. Riedel, M. Schick, Phys. Rev. B27, 5625 (1983)

[28] J. Cardy, Nucl. Phys. B270, 186 (1986).

[29] V. Pasquier, J. Phys. A20, L1229 (1987).

[30] P. Di Francesco, H. Saleur, J.B. Zuber, Nucl. Phys. B300, 393 (1988).

[31] V. Pasquier, Nucl. Phys. B295, 491 (1987).

[32] I. Kostov, Nucl. Phys. B300, 559 (1987).

[33] V.A. Fateev, A.B. Zamolodchikov, Sov. J. Nucl. Phys. 32, 298 (1980).

[34] H. Saleur, to appear in proceedings of "Annecy Conference on Conformal Field Theories"

[35] A. Cappelli, Phys. Lett. 185B, 82 (1987).

[36] V. Pasquier, H. Saleur, Preprint Saclay

[37] L.D. Faddeev, N. Yu Reshetikhin, L.A. Takhtajan, LOMI preprint E-14-84 and Ref. therein.

[38] M. Jimbo, Lett. in Math. Phys. 10, 63 (1985).

[39] V.G. Drinfeld, Dokl. Akad. Nauk. SSSR 283, 1060 (1985).

[40] F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter, G.R.W. Quispel, J. Phys. A20, 6397 (1987).
[41] H. Saleur, M. Bauer, to appear in Nucl. Phys.
[42] A. Rocha Caridi, in "Vertex Operators in Math. and Phys.", Springer Verlag (1985).
[43] V. Pasquier, Comm. in Math. Phys. 118, 355 (1988).
[44] B.L. Feigin, D.B. Fuchs, Moscow Preprint (1983) (unpublished).
[45] V.I. Dotsenko, V.A. Fateev, Nucl. Phys. B240, 312 (1984).
[47] P.J. Forrester, J. Phys. A19, L143 (1986)
[48] H. Saleur, To appear in J. Phys. A Letters.
[49] J. Cardy, Nucl. Phys. B275, 200 (1987).

FIGURE CAPTIONS

Fig.1 : A typical graph in the high temperature expansion of $\mathfrak{Z}_Q$ (2.2) and its alternative polygon representation. After arbitrary orientation, the polygons are considered as walls between regions of constant height in a solid on solid model.

Fig.2 : Vertices of the 6-vertex model.

Fig.3 : These two additional vertices correspond to vortex configurations in the SOS language, with multi valuedness of φ along C.

Figs.4,5 : The local polygon weights are sensitive to curvature. Here a loop around a corner (or a cylinder) does not have the weight $\sqrt{Q}$ but $2 \cos 3\alpha/4$ (2).

Fig.6 : The weights of the SOS model give additional phase factors for polygons encircling extremities.

Fig.7 : Schematic representation of a cluster with cross topology.

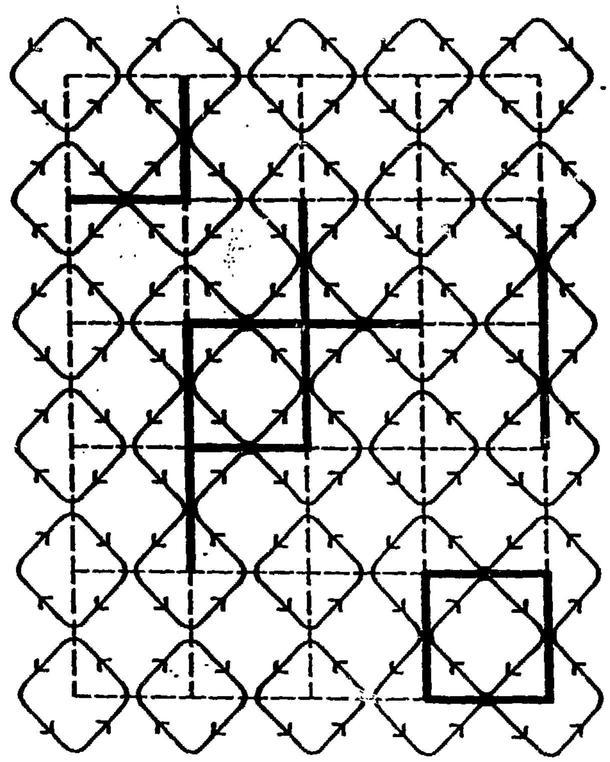


Figure 1

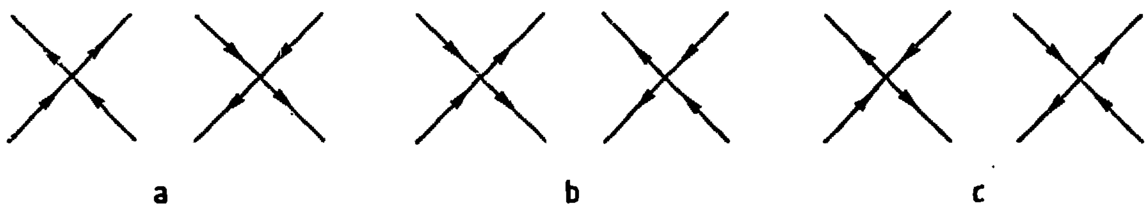
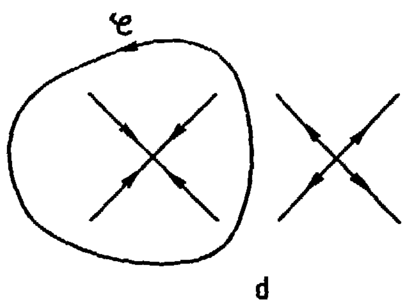


Figure 2



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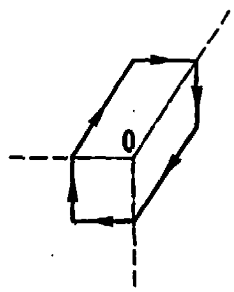

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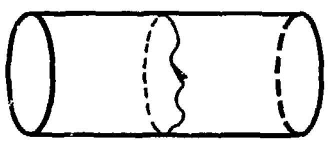

Figure 5

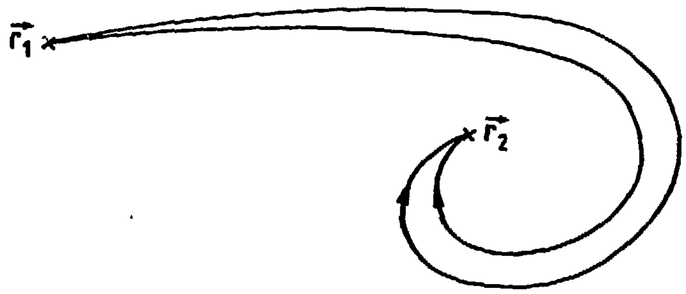

Figure 6

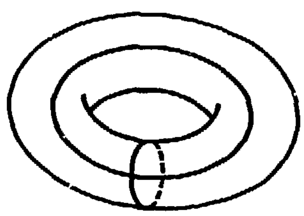

Figure 7