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On Nodes of Local Solutions to Schrödinger Equations

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We shall consider the behaviour of a real valued solution ψ of a Schrödinger equation in the neighbourhood of a zero. Let

$$(-\Delta + V)\psi = 0 \text{ in } \Omega,$$

where Ω is a domain in $\mathbf{R}^n$, $n \geq 2$ and $V \in C^\infty(\Omega)$ with V real valued. (1)

Without loss we assume $O \in \Omega$ and $\psi(O) = 0$.

Note that $\psi \in C^\infty(\Omega)$ by elliptic regularity. Further let $\varepsilon > 0$, $B_\varepsilon = \{x \in \mathbf{R}^n \| x| < \varepsilon\}$ such that $B_\varepsilon \subset \Omega$. In the following we present a survey of results recently obtained in [8] on the behaviour of the nodal set and the nodal domains of ψ in B_ε. Let $\mathcal{N}_\varepsilon = \{x \in B_\varepsilon | \psi(x) = 0\}$, a component D of $B_\varepsilon \setminus \mathcal{N}_\varepsilon$ will be called a *local nodal domain (l.n.d.) of ψ in B_ε* and we define

$$\mathcal{D}_\varepsilon = \{\text{l.n.d. } D \text{ of } \psi \text{ in } B_\varepsilon \text{ with } O \in \partial D\}.$$

Let $C_\varepsilon = \{x \in B_\varepsilon | \psi(x) = 0, \nabla\psi(x) = 0\}$. It is known (see [2,3,4]) that under the above assumptions the manifold $\mathcal{N}_\varepsilon \setminus C_\varepsilon$ is as regular as the solution ψ, and that the Hausdorff dimension of $C_\varepsilon \leq n-2$. So clearly the case $O \in C_\varepsilon$ is the interesting one, and given $D \in \mathcal{D}_\varepsilon$ one may ask whether it satisfies an interior cone condition. Furthermore we shall also deal with the question of the cardinality of $\mathcal{D}_\varepsilon$. To investigate such problems we rely heavily on a result of Bers [1]:

Proposition: Let ψ satisfy (1). Then there exists a harmonic homogeneous polynomial $P_M(x) \not\equiv 0$ of degree $M \geq 1$ such that for $0 < \nu < 1$

$$\frac{\partial^\ell(\psi - P_M)(x)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} = O(|x|^{M-\ell+\nu}) \quad \text{for } |x| \to 0 \tag{2}$$

for $\ell = 0, 1, \ldots M$, where $\sum_{j=1}^n i_j = \ell$.

Using polar coordinates $x = ry$ with $r = |x|$ and $y = x/|x| \in S^{n-1}$, S^{n-1} the $n-1$-dimensional unit sphere, we can write $P_M(ry) = r^M Y_M(y)$ with Y_M some surface harmonic, and (2) implies

$$r^{-M}\psi(ry) \to Y_M(y) \quad \text{for } r \to 0, \quad \text{for } y \in S^{n-1}. \tag{3}$$

We denote the nodal set of Y_M with $\mathcal{N}(Y_M)$ and the set of nodal domains of Y_M with $\mathcal{U}(Y_M)$. (The components of $S^{n-1} \setminus \mathcal{N}(Y_M)$ are called nodal domains of Y_M.)

For dimension $n = 2$ Cheng [3] showed that the nodal lines of ψ look locally as the nodal lines of P_M, which are straight lines intersecting in O and forming an equiangular system. So for ε small, $\mathcal{D}_\varepsilon$ and $\mathcal{U}(Y_M)$ have the same number of elements.

For dimensions $n \geq 3$ the situation is more delicate as can be seen from the following harmonic function in $\mathbf{R}^3$

$$\psi(x_1, x_2, x_3) = x_1x_2 + x_1^2x_3 - x_3^3/3$$

which has a zero in $\mathcal{O}$ of order 2. The corresponding harmonic homogeneous polynomial is $P_2(x_1, x_2, x_3) = x_1x_2$, so obviously $\#\mathcal{U}(Y_2) = 4$. On the other hand studying the intersection of the nodal set of ψ with planes $x_3 = c$ with $c < 0$, $= 0$ and > 0, it is easily seen that ψ has only 2 nodal domains.

To state our results we need the following definition: For $D \in \mathcal{D}_\epsilon$, D arbitrary but fixed, let

$$S(r) = \{y \in S^{n-1} | ry \in D\} \quad \text{for } 0 < r < \varepsilon.$$

Further we denote $|\{\cdot\}| = \int_{\{\cdot\}} d\sigma$, with $d\sigma$ the surface measure on S^{n-1}.

Theorem 1: Let $n \geq 3$, suppose ψ satisfies (1), and let Y_M denote the surface harmonic for which (3) holds. Let $D \in \mathcal{D}_\epsilon$, D arbitrary but fixed, with $S(r)$, $0 < r < \varepsilon$ given as above.

Let $\mathcal{M} \subset S^{n-1}$ denote the union of all nodal domains U of Y_M with the property that there exists a sequence $\{r_m y_m\}$ with $r_m y_m \in D$ for all m, $r_m \to 0$ and $y_m \to \bar{y}$ for $m \to \infty$ for some $\bar{y} \in U$.

Then $\mathcal{M} \neq \emptyset$ and $|\mathcal{M} \setminus S(r) \cup S(r) \setminus \mathcal{M}| \to 0 \quad$ for $r \to 0$.

This result, which implies $|S(r)| \to |\mathcal{M}| \neq 0$ for $r \to 0$ in particular rules out that $S(r)$ "shrinks" for $r \to 0$ into a subset of $\mathcal{N}(Y_M)$. There are some rather immediate consequences:

Corollary:

(i) There exists a cone K with vertex $\mathcal{O}$ and $K \subset D$.

(ii) $\#\mathcal{D}_\epsilon \leq \#\mathcal{U}(Y_M)$, furthermore $\#\mathcal{D}_\epsilon$ is constant for ϵ small enough. ($\#\{\cdot\}$ denotes the cardinality of $\{\cdot\}$.)

(iii) Let $\psi_0^2(r) = \int_{S(r)} \psi^2 d\sigma$ and $\psi_{av}^2(r) = \int_{S^{n-1}} \psi^2 d\sigma$, then

$$r^{-M}\psi_0 \to \left(\int_{\mathcal{M}} Y_M^2 d\sigma\right)^{1/2} > 0$$

and

$$\psi_0/\psi_{av} \to \left(\int_{\mathcal{M}} Y_M^2 d\sigma \Big/ \int_{S^{n-1}} Y_M^2 d\sigma\right)^{1/2} \quad \text{for } r \to 0.$$

These findings show that the local properties of ψ in the neighborhood of a zero are determined to a certain extent by global properties of the nodal set of the corresponding surface harmonic.

Of course it would be desirable to study the local behaviour of the nodal domains of ψ with weaker regularity assumptions on V. It would be also of interest to extend the foregoing results appropriately to the case where the Laplacian is replaced by more general elliptic operators.

Let us sketch the main idea of the proof of Theorem 1 (for the full proof see [8]): The difficult part of the proof is to verify that $\mathcal{M} \neq \emptyset$. For this purpose we investigate the asymptotic behaviour of $\psi_0(r) = (\int_{S(r)} \psi^2 d\sigma)^{1/2}$ for $r \to 0$ and proceed similarly as in [6] where we studied the asymptotics of a solution ψ of a Schrödinger equation for $r \to \infty$. We suppose indirectly that $\mathcal{M} = \emptyset$, which implies

$$|S(r)| \to 0 \quad \text{for } r \to 0. \tag{6}$$

Let for $0 < r < \varepsilon$, ε small

$$\lambda^2(r) = \inf_{\varphi \in C_0^\infty(S(r))} \int |L\varphi|^2 d\sigma / \int |\varphi|^2 d\sigma$$

where $-L^2$ denotes the Laplace-Beltrami operator on S^{n-1}, then we obtain from (6)

$$\lambda^2(r) \to \infty \quad \text{for } r \to 0. \tag{7}$$

It can be shown that $\tilde{\psi}_0 = r^{(n-1)/2}\psi_0$ satisfies

$$\left(-\frac{d^2}{dr^2} + \inf_{\nu \in S^{n-1}} V + \frac{(n-1)(n-3)}{4r^2} + \frac{\lambda^2(r)}{r^2}\right)\tilde{\psi}_0 \leq 0 \quad \text{for } 0 < r < \varepsilon \tag{8}$$

in the distributional sense. The proof of the following inequality is rather involved (compare Lemma 3.2 in [6]) and we note that the C^∞-assumption on V plays an essential role here. We have for some $C > 0$

$$\begin{aligned} &\tilde{\psi}_0 \geq C\left(\frac{r}{\lambda(r)}\right)^{2\gamma} \quad \text{for } r < \varepsilon \\ &\text{with } 2\gamma = M + \frac{n-1}{2}. \end{aligned} \tag{9}$$

Now we take into account (7) and obtain from inequality (8) by standard comparison techniques that

$$\tilde{\psi}_0 = O(r^m) \quad \text{for } r \to 0 \text{ for all } m \in \mathbb{N}. \tag{10}$$

On the other hand combination of (8) and (9) yields

$$-\tilde{\psi}_0'' + \tilde{\psi}_0^{1-2\alpha} \leq 0 \quad \text{for } 0 < r < R_\alpha \tag{11}$$

for some $\alpha < (2\gamma)^{-1}$ and some $R_\alpha < \varepsilon$. Finally a further investigation of this nonlinear differential inequality shows that $\tilde{\psi}_0$ vanishes polynomially in O contradicting (10). Hence $\mathcal{M} \neq \emptyset$.

In connection with the results given here more detailed questions about the local behaviour of nodal sets arise in a natural way. For instance one might ask whether $S(r)$ is connected for $r > 0$ sufficiently small. This we could not answer, but for dimension $n = 3$ we obtained more detailed results about the set $\mathcal{M}$.

Theorem 2: Let $n = 3$ and suppose that the assumptions of Theorem 1 hold. Let $\{r_m y_m\}$ be a sequence with $r_m y_m \in D$, $\forall m$ and $r_m \to 0$, $y_m \to \bar{y}$ for $m \to \infty$ for some $\bar{y} \in S^{n-1}$, then $\bar{y} \in \bar{\mathcal{M}}$. Furthermore $\bar{\mathcal{M}}$ is connected ($\bar{\mathcal{M}}$ denotes the closure of $\mathcal{M}$).

Note that the case $Y_M(\bar{y}) \neq 0$ is trivial since per definition $\bar{y} \in \mathcal{M}$. To verify that $\bar{y} \in \bar{\mathcal{M}}$, if $Y_M(\bar{y}) = 0$ we used the fact that on S^2 there are only finitely many zeros of order greater one of Y_M.

Though we could not prove it we believe that Theorem 2 holds also for dimensions > 3.

As already noted the methods for the main part of the proof of Theorem 1 have been developed in [6] to investigate the asymptotic behaviour of nodes of solutions of Schrödinger equations in exterior domains for dimension $n \geq 3$. We remark that there the situation is much more complex than here, even for the 2-dimensional case it is rather delicate (see [5]). For a survey on these results see also [7].

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