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Bounds on the Order of Vanishing of Eigenfunctions of Schrödinger Operators

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1 Introduction

We consider real valued L^2-solutions ψ, $\psi \not\equiv 0$, of the Schrödinger equation

$$(-\Delta + V - E)\psi = 0 \quad \text{in } \mathbf{R}^n, \quad n \geq 2 \tag{1.1}$$

with suitable assumptions on $V : \mathbf{R}^n \to \mathbf{R}$ and with E real. For $\psi \in L^2_{loc}(\mathbf{R}^n) \cap C^0(\mathbf{R}^n)$ with $\psi(x_0) = 0$ for some $x_0 \in \mathbf{R}^n$ we define the order of vanishing of ψ in x_0 by

$$\ell_{x_0}(\psi) := \sup\{\alpha : \overline{\lim_{R\downarrow 0}}\, R^{-2\alpha-n} \int_{B_R(x_0)} \psi^2 dx < \infty\}, \tag{1.2}$$

where $B_R(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < R\}$. If ψ is C^∞ in a neighbourhood of x_0, then this definition coincides with the usual one, namely

$$\ell_{x_0}(\psi) = \inf\{|\beta| : (D^\beta \psi)(x_0) \neq 0\},$$

where β denotes a multiindex $(\beta_1, \dots, \beta_n)$, $|\beta| = \sum_{i=1}^n \beta_i$ and $D^\beta = \partial^{|\beta|}/(\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n})$.

From unique continuation theorems it is known [11] for a very general class of potentials that such solutions ψ do not vanish locally of infinte order, so that $\ell_{x_0}(\psi) < \infty$. Hence bounds to $\ell_{x_0}(\psi)$ might be considered as quantitative versions of unique continuation results. Such estimates were recently obtained by H. Donnelly and Ch. Fefferman [8] for Laplacians on compact Riemannian manifolds. They consider $-\Delta u_k = \lambda_k u_k$ on such a manifold M, where $\lambda_k \to \infty$ for $k \to \infty$, and obtain the asymptotically optimal estimate

$$\sup_{x \in M} \ell_x(u_k) \leq C\sqrt{\lambda_k} \tag{1.3}$$

with some constant $C = C(M) < \infty$. In their proof the compactness of M is used. For solutions to the Schrödinger equation in all of $\mathbf{R}^n$ the situation is quite different. In particular it is not even clear when $\sup\{\ell_x(\psi) : x \in \mathbf{R}^n\}$ is bounded for a fixed solution ψ of (1.1).

To get some intuition we consider the Schrödinger equation of the Hydrogen atom (in suitable units)

$$(-\Delta - \frac{2}{|x|} - E_k)\psi_k = 0 \quad \text{in } \mathbf{R}^3, \tag{1.4}$$

where $E_k = -k^{-2}$, $k \in \mathbf{N}$ and the eigenvalues E_k are k^2-fold degenerate. The corresponding real valued eigenfunctions can be written as (see any textbook in quantum mechanics)

$$\psi_k(x) = \sum_{\ell=0}^{k-1} \sum_{m=-\ell}^{\ell} c_{m,\ell,k} f_{\ell,k}(r) Y_{\ell,m}(\frac{x}{r})$$

with $r = |x|$, where the $Y_{\ell,m}$ are the usual orthonormalized surface harmonics. The $f_{\ell,k}$ satisfy in $\mathbf{R}^+$

$$-f''_{\ell,k} - \frac{2}{r} f'_{\ell,k} + (\frac{\ell(\ell+1)}{r^2} - \frac{2}{r} + k^{-2}) f_{\ell,k} = 0 \tag{1.5}$$

($'$ denoting d/dr), and $f_{\ell,k} \sim r^\ell$ for $r \to 0$. The $c_{m,\ell,k}$ are real constants.

Clearly we have for $x_0 = \mathcal{O}$

$$\ell_{\mathcal{O}}(\psi_k) \le k - 1 = \frac{1}{\sqrt{|E_k|}} - 1 \quad \text{for } k \in \mathbf{N}. \tag{1.6}$$

Since $E_k \to 0$ for $k \to \infty$ the above particularly implies that $\ell_{\mathcal{O}}(\psi_k)$ may tend to infinity for $k \to \infty$. Another explicit example showing such a behaviour, but with $E_k \to \infty$ for $k \to \infty$, is the Schrödinger equation for the n-dimensional harmonic oscillator $(-\Delta + |x|^2 - E_k)\psi_k = 0$. Note that for any reasonable potential $V = V(|x|)$ for which $-\Delta + V$ has infinite discrete spectrum $\{E_k, k \in \mathbf{N}\}$, eigenfunctions ψ_k, $k \in \mathbf{N}$ exist, such that $\ell_{\mathcal{O}}(\psi_k) \to \infty$ for $k \to \infty$. The hydrogenic case is a particularly illustrative one and it will be referred to below.

In Section 2 we study L^2-solutions of (1.1) and derive a sharp upper bound to $\ell_{x_0}(\psi)$ in terms of x_0, E and V, under unfortunately rather restrictive assumptions on V. The proof is partly based on methods developed in [9]. In section 3 we show for V smooth that an upper bound to $\ell_{x_0}(\psi)$ implies an upper bound to the dimension of the eigenspace associated to E.

2 Statement of the Results

From now on we assume that $V : \mathbf{R}^n \to \mathbf{R}$, $n \ge 2$ has the following properties:

$$\left.\begin{array}{l} \text{For some } x_0 \in \mathbf{R}^n,\ V \in C^1(\mathbf{R}^n \setminus \{x_0\}) \text{ and} \\ |x - x_0|^{p-\delta} V,\ |x - x_0|^{p-\delta}(x - x_0)\cdot\nabla V \in L^\infty_{loc}(\mathbf{R}^n) \\ \text{for some } \delta > 0, \text{ where } p = 2 \text{ for } n \ge 4 \\ \text{and } p = n/2 \text{ for } n = 2, 3. \end{array}\right\} \tag{A}$$

In addition we assume that either

$$V \to 0, \quad r|\nabla V| \to 0 \quad \text{for } r \to \infty \tag{$B1$}$$

or

$$\left.\begin{array}{l} \text{for some } \beta_0, \beta > 0 \text{ with } \beta_0 \le \beta \text{ there is an } R > 0 \\ \text{such that } C_1 r^{\beta_0} \le V \le C_2 r^{\beta} \text{ and} \\ C_1 r^{\beta_0} \le |(x - x_0)\cdot\nabla V| \le C_2 r^{\beta} \text{ for } r \ge R \\ \text{with some } C_1, C_2 > 0. \end{array}\right\} \tag{$B2$}$$

Our assumptions on V imply (via quadratic form techniques) that there is a unique semibounded self adjoint operator H associated to $-\Delta + V$ with core $C_0^\infty(\mathbf{R}^n)$ (see e.g. [16]). Note that if V obeys (B1) then clearly $\sigma_{ess} H = [0, \infty)$ and (1.1) has no L^2-solutions with $E > 0$, and if V obeys (B2) then H has only discrete spectrum (see e.g. [15]).

We remark that each eigenfunction ψ of H is Hölder continuous, $\psi \in W^{2,2}(\mathbf{R}^n)$ and

$$e^{\alpha r}\psi \in L^2(\mathbf{R}^n) \quad \text{for some } \alpha > 0 \tag{2.1}$$

(see [15,16]). This and (1.1) implies

$$x \cdot \nabla\psi \in L^2(\mathbf{R}^n). \tag{2.2}$$

In addition our assumptions on V imply via unique continuation [11] that

$$\ell_{x_0}(\psi) < \infty \quad \text{for } x_0 \in \mathbf{R}^n. \tag{2.3}$$

Theorem 2.1 Let $V : \mathbf{R}^n \to \mathbf{R}$, $n \geq 2$ satisfy condition (A) and either (B1) or (B2). Assume that $\psi \in L^2(\mathbf{R}^n)$, $\psi \not\equiv 0$, is a real valued solution of (1.1).

Define for $k \in \mathbf{R}^+$

$$\lambda_{x_0}(k) = \inf_{\varphi \in C_0^\infty(\mathbf{R}^2)} \left(\int_{\mathbf{R}^2} (k|\nabla\varphi|^2 + W_{x_0}(|x|)|\varphi|^2)dx \Big/ \int_{\mathbf{R}^2} |\varphi|^2 dx \right) \tag{2.4}$$

where $W_{x_0} : [0,\infty) \to \mathbf{R}$ satisfies the following conditions:

$$\left.\begin{array}{l} r^{2-\delta}W_{x_0}(r) \to 0 \text{ for } r \to 0 \text{ for some } \delta > 0, \\ \sup(-W_{x_0}, 0) \text{ is bounded for } r > 0, \text{ and} \\ W_{x_0}(|x - x_0|) \leq V(x) + \frac{1}{2}(x - x_0) \cdot \nabla V(x) \text{ for } x \in \mathbf{R}^n. \end{array}\right\} \tag{2.5}$$

Suppose that $\ell_{x_0}(\psi) > 0$, then

(i) there exists a unique $k_0 > n - 2$ such that

$$\lambda_{x_0}(k_0) = E \quad \text{and} \quad \ell_{x_0}(\psi) \leq \frac{1}{2}(k_0 - n + 2). \tag{2.6}$$

(ii) Inequality (2.6) is best possible in the following sense: Let for $\ell \in \mathbf{N}$ and $x \in \mathbf{R}^n$, $r = |x|$

$$\left.\begin{array}{l} V(r) = (2\ell + n - 2)^2(r^{4\ell+2n-6} - 2r^{2\ell+n-4}), \\ E = 0 \text{ and } W_O = V + \frac{1}{2}rV' \end{array}\right\} \tag{2.7}$$

then

$$\psi(x) = r^\ell \exp(-r^{2\ell+n-2})Y_\ell(\frac{x}{r}) \tag{2.8}$$

(with Y_ℓ a spherical harmonic of degree ℓ) is an L^2-solution of (1.1) with V and E given in (2.7),

$$\ell_O(\psi) = \ell \text{ and } \lambda_O(k_0) = 0 \text{ with } k_0 = 2\ell + n - 2. \tag{2.9}$$

Remark 2.1

a) The assumptions on V guarantee the existence of a W_{x_0} obeying (2.5).

b) Theorem 2.1 particularly implies

$$\lambda_{x_0}(2\ell_{x_0}(\psi) + n - 2) \leq E. \tag{2.10}$$

Since it is straightforward and might be illustrative we now give the

Proof of Theorem 2.1 (ii): Let $\varphi(r) = \exp(-r^{2\ell+n-2})$ and note that $L^2Y_\ell = \ell(\ell + n-2)Y_\ell$, where L^2 denotes the Laplace-Beltrami operator on the unit sphere S^{n-1}. Then with $\psi = r^\ell \varphi Y_\ell$ according to (2.8),

$$\Delta\psi = (\varphi'' + \frac{n-1+2\ell}{r}\varphi')\varphi^{-1}\psi$$

where with $k_0 = 2\ell + n - 2$

$$\varphi' = -k_0 r^{k_0-1}\varphi, \text{ and } \varphi'' = k_0(k_0 r^{2k_0-2} - (k_0-1)r^{k_0-2})\varphi.$$

Therefrom $\Delta\psi = V\psi$ follows with V given in (2.7). On the other hand it is easily seen that

$$-\varphi'' - \frac{1}{r}\varphi' + \frac{1}{k_0}W_O\varphi = 0$$

implying (2.9). □

Remark 2.2

a) Away from x_0 we could have allowed for rather mild singularities of V and hence of W_{x_0} (compare [9]). But since we were not able to handle the physically interesting case of a one-electron molecule with fixed nuclei, we refrained from doing so.

b) For dimension $n = 2, 3$ the assumption (A) could be probably relaxed allowing for an $|x - x_0|^{-2+\delta}$ singularity, but then the approximation arguments in the proof of (2.6) would become more involved.

Remark 2.3: Even for $V \in C^\infty(\mathbf{R}^n)$ our methods do not lead to a bound to $\sup\{\ell_{x_0}(\psi) : x_0 \in \mathbf{R}^n\}$. Such global estimates seem to be out of reach at least for $n \geq 3$.

Remark 2.4: For the 2-dimensional case there is for smooth V another, topological approach, which leads to an upper bound to $\ell_{x_0}(\psi)$: By a result of L. Bers [2] and via a suitable version of Euler's theorem on polyhedra [1,12] the number of nodal domains of ψ is greater than or equal to $\ell_{x_0}(\psi) + 1$. This together with Courant's nodal theorem [7,11] implies

$$\ell_{x_0}(\psi_j) \leq j - 1 \quad \text{for } x_0 \in \mathbf{R}^n, \qquad j \in \mathbb{N} \tag{2.11}$$

with the corresponding eigenvalues E_j ordered in a nondecreasing sequence.

Next we consider Theorem 2.1 for potentials which can be estimated by some polynomial r^α and obtain in (2.6) an explicit dependence on the eigenvalue:

Corollary 2.1: Let the assumptions of Theorem 2.1 hold. Let $x_0 = O$ and assume that in $\mathbf{R}^n$

$$\left.\begin{array}{l} V + \frac{1}{2}x \cdot \nabla V \geq c_\alpha r^\alpha \text{ for some } \alpha > -2,\ \alpha \neq 0 \\ \text{and some } c_\alpha \text{ with } \operatorname{sgn} c_\alpha = \operatorname{sgn} \alpha. \end{array}\right\} \tag{2.12}$$

Denote

$$\mu_\alpha = \inf_{\varphi \in C_0^\infty(\mathbf{R}^2)} \left(\int_{\mathbf{R}^2} (|\nabla\varphi|^2 + c_\alpha r^\alpha |\varphi|^2) dx \Big/ \int_{\mathbf{R}^2} |\varphi|^2 dx \right) \tag{2.13}$$

and suppose that sgn μ_α = sgn E. Then

$$\ell_O(\psi) \leq \frac{1}{2}\left(\left| \frac{E}{\mu_\alpha} \right|^{1+2\alpha^{-1}} - n + 2 \right). \tag{2.14}$$

Proof of Corollary 2.1: Define for $k \in \mathbf{R}^+$

$$\mu_\alpha(k) = \inf_{\varphi \in C_0^\infty(\mathbf{R}^2)} \int_{\mathbf{R}^2} (k|\nabla\varphi|^2 + c_\alpha r^\alpha |\varphi|^2) dx \Big/ \int_{\mathbf{R}^2} |\varphi|^2 dx,$$

then $\mu_\alpha(1) = \mu_\alpha$. By scaling we obtain

$$\mu_\alpha(k) = k^{\alpha/(\alpha+2)} \mu_\alpha.$$

So for $k = k_0$ with $\mu_\alpha(k_0) = E$,

$$k_0 = \left| \frac{E}{\mu_\alpha} \right|^{1+2\alpha^{-1}}$$

follows, if sgn μ_α = sgn E. Inserting into (2.6) yields (2.14). □

Remark 2.5: Suppose we know that $\ell_{x_0}(\psi)$ is integer, then the upper bounds to $\ell_{x_0}(\psi)$ in (2.6) and (2.14) can be replaced by their integer parts. Though this seems likely for a very general class of potentials, we are just aware of a result of L. Caffarelli and A. Friedman [4] which implies that $\ell_{x_0}(\psi)$ is integer under the assumption that

$$|\Delta\psi| \leq C_1|\psi|^\alpha + C_2|\nabla\psi|^\gamma$$

in a neighbourhood of x_0, for some $C_1, C_2 \geq 0$ and $\alpha, \gamma \geq 1$.

Remark 2.6: For the hydrogenic case (compare Section 1) (2.14) implies $\ell_O(\psi_E) = O(1/|E|)$ for $|E| \to 0$ instead of $O(1/\sqrt{|E|})$. In fact if we consider

$$(-\Delta + c_\alpha r^\alpha - E)\psi_E = 0 \quad \text{in } \mathbf{R}^n$$

with sgn α = sgn c_α, $\alpha \neq 0$, $\alpha > -2$, a simple ODE-analysis shows that

$$\ell_O(\psi_E) = O(|E|^{\frac{1}{2}+\frac{1}{\alpha}})$$

for $E \nearrow 0$ if $\alpha < 0$ resp. for $E \nearrow \infty$ if $\alpha > 0$. This asymptotic behaviour is different from that in (2.14) and we do not know (but doubt it) whether there are potentials such that (2.14) shows the correct asymptotics.

Now we give the

Proof of Theorem 2.1(i): Without loss we take $x_0 = O$ and suppose that $\ell_O(\psi) = \ell > 0$.

We first show

Proposition 2.1: For all $\gamma > 0$

$$r^{-\ell-n/2+\gamma}\psi, \quad r^{-\ell-n/2+1+\gamma}|\nabla\psi| \quad \in L^2(\mathbf{R}^n). \tag{2.15}$$

Proof of Proposition2.1: Since $\ell_O(\psi) = \ell > 0$ we have according to (1.2)

$$\overline{\lim_{R\downarrow 0}}\, R^{-2\ell-n+2\gamma}\int_{B_R}\psi^2 dx < \infty \quad \text{for all } \gamma > 0 \tag{2.16}$$

with $B_R = \{x \in \mathbf{R}^n : |x| < R\}$. Let $2\gamma \neq 2\ell + n$ and choose $0 < \nu < \varepsilon$, then we obtain by partial integration

$$\int_\nu^\varepsilon R^{-2\ell-n+2\gamma-1}\int_{B_R}\psi^2 dx dR + \frac{R^{-2\ell-n+2\gamma}}{2\ell+n-2\gamma}\int_{B_R}\psi^2 dx\Big|_\nu^\varepsilon =$$

$$= \int_\nu^\varepsilon \frac{R^{-2\ell-n+2\gamma}}{2\ell+n-2\gamma}\Big(\int_{S^{n-1}}\psi^2 d\sigma\Big)(R)R^{n-1}dR.$$

Taking the limit $\nu \to 0$

$$\int_{B_\varepsilon}\psi^2 r^{-2\ell-n+2\gamma}dx \le c(\varepsilon,\ell,\gamma) < \infty \quad \text{for } \gamma > 0 \tag{2.17}$$

results because of (2.16). For $2\gamma = 2\ell + n$ (2.17) is trivial. (2.17) together with the exponential decay of ψ (compare (2.1)) implies $r^{-\ell-n/2+\gamma}\psi \in L^2(\mathbf{R}^n)$ for $\gamma > 0$, verifying the first part of Proposition 2.1.

Now let $f \in C^2(\mathbf{R}^n)$ with $f \ge 0$, radially symmetric, where f, $|\nabla f|$ and $|\Delta f|$ are polynomially bounded for $r \to \infty$. Then taking into account (1.1) we easily obtain by partial integration

$$\int f\psi(-\Delta + V - E)f\psi dx = \|f'\psi\|^2$$

and further

$$\|\nabla f\psi\|^2 = \|f'\psi\|^2 + \int (E-V)f^2\psi^2 dx. \tag{2.18}$$

But by Cauchy-Schwarz and the arithmetic-geometric inequality

$$\|\nabla f\psi\|^2 \ge \frac{1}{2}\|f\nabla\psi\|^2 - \|f'\psi\|^2 \tag{2.19}$$

so that

$$\|f\nabla\psi\|^2 \le 4\|f'\psi\|^2 + 2\int (E-V)f^2\psi^2 dx \tag{2.20}$$

results. Now we choose for f functions f_μ, $\mu > 0$ defined in the following

$$\left.\begin{array}{l} f_\mu(r) = (r^2+\mu^2)^{-m/2}\chi(r) \text{ with } m = \ell + \frac{n}{2} - 1 - \gamma \\ \text{where } \chi \in C_0^\infty(B_2),\ \chi = 1 \text{ in } B_1,\ \chi > 0 \text{ and} \\ \text{radially symmetric.} \end{array}\right\} \tag{2.21}$$

Then (2.20) yields for $\mu > 0$

$$\begin{aligned}\|f_\mu \nabla\psi\|^2 &\leq 8(\|\chi'(r^2+\mu^2)^{-m/2}\psi\|^2 + m^2\|r(r^2+\mu^2)^{-m/2-1}\chi\psi\|^2) + \\ &\quad + 2\int (E-V) f_\mu^2 \psi^2 dx \leq \\ &\leq 8(\|\chi' r^{-m}\psi\|^2 + m^2\|r^{-m-1}\chi\psi\|^2) + 2\|\sqrt{|E-V|}r^{-m}\chi\psi\|^2 \leq c(m) < \infty\end{aligned}$$

where we used assumption (A) on V and (2.17). For $\mu \downarrow 0$ we obtain therefrom by the monotone convergence theorem $r^{-m}\chi\nabla\psi \in L^2(B_2)$ and hence

$$r^{-\ell-n/2+1-\gamma}|\nabla\psi| \in L^2_{loc}(\mathbf{R}^n) \quad \text{for } \gamma > 0.$$

Finally to verify that

$$r^{-\ell-n/2+1+\gamma}|\nabla\psi| \in L^2(\mathbf{R}^n) \quad \text{for } \gamma > 0$$

we choose

$$\left.\begin{array}{l} f_\mu(r) = (\frac{r}{1+\mu^2 r})^m \chi(r) \text{ with } m > 0 \text{ and} \\ \chi \in C^\infty(\mathbf{R}^n), \text{ bounded, radially symmetric}, \chi \geq 0 \\ \text{supp}\, \chi \subset (\mathbf{R}^n \setminus B_1) \text{ and } \chi = 1 \text{ for } r > R > 1. \end{array}\right\} \tag{2.22}$$

Again inserting into (2.20) we arrive at

$$\begin{aligned}\|f_\mu\nabla\psi\|^2 &\leq 4\|(\chi' + \chi(1+\mu^2 r)^{-1})(\frac{r}{1+\mu^2 r})^m\psi\|^2 + 2\|\sqrt{|E-V|}f_\mu\psi\|^2 \leq \\ &\leq C(R)(\|\psi\|^2 + \|r^m\chi\psi\|^2) + 2\|\sqrt{|E-V|}r^m\chi\psi\|^2 \leq const < \infty\end{aligned}$$

where we used assumption (B1) resp. (B2) on V and (2.1). By the monotone convergence theorem we obtain for $\mu \to 0$ the desired result. □

Our main tool to verify Theorem 2.1 is the following

Proposition 2.2: Let $A = x \cdot \nabla + n/2$ be defined on $C_0^\infty(\mathbf{R}^n)$ and let $f \in C^\infty(\mathbf{R}^n)$ be strictly positive and radially symmetric. Suppose that V satisfies condition (A). Then for $\phi \in C_0^\infty(\mathbf{R}^n)$

$$\begin{aligned}&\text{Re}\,((A-1)f\phi, f(-\Delta+V-E)\phi) = \\ &= 2((r^{(n-2)/2}f\phi)', r^{3-n}f'f^{-1}(r^{(n-2)/2}f\phi)') + \\ &+\tfrac{1}{2}(\phi, (-4rf'^3f^{-1} - rff''' + 5rf'f'' - 3ff'' + 5f'^2 - r^{-1}ff')\phi) - \\ &-(f\phi, (V + \tfrac{1}{2}x\cdot\nabla V)f\phi) + E(f\phi, f\phi)\end{aligned} \tag{2.23}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbf{R}^n)$.

Proof of Proposition 2.2: In [9] identity (2.23) was derived for $\phi \in C_0^\infty(\mathbf{R}^n \setminus B_R)$, $R > 0$ under the assumptions $V, x \cdot \nabla V \in L^1_{loc}(\mathbf{R}^n)$. Taking into account assumption (A) with $x_0 = \mathcal{O}$ it is straightforward to verify that (2.23) holds for $\phi \in C_0^\infty(\mathbf{R}^n)$. □

Now suppose that identity (2.23) still holds if we insert formally $\phi = \psi$ and $f(r) = r^{-(2\ell+n-2-\nu)/2}$ with $\nu > 0$ small. Then we have

Lemma 2.1:

$$\left.\begin{aligned} &((-A+1)r^{-k/2}\psi, r^{-k/2}(-\Delta + V - E)\psi) = 0 = \\ &= k(y', r^{2-n}y') + (y, (V - E + \tfrac{1}{2}x \cdot \nabla V)r^{-n+2}y) \\ &\text{where } k = 2\ell + n - 2 - \nu \ (\nu > 0 \text{ small}), \\ &y = r^{(n-2-k)/2}\psi, \text{ and }' \text{ denotes } \partial/\partial r \end{aligned}\right\} \tag{2.24}$$

The proof of Lemma 2.1, given in an Appendix, follows by standard approximation arguments from Proposition 2.2.

Rewriting (2.24) in polar coordinates gives

$$\int_{S^{n-1}} \int_0^\infty (ky'^2 + (V - E + \frac{1}{2}rV')y^2)r\,dr\,d\sigma = 0 \tag{2.25}$$

where $d\sigma$ denotes integration over the unit sphere S^{n-1}.

That the above integrals are finite can be easily seen: The assumptions on V imply that

$$\begin{aligned} &\textstyle\int_{S^{n-1}} \int_0^\infty (ky'^2 + |V - E + \frac{1}{2}rV'|y^2)r\,dr\,d\sigma \le \\ &\le const \textstyle\int_{\mathbf{R}^n} [r^{-k-2}(\psi^2 + r^2\psi'^2) + r^{-k}(r^{-p+\delta} + r^\beta + |E|)\psi^2]dx \end{aligned} \tag{2.26}$$

and the r.h.s. is finite due to Proposition 2.1.

Now we define

$$y_{av}^2(r) = (\int_{S^{n-1}} y^2 d\sigma)(r)$$

and note that (because of Cauchy-Schwarz)

$$y'_{av}{}^2 \le \int_{S^{n-1}} y'^2 d\sigma. \tag{2.27}$$

Application of (2.27) and (2.5) to equation (2.25) leads to

$$\int_0^\infty (ky'_{av}{}^2 + (W_{\mathcal{O}} - E)y_{av}^2)r\,dr \le 0.$$

Let $\eta : \mathbf{R}^2 \to \mathbf{R}$ be given by $\eta(x) = y_{av}(|x|)$, then clearly

$$\lambda_{\mathcal{O}}(k) \le \int_{\mathbf{R}^2} (k|\nabla\eta|^2 + W_{\mathcal{O}}(|x|)\eta^2)dx \Big/ \int_{\mathbf{R}^2} |\eta|^2 dx \le E$$

follows, and therefore $\lambda_{\mathcal{O}}(2\ell + n - 2 - \nu) \le E$ for all $\nu > 0$ small enough. The strict monotonicity of $\lambda_{\mathcal{O}}(k)$ in k implies that there is a unique k_0 with $\lambda_{\mathcal{O}}(k_0) = E$ and $2\ell + n - 2 \le k_0$. □

3 Multiplicity of Eigenvalues and Order of Vanishing of Eigenfunctions

We consider again L^2-solutions of (1.1), but assume now that $V \in C^\infty(\mathbf{R}^n)$ in addition to the assumptions (A) and (B1) or (B2). This implies that $\psi \in C^\infty(\mathbf{R}^n)$. Let $E \in \sigma_{disc}H$ and let N_E denote the corresponding eigenspace with multiplicity ν_E. If $\psi \in N_E$, then $\ell_{x_0}(\psi)$ and ν_E are related by a simple inequality. This will follow from the observation that one can construct an eigenfunction ψ by linear combinations of $\psi_i \in N_E$, $1 \le i \le \nu_E$, whose order of vanishing at a given arbitrary x_0 depends on ν_E.

Theorem 3.1: Let V and E satisfy the above assumptions and define for $x_0 \in \mathbf{R}^n$

$$\ell(x_0; E) = \sup\{\ell \in \mathbf{N} \mid \exists \psi \in N_E \text{ with } \ell_{x_0}(\psi) = \ell\}, \tag{3.1}$$

then $\ell(x_0; E)$ is finite and

$$\nu_E \le \inf_{x_0 \in \mathbf{R}^n} \mathcal{C}(\ell(x_0; E)) \tag{3.2}$$

where

$$\mathcal{C}(\ell) = \frac{(n+\ell-2)!(n+2\ell-1)}{(n-1)!\ell!}. \tag{3.3}$$

Vice versa, if $M \in \mathbf{N}$ is chosen such that

$$\mathcal{C}(M-1) \le \nu_E, \tag{3.4}$$

then for each x_0 there is a $\psi \in N_E$ with

$$\ell_{x_0}(\psi) = M. \tag{3.5}$$

Remark 3.1:

a) It might be possible that such observations already exist in the literature since the proofs are elementary.

b) Of course, bounds like (3.2) hold under less restrictive conditions on V. For instance, if V is only smooth in a subset $\Omega \subset \mathbf{R}^n$, then (3.2) still holds if we take the infimum over Ω.

c) For the hydrogenic case considered in Section 1 it can be easily seen that we have equality in (3.2), though Theorem 3.1 does not apply for $x_0 = \mathcal{O}$.

Naturally we can replace $\ell(x_0; E)$ in (3.2) by an upper bound. Specifically we have from Corollary 2.1

Corollary 3.1: Suppose that V and E satisfy the conditions of Corollary 2.1 and that V is C^∞ in a neighbourhood of the origin, then

$$\nu_E \leq C([\frac{1}{2}\left|\frac{E}{\mu_\alpha}\right|^{1+2\alpha^{-1}} - n + 2]) \tag{3.6}$$

with $C(\cdot)$ defined in (3.3), and where $[\cdot]$ denotes integer part of $\cdot$.

Remark 3.2: If $\sigma_{disc}H$ consists of infinitely many points, then (3.6) implies

$$\nu_E = O(|E|^{(2+\alpha)(n-1)/\alpha}) \tag{3.7}$$

as $E \nearrow \infty$ for $\alpha > 0$, or as $E \nearrow 0$ for $\alpha < 0$, and this asymptotic bound shows the same shortcomings as the bound to $\ell_{x_0}(\psi)$ in Corollary 2.1, which was discussed in Remark 2.6.

There is a rich literature on bounds to multiplicities of eigenvalues (a recent reference is for instance [3]), and the 2-dimensional case is special (see [3,6,14]). For n-dimensional compact Riemannian manifolds there is a bound due to Li (see [13,5]) which is a little in the spirit of our findings.

Proof of Theorem 3.1: We first show that (3.4) implies (3.5): Let $\{\psi_1, \ldots, \psi_{\nu_E}\}$ span N_E, fix $x_0 \in \mathbf{R}^n$ and suppose that M satisfies (3.4). Obviously it suffices to show that the homogeneous system of linear equations

$$\left.\begin{aligned} &\textstyle\sum_{i=1}^{\nu_E} c_i (D^\alpha \psi_i)(x_0) = 0, \quad \forall\, \alpha \in I_m, \quad 0 \leq m \leq M-1 \\ &\text{where } I_m = \{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbf{N}_0 \ \forall\, i \text{ with } \textstyle\sum_{i=1}^n \alpha_i = m\} \end{aligned}\right\} \tag{3.8}$$

has a non trivial solution $(c_1, \ldots, c_{\nu_E}) \in \mathbf{R}^{\nu_E}$. The cardinality of I_m, denoted by d_m, is given by

$$d_m = \binom{n+m-1}{n-1}, \tag{3.9}$$

and we have therefore $\sum_{m=0}^{M-1} d_m$ equations in (3.8). We will show now that for $m \geq 2$ the number of equations can be reduced because some of them are linearly dependent, since

$$(-\Delta + V - E)\sum_{i=1}^{\nu_E} c_i \psi_i = 0 \quad \text{in } \mathbf{R}^n. \tag{3.10}$$

Hence for $m = 2$ the number of equations in (3.8) reduces from d_2 to $d_2 - d_0$ ($d_0 = 1$). Now let $\alpha \in I_{m-2}$, $3 \leq m \leq M-1$, then we obtain from (3.10)

$$-\sum_{j=1}^{n}\sum_{i=1}^{\nu_E} c_i (\frac{\partial^2}{\partial x_j^2} D^\alpha \psi_i)(x_0) + \sum_{i=1}^{\nu_E} c_i (D^\alpha (V\psi_i))(x_0) = 0, \tag{3.11}$$

where the second term vanishes since

$$\sum_{i=1}^{\nu_E} c_i(D^\beta\psi_i)(x_0) = 0 \quad \text{for } \beta \in I_k \text{ with } k \leq m.$$

Hence also the first term in (3.11) vanishes. This implies that for each m with $3 \leq m \leq M-1$ it suffices to consider instead of d_m only $d_m - d_{m-2}$ equations. Therefore the system (3.8) is reduced to a homogeneous system of

$$\sum_{m=2}^{M-1}(d_m - d_{m-2}) + n + 1 = d_{M-1} + d_{M-2}$$

linear equations in ν_E variables. But (compare (3.3))

$$d_{M-1} + d_{M-2} = C(M-1)$$

and because of (3.4) the system has a non trivial solution verifying (3.5).

The proof of (3.2) is now immediate. Suppose first that for some x_0, $\ell(x_0; E)$ is infinite. Then there are sequences $\psi^{(m)} = \sum_{i=1}^{\nu_E} c_i^{(m)}\psi_i$, $\psi_i \in N_E$, $\sum_{i=1}^{\nu_E} c_i^{(m)2} = 1$, so that these $\psi^{(m)}$ have zeros of order $M(m)$ at x_0 with $M(m) \to \infty$ for $m \to \infty$. This implies that there is a subsequence $\{(c_1^{(m')}, \ldots, c_{\nu_E}^{(m')})\}$ of $\{(c_1^{(m)}, \ldots, c_{\nu_E}^{(m)})\}$ which converges for $m' \to \infty$ to some $(c_1', \ldots, c_{\nu_E}') \in S^{\nu_E - 1}$. Clearly $\psi = \sum_{i=1}^{\nu_E} c_i'\psi_i$ has a zero of infinite order in x_0, and this contradicts unique continuation. Hence $\ell(x_0, E)$ is finite.

(3.2) follows now from the fact that $C(\ell)$ is monotonically increasing in ℓ, and that (3.4) implies (3.5), where we used that ν_E is finite. □

It seems likely that Theorem 3.1 still holds under weaker smoothness conditions on V, but then substantial modifications of the proof are certainly necessary.

Appendix

The proof of Lemma 2.1 is based on Proposition 2.2. We first note that it is straightforward to show that

$$(r^2+1)^{\gamma/2}\psi \in W^{2,2}(\mathbf{R}^n) \quad \forall \gamma > 0 \tag{1}$$

by taking into account (1.1) and Proposition 2.1. Pick $\gamma \geq \max(\beta, 1)$ and let

$$\eta = (r^2+1)^{\gamma/2}\psi \quad \text{and} \quad \eta_h = \eta * \rho_h, \qquad h > 0, \tag{2}$$

where $\rho_h \in C_0^\infty(\mathbf{R}^n)$ is a mollifier, such that η_h converges pointwise uniformly to η for $h \to 0$ in every compact set and

$$\|\eta_h - \eta\|_{W^{2,2}} \to 0 \quad \text{for } h \to 0 \tag{3}$$

(see e.g. [10]). Further let

$$\varphi_h = (r^2+1)^{-\gamma/2}\eta_h. \tag{4}$$

Then φ_h converges pointwise uniformly to ψ for $h \to 0$ in every compact set. Since $(r^2+1)^{-\gamma/2}$, $|\nabla(r^2+1)^{-\gamma/2}|$ and $\Delta(r^2+1)^{-\gamma/2}$ are bounded we easily conclude from (3)

$$\|\varphi_h - \psi\|_{W^{2,2}} \to 0 \quad \text{for } h \to 0. \tag{5}$$

Further obviously

$$\begin{aligned} &\|x \cdot \nabla[(r^2+1)^{-\gamma/2}(\eta_h - \eta)]\| \le \\ &\le \gamma\|\eta_h - \eta\| + \|(r^2+1)^{-\gamma/2} r \frac{\partial}{\partial r}(\eta_h - \eta)\| \le \\ &\le \gamma\|\eta_h - \eta\| + \|\nabla(\eta_h - \eta)\| \to 0 \quad \text{for } h \to 0 \end{aligned}$$

because of (3). Hence

$$\|x \cdot \nabla(\varphi_h - \psi)\| \to 0 \quad \text{for } h \to 0. \tag{6}$$

Next we choose $\phi = \varphi_h$ and $f = f_\varepsilon$ with

$$f_\varepsilon = (r^2 + \varepsilon^2)^{-k/4}, \qquad k = 2\ell + n - 2 - \nu \qquad (\nu > 0 \text{ small}) \tag{7}$$

and define

$$y_{\varepsilon,h} = r^{(n-2)/2} f_\varepsilon \varphi_h. \tag{8}$$

Inserting into (2.23) we obtain

$$\left.\begin{aligned} &((-A+1)f_\varepsilon\varphi_h, f_\varepsilon(-\Delta + V - E)\varphi_h) = \\ &= k(y'_{\varepsilon,h}, \frac{r^{4-n}}{r^2+\varepsilon^2} y'_{\varepsilon,h}) + (f_\varepsilon\varphi_h, (V - E + \frac{1}{2}rV')f_\varepsilon\varphi_h) + \\ &+ \frac{1}{2}(f_\varepsilon\varphi_h, r^{-2}G_\varepsilon f_\varepsilon\varphi_h) \\ &\text{where } G_\varepsilon(r) = \frac{(k^2-4k)r^6}{(r^2+\varepsilon^2)^3} + \frac{(6k-k^2)r^4}{(r^2+\varepsilon^2)^2} - \frac{2kr^2}{r^2+\varepsilon^2}. \end{aligned}\right\} \tag{9}$$

We show that

$$\text{l.h.s. of (9)} \to 0 \text{ for } h \to 0: \tag{10}$$

Obviously we have

$$|\text{l.h.s. of (9)}| \le (\|(A-1)f_\varepsilon(\varphi_h - \psi)\| + \|(A-1)f_\varepsilon\psi\|) \cdot \|f_\varepsilon(-\Delta + V - E)(\varphi_h - \psi)\|. \tag{11}$$

Since

$$f'_\varepsilon = -\frac{k}{2}\frac{r}{r^2+\varepsilon^2} f_\varepsilon \quad \text{and} \quad |f_\varepsilon| \le 1, \tag{12}$$

and (5) and (6) hold we immediately obtain for $h \to 0$

$$\begin{aligned} &\|(A-1)f_\varepsilon(\varphi_h - \psi)\| + \|(A-1)f_\varepsilon\psi\| \le \\ &\le c_0(k)(\|f_\varepsilon(\varphi_h - \psi)\| + \|f_\varepsilon\psi\| + \|f_\varepsilon x \cdot \nabla(\varphi_h - \psi)\| + \|f_\varepsilon x \cdot \nabla\psi\|) \le \\ &\le c(k) < \infty. \end{aligned} \tag{13}$$

(11) together with (13) gives

$$|\text{l.h.s. of (9)}| \le c(k)\|f_\varepsilon(-\Delta + V - E)(\varphi_h - \psi)\|. \tag{14}$$

Due to assumption (A) and (B) on V, and because $\gamma \ge \max(1, \beta)$ we have

$$\begin{aligned} \|f_\varepsilon V(\varphi_h - \psi)\| &\le const(\|r^{-p+\delta}(\varphi_h - \psi)\|_{B_1} + \|r^\beta(\varphi_h - \psi)\|) \le \\ &\le const(\|r^{-p+\delta}(\varphi_h - \psi)\|_{B_1} + \|\eta_h - \eta\|). \end{aligned} \tag{15}$$

But $r^{-p+\delta} \in L^1(B_1)$ and $\sup_{B_1} |\varphi_h - \psi| \to 0$ for $h \to 0$, together with (3) lead from (15) to

$$\|f_\varepsilon V(\varphi_h - \psi)\| \to 0 \qquad \text{for } h \to 0. \tag{16}$$

(14), (16) and (5) finally imply

$$|\text{l.h.s. of (9)}| \to 0 \quad \text{for } h \to 0. \tag{17}$$

Next we have to investigate the r.h.s. of (9) for $h \to 0$: Let

$$y_\varepsilon = r^{(n-2)/2} f_\varepsilon \psi \quad \text{with } f_\varepsilon \text{ defined in (7)}, \tag{18}$$

then by (12)

$$y_\varepsilon'^2 = r^{n-4} f_\varepsilon^2 (d(r)\psi + r\psi'^2) \quad \text{with } d(r) = \frac{1}{2}(n - 2 - \frac{kr^2}{r^2 + \varepsilon^2}) \tag{19}$$

and further

$$\begin{aligned} (y'_\varepsilon, \frac{r^{4-n}}{r^2+\varepsilon^2} y'_\varepsilon)^{1/2} &= \|\frac{r^{2-n/2}}{\sqrt{r^2+\varepsilon^2}} y'_\varepsilon\| = \|\frac{f_\varepsilon}{\sqrt{r^2+\varepsilon^2}}(d\psi + r\psi')\| \le \\ &\le c(\varepsilon, n, k)(\|\psi\| + \|x \cdot \nabla\psi\|) < \infty \end{aligned}$$

where we used (2.2). Analoguously

$$(y'_{\varepsilon,h}, \frac{r^{4-n}}{r^2+\varepsilon^2} y'_{\varepsilon,h}) < \infty \quad \text{with } y_{\varepsilon,h} \text{ given in (8)}.$$

Therefore

$$\begin{aligned} &\left| \|\frac{r^{2-n/2}}{\sqrt{r^2+\varepsilon^2}} y'_{\varepsilon,h}\| - \|\frac{r^{2-n/2}}{\sqrt{r^2+\varepsilon^2}} y'_\varepsilon\| \right| \le \\ &\le \|\frac{f_\varepsilon}{\sqrt{r^2+\varepsilon^2}}(d(\psi - \varphi_h) + r(\psi - \varphi_h)')\| \le \\ &\le c(\varepsilon, n, k)(\|\psi - \varphi_h\| + \|x \cdot \nabla(\psi - \varphi_h)\| \to 0 \quad \text{for } h \to 0 \end{aligned} \tag{20}$$

because of (5) and (6).

Next we note that

$$\left.\begin{array}{l}(\varphi_h, U f_\varepsilon^2 \varphi_h) \to (\psi, U f_\varepsilon^2 \psi) \quad \text{for } h \to 0 \\ \text{with } U = V - E + \frac{1}{2} x \cdot \nabla V\end{array}\right\} \tag{21}$$

which can be seen from the following: Clearly

$$|(\varphi_h, f_\varepsilon^2 U \varphi_h) - (\psi, f_\varepsilon^2 U \psi)| = |(\varphi_h - \psi, f_\varepsilon^2 U(\varphi_h - \psi)) + 2(\psi, f_\varepsilon^2 U(\varphi_h - \psi))|.$$

Therefrom we proceed as from (15) to (16) and arrive at (21).

It is easily seen that for all $\varepsilon > 0$, $r^{-2} G_\varepsilon f_\varepsilon^2$ is bounded for $r \geq 0$ and therefore

$$\begin{aligned} &|(\varphi_h, f_\varepsilon^2 r^{-2} G_\varepsilon \varphi_h) - (\psi, f_\varepsilon^2 r^{-2} G_\varepsilon \psi)| = \\ &= |(\varphi_h - \psi, f^2 r^{-2} G_\varepsilon(\varphi_h - \psi)) + 2(\varphi_h - \psi, f_\varepsilon^2 r^{-2} G_\varepsilon \psi)| \leq \\ &\leq c(\varepsilon)(\|\varphi_h - \psi\|^2 + \|\psi\| \cdot \|\varphi_h - \psi\|) \to 0 \quad \text{for } h \to 0. \end{aligned} \tag{22}$$

(20), (21) and (22) together yield

$$\text{r.h.s. of (9)} \to k(y_\varepsilon', \frac{r^{4-n}}{r^2 + \varepsilon^2} y_\varepsilon') + (\psi, f_\varepsilon^2 U \psi) + \frac{1}{2}(\psi, f_\varepsilon^2 r^{-2} G_\varepsilon \psi) \quad \text{for } h \to 0. \tag{23}$$

Combining (17) and (23) we arrive at

$$0 = k(y_\varepsilon', \frac{r^{4-n}}{r^2 + \varepsilon^2} y_\varepsilon') + (\psi, f_\varepsilon^2 U \psi) + \frac{1}{2}(\psi, f_\varepsilon^2 r^{-2} G_\varepsilon \psi). \tag{24}$$

Finally we have to investigate (24) for $\varepsilon \to 0$: With y defined as in (2.24) we clearly have for $\varepsilon \to 0$, $y_\varepsilon \to y$ and $y_\varepsilon' \to y'$ pointwise in $\mathbf{R}^n \setminus \{0\}$. From (19)

$$y_\varepsilon'^2 \frac{r^{4-n}}{r^2 + \varepsilon^2} \leq c(k, n) r^{-2-k} (\psi^2 + r^2 \psi'^2)$$

follows, and $r^{-2-k}(\psi^2 + r^2\psi'^2) \in L^1(\mathbf{R}^n)$ due to Proposition 2.1. Hence we conclude by Lebesgue's dominated convergence theorem that

$$(y_\varepsilon', \frac{r^{4-n}}{r^2 + \varepsilon^2} y_\varepsilon') \to (y', r^{2-n} y') \quad \text{for } \varepsilon \to 0. \tag{25}$$

Since $f_\varepsilon^2 U \psi^2 \to r^{-k} U \psi^2$ for $\varepsilon \to 0$ in $\mathbf{R}^n \setminus \{0\}$, $f_\varepsilon \leq 1$, and $r^{-k}\psi^2 |U| \in L^1(\mathbf{R}^n)$ due to our assumptions on V (compare (2.26)) and Proposition 2.1, we obtain also

$$(f_\varepsilon \psi, U f_\varepsilon \psi) \to (\psi, r^{-k} U \psi) \quad \text{for } \varepsilon \to 0. \tag{26}$$

Finally we show that

$$(\psi, r^{-2} f_\varepsilon^2 G_\varepsilon \psi) \to 0 \quad \text{for } \varepsilon \to 0: \tag{27}$$

It is easily seen that
$G_\varepsilon(r) \to 0$ for $\varepsilon \to 0$, for $r \in [0, \infty)$, and therefore $\psi^2 r^{-2}(r^2 + \varepsilon^2)^{-k/2} G_\varepsilon \to 0$ for $\varepsilon \to 0$ in $\mathbf{R}^n \setminus \{0\}$. Further for $\varepsilon > 0$ we have for some $C < \infty$ not depending on ε, $\psi^2 r^{-2}(r^2 + \varepsilon^2)^{-k/2} |G_\varepsilon| \leq C \psi^2 r^{-2-k}$ in $\mathbf{R}^n \setminus \{0\}$, but $\psi^2 r^{-2-k} \in L^1(\mathbf{R}^n)$ because of Proposition 2.1. This implies via Lebesgue's convergence theorem (27). Combination of (25), (26) and (27) completes the proof of Lemma 2.1. □

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