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SUPERCONFORMAL MODELS AND THE SUPERSYMMETRIC COULOMB GAS

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SUPERCONFORMAL MODELS AND THE SUPERSYMMETRIC COULOMB GAS

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1. Introduction

The recent period has seen an intense activity in the construction of conformal invariant theories and their classification /1/. Modular invariance on the torus has proved to be an effective constraint for this purpose.

On the other hand, according to ideas /2/ prior to modern developments in conformal invariance, most simple two dimensional critical models are expected to derive from free bosonic theories. Steps have been taken to establish links between these two approaches /3-5/. In ref./5/ it was shown that all minimal partition functions classified in /6/ are linear combinations of gaussian ones /7/. A direct derivation of these expressions from ADE restricted solid on solid /8a/ lattice models was proposed in /5,8b/.

The fact that minimal models can be described by a "decorated" free bosonic field is related to the underlying 6-vertex model. The latter presents a gaussian line (as in the low temperature phase of the XY model) and the other critical models /8/ are then obtained by a restriction procedure at special points of this line.

We explain in this paper how a similar scheme applies in the super minimal case /9/. The underlying theory is now the 19-vertex model /10/, the critical line of which is described by a free superfield, and related to the fully frustrated XY model. Partition functions on the torus /11/ are then recovered by decorating the super gaussian ones. They describe the "fused" lattice models of ref./12/.

These results are generalizable /13/ to the general hierarchy of models obtained by SU(2) coset construction /12,14/.

2. The supersymmetric 19-vertex model

The 19-vertex model /10/ is a natural generalization of the much studied 6-vertex model /15/, obtained by associating three possible states (represented on fig.1 by in-out arrows or a dot) to each bond of the square lattice, with the constraint that the current is conserved at each node. In the rather large space of parameters, an integrable set of Boltzmann weights was obtained by Zamolodchikov Fateev as

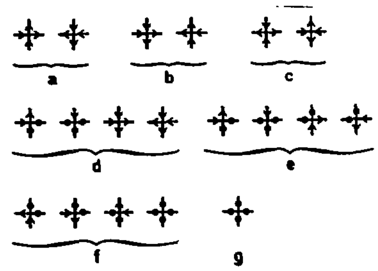

Fig.1

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$$\begin{cases} a = \sin\lambda(u+1)\,\sin\lambda(u+2) \\ b = \sin\lambda u\,\sin\lambda(u+1) \\ c = \sin\lambda\,\sin 2\lambda \\ d = \sin\lambda u\,\sin\lambda(u+1) \\ e = \sin 2\lambda\,\sin\lambda(u+1) \\ f = \sin 2\lambda\,\sin\lambda u \\ g = \sin\lambda\,\sin 2\lambda+\sin\lambda u\,\sin\lambda(u+1) \end{cases} \tag{2.1}$$

In (2.1) u plays the role of a spectral parameter, and transfer matrices with the same value of λ commute. In the very anisotropic limit, the model is equivalent to a one-dimensional spin 1 antiferromagnetic chain with hamiltonian

$$H = \sum_i \vec{S}_i\cdot\vec{S}_{i+1} - \left(\vec{S}_i\cdot\vec{S}_{i+1}\right)^2 - 2(\cos\lambda-1) \left[\left(S^z_i S^z_{i+1}\right)\left(S^x_i S^x_{i+1}+S^y_i S^y_{i+1}\right) + \left(S^x_i S^x_{i+1}+S^y_i S^y_{i+1}\right)\left(S^z_i S^z_{i+1}\right)\right] + 2\sin^2\lambda\left[S^z_i S^z_{i+1} - \left(S^z_i S^z_{i+1}\right)^2 + 2\left(S^z_i\right)^2\right] \tag{2.2}$$

In fact, the 19-vertex model (2.1) can be obtained starting from the 6-vertex model (fig.2) with weights

fig.2

$$\begin{cases} \alpha = \sin\lambda(u+1) \\ \beta = \sin\lambda u \\ \gamma = \sin\lambda \end{cases} \tag{2.3}$$

by a "fusion procedure", which is some kind /16/ of block-spin renormalization. One starts for this with a system of four sites in the 6-vertex model, with inhomogeneous values of the spectral parameter u, u-1, u, u+1. External arrows on the four sides are combined and projected on symmetric space (as in the addition of two spin 1/2 representations and projection onto spin 1) which defines new bond states

$$\rightarrow = (⇉)\ ;\ \leftarrow = (⇇)\ ;\ 0 = \frac{1}{\sqrt{2}}\,(⇄) + \frac{1}{\sqrt{2}}\,(⇆) \tag{2.4}$$

One gets in this way the 19-vertices of fig.1. Now weights are calculated by summing over all internal arrows, as well as the external ones on the south and west sides. One can check that the result does not depend on the configuration at the North or east sides, which is in fact a consequence of Yang Baxter equations and the special set of spectral parameter chosen. To illustrate this construction, we calculate explicitly some of the Boltzmann weights. The simplest case is a in (2.1) which can result from only one 6-vertex configuration

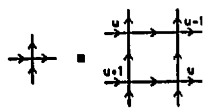

fig.3 (2.5)

thus

$$a(u) = \alpha(u-1)\alpha^2(u)\alpha(u+1) = \sin\lambda u\,\sin^2\lambda(u+1)\sin\lambda(u+2) \tag{2.6}$$

The most complicated is g

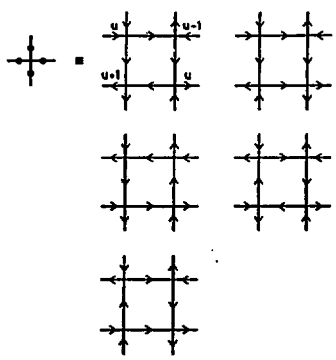

fig.4 (2.7)

for which

$$g(u)=\gamma^2[2\alpha(u+1)\beta(u)+\beta^2(u)+\gamma^2] + \alpha^2(u)\beta(u-1)\beta(u+1) \qquad (2.8)$$

After simple calculation g is rewritten

$$g(u)=\sin\lambda\sin2\lambda\sin\lambda u\sin\lambda(u+1)+\sin^2\lambda u\sin^2\lambda(u+1) \qquad (2.9)$$

We notice also that the configuration of spectral parameters in (2.5) is non invariant under the symmetry with respect to the NW-SE diagonal, while weights (2.1) are. In fact the above procedure gives for instance

weight of  $=\omega=2\sin\lambda\sin^2\lambda u\sin\lambda(u+1)$ (2.10)

weight of  $=\bar{\omega}=2\sin\lambda\cos^2\lambda\sin^2\lambda u \sin\lambda(u+1)$

Each of these two weights can be replaced by $(\omega\bar{\omega})^{1/2}$ using a "gauge transformation", $w(ijkl)\rightarrow \frac{f(l)\bar{f}(i)}{f(k)\bar{f}(j)} w(ijkl)$, which does not change the partition function an a torus. Dividing finally all weights by the irrelevant factor $\sin\lambda u \sin\lambda(u+1)$, one gets indeed (2.1).

Consider now a configuration of the 19-vertex model. Each vertex can be thought of as the superposition of various configurations for the four sites block in the 6-vertex model, with any possible choice of the arrows at the North and East sides. Because of this property one can now identify the arrows at West (resp. South) sides of a given block with the arrows at East (resp. North) of the West (resp. South) neighbouring blocks without changing Boltzmann weights /17/. The 19-vertex model is thus equivalent to an inhomogeneous 6-vertex model. This however does not work on a torus because of the non contractible cycles in the correspondence.

For a given configuration of the 19-vertex model, the 0-current bond define loops on the square lattice with possible crossings, exactly as in the say low temperature graph expansion of the Ising model (fig.5). On the other hand, one can introduce height variables φ on the faces of the lattice such that neighbouring φ differ by 0, $\pm2\pi$ depending on the bond which

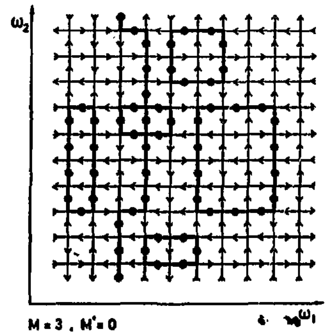

fig.5

separates them. These two aspects give rise respectively to a Z_2 and U(1) symmetry, and one can expect the 19-vertex model to have in the generic case either an Ising type or a Kosterlitz Thouless type phase transition. For the special (integrable) set of weights (2.1), these two kind of behaviours merge, and it turns out /13/ that the model has a critical line described by a superfree field with action

$$\mathcal{A}_{SC}= \mathcal{A}_B+ \mathcal{A}_F= \frac{g}{2\pi}\int [\partial\varphi\bar{\partial}\varphi-\psi\bar{\partial}\psi-\bar{\psi}\partial\bar{\psi}]dzd\bar{z} \qquad (2.11)$$

To justify this result, we notice first that in the limit $\lambda\rightarrow0$ where the weights (2.1) become rational functions, the model acquires an additional SU(2) symmetry.

This is most easily seen on the chain (2.2) which becomes simply in this case

$$H = \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1}\right) - \left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 \qquad (2.12)$$

The hamiltonian (2.12) was partly studied in /18/ using Bethe ansatz technique. Its spectrum is massless and corresponds to a central charge $c = 3/2$ (the latter being deduced from "finite temperature" properties). This agrees with non abelian bosonization arguments /19/ suggesting that (2.12) is critical and described by the SU(2) level 2 Wess Zumino Witten theory. The corresponding partition function on a torus has been obtained /6/, and can be rewritten

$$Z_{SU(2)\,level\,2} = Z_{SC}\left(g=\frac{1}{2}\right) = \sum_{r,s=0,1} \mathfrak{Z}_2(r,s) \sum_{\substack{M = r \bmod 2 \\ M' = S \bmod 2}} Z_{MM'}\left(g=\frac{1}{2}\right) \qquad (2.12)$$

In this expression $\mathfrak{Z}_2(r,s)$ is the partition function of the Ising model with twisted boundary conditions $(-)^r$, $(-)^s$ on the spin variable, while $Z_{MM'}(g=\frac{1}{2})$ is the partition function of the free bosonic field with shifted boundary conditions $\varphi(z+1)=\varphi(z)+2\pi M$, $\varphi(z+\tau)=\varphi(z)+2\pi M'$ is

$$Z_{MM'}(g) = \frac{\sqrt{g/\mathrm{Im}\tau}}{|\eta|^2} \exp -\pi g \frac{|M-M'\tau|^2}{\mathrm{Im}\tau} \qquad (2.13)$$

where τ is the modular parameter, $q=e^{2i\pi\tau}$, $\eta(q)=q^{1/24}\prod_1^\infty(1-q^n)$. Using the expressions of $\mathfrak{Z}_2$, (2.12) reads also

$$Z_{SC}(g) = \sum_{r,s=0,1} \sum_{M,M'\in\mathbb{Z}} \epsilon^{rs}_{MM'} \int_{\substack{\psi(z+1)=e^{i\pi r}\psi(z) \\ \psi(z+\tau)=e^{i\pi s}\psi(z) \\ \text{same for } \bar\psi}} d[\psi\bar\psi] \int_{\substack{\varphi(z+1)=\varphi(z)+2\pi M \\ \varphi(z+\tau)=\varphi(z)+2\pi M'}} [d\varphi]\, e^{-\mathcal{A}_{SC}} \qquad (2.14)$$

where ϵ is equal /13/ to ± 1.

The form of (2.12) is easily explained using the above arguments ; the Ising like degrees of freedom contribute to $\mathfrak{Z}_2$ and the SOS ones to $Z_{MM'}$, the shifts M, M' for the field φ originating in its local definition which cannot be consistently achieved on the torus /5/. The only coupling between Z_2 and U(1) degrees of freedom in (2.12) is due to boundary conditions. On a even x even lattice to which (2.12) indeed corresponds /13/, the algebraic number of arrows crossed along the periods M, M', and the number of 0-current bonds crossed, which determine r, s, are of the same parity.

Away from $\lambda = 0$, the SU(2) symmetry is broken into $Z_2\times U(1)$, and it is natural to expect that the corresponding partition function is still given by (2.12) with a varying coupling g. By analogy with the 6-vertex model case, the precise dependence

$$g = \frac{1}{2} - \lambda/\pi \qquad (2.15)$$

was proposed and numerically checked /13/. For $\lambda > \frac{\pi}{2}$, the model is antiferroelectrically ordered.

To exhibit the operator content of (2.12), it is useful to perform a Poisson transformation which gives

$$Z_{SC}(g) = \frac{1}{|\eta|^2} \left\{ \left(|\chi_0|^2 + |\chi_{1/2}|^2\right) \sum_{E,M \text{ even}} + \left(\chi_0 \chi_{1/2}^* + c.c\right) \sum_{E,M \text{ odd}} + |\chi_{1/16}|^2 \sum_{E-M \text{ odd}} \right\}$$

$$q^{(E/\sqrt{2g} + M\sqrt{2g})^2/8} \, \bar{q}^{(E/\sqrt{2g} - M\sqrt{2g})^2/8} \qquad (2.16)$$

where the χ's are characters of the Ising model. One can check that (2.15) decomposes properly an C = 3/2 SUSY characters /13/.

In the same way than the 6-vertex model gaussian critical line corresponds - after a duality transformation - to the low temperature phase of the XY model, one expects that the 19-vertex model super gaussian line describes the fully frustrated XY model /20/.

Of course various other SUSY critical lines can be constructed /21/. The simplest one involving completely decoupled fermionic and bosonic degrees of freedom, is

$$\tilde{Z}_{SC}(g) = \vartheta_2 \; Z_C(g) \qquad (2.17)$$

where $Z_C(g) = \sum_{MM' \in \mathbb{Z}} Z_{MM'}(g)$.

3. Superconformal minimal models

The construction of the 19-vertex starting from an inhomogeneous 6-vertex model can be reproduced as well for the restricted solid on solid models, variables of which are associated to the nodes of A_N Dynkin diagram (fig.6). After summing over internal states in each four faces block, one obtains the Boltzmann weights given in /12/, while neighbouring heights differ now by 0,±2. In the associated vertex model, the parameter λ must be adjusted to $\lambda = \pi/H$, where $H = N+1$ is the Coxeter number of A_N, thus $g = (H-2)/2H$.

To calculate the partition function on the torus, one must complement (2.12) by some topological terms. They depend on the number of N steps randow walks on the Dynkin diagram with incidence matrix G /8/

$$\Gamma(N) = \mathrm{Tr}\; G^N = \sum_{n \in \exp} \left(2\cos\frac{n\pi}{H}\right)^N \qquad (3.1)$$

where n belongs to the exponents of the algebra. In the fused models, neigbouring heights differing by 0,±2 one can consider two decoupled models depending on the parity of the heights over the lattice /12/. However, using the explicit form of the eigenvectors of G one checks that

$$\sum_{h_{odd}} \langle h | G^N | h \rangle = \sum_{h_{even}} \langle h | G^N | h \rangle = \frac{1}{2}\,\mathrm{Tr}\; G^N \qquad (3.2)$$

i.e. the two models in fact have the same partition function. One finds then

$$Z = \sum_{r,s=0,1} \vartheta_2(r,s) \sum_{\substack{M = r \bmod 2 \\ M' = s \bmod 2}} Z_{MM'}\left(\frac{H-2}{2H}\right) \sum_{n \in \exp} \cos\left(\frac{2\pi n}{H} M \wedge M'\right) \qquad (3.3)$$

where $\wedge$ denotes the greatest common divisor. The associated central charge is given by the lowest exponent

$$c = \frac{3}{2} - \frac{6(1/H)^2}{g} = \frac{3}{2} - \frac{12}{H(H-2)} \qquad (3.4)$$

Formula (3.3) generalizes to a tricritical branch with the other determination: $g = (H+2)/2H$, and to D_N, E_6, E_7, E_8 models.

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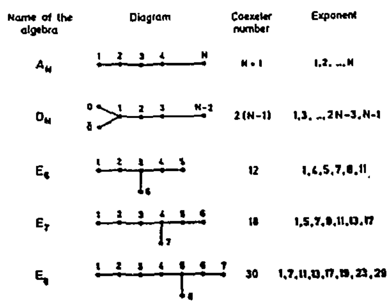

fig.6

After some work one checks that one gets in this case all the unitary modular invariant partition function of ref./11/ that are classified by a pair of Lie algebras one of which is of A type. The exceptional invariants (D_6,E_6) and (D_8,E_6) however are not obtained, and their lattice realisation is still unknown to us.

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References

/1/ C.Itzykson, H.Saleur, J.B.Zuber "Conformal Invariance and Applications to Statistical Mechanics", World Scientific, to appear.

/2/ L.P.Kadanoff, J.Phys. A11 (1978) 1399.

/3/ VP.S.Dotsenko, V.A.Fateev, Nucl.Phys. B240 (1984) 312.

/4/ J.L.Gervais, A.Neveu, Comm.Math.Phys. 100 (1985) 15.

/5/ P.Di Francesco, H.Saleur, J.B.Zuber, J.Stat.Phys. 49 (1987) 57.

/6/ A.Cappelli, C.Itzykson, J.B.Zuber, Comm.Math.Phys. 113 (1987) 1.
A.Kato, Mod.Phys.Lett. A2 (1987) 585.

/7/ D.J.Gross, J.A.Harvey, E.Martinec, R.Rohm, Nucl.Phys. B256 (1985) 253.
P. Di Francesco, H.Saleur, J.B.Zuber, Nucl.Phys. B285 (1987) 454.
S.K.Yang, Nucl.Phys. B285 (1987) 183.

/8/ a) G.Andrews; R.J.Baxter, P.J.Forrester, J.Stat.Phys. 35 (1984) 193.
V.Pasquier, Nucl.Phys. B285 (1987) 162.
b) V.Pasquier, J.Phys. A20 (1987) L1229.

/9/ D.Friedan, Z.Qiu, S.Shenker, Phys.Lett. 151B (1985) 37.

/10/ A.B.Zamolodchikov, V.A.Fateev, Sov.J.Nucl.Phys. 32 (1980) 298.
T.Sogo, Y.Akutsu, T.Abe, Prog.Th.Phys. 70 (1983) 730, 739.

/11/ A.Cappelli, Phys.Lett. 185B (1987) 82.

/12/ E.Date, M.Jimbo, A.Kuniba, T.Miwa, M.Okado, Nucl.Phys. B290 (1987) 231.
E.Date, M.Jimbo, T.Miwa, M.Okado, Lett.Math.Phys. 12 (1986) 209.

/13/ P. Di Francesco, H.Saleur, J.B.Zuber, Preprint Saclay 88.

/14/ F.Ravanini, Mod.Phys.Lett. A3 (1988) 271 and Nordita preprint 87.
D.Kastor, E.Martinec and Z.Qiu, Phys.Lett. 200B (1988) 434.
J.Bagger, D.Nemeschansky and S.Yankielowicz, Phys.Rev.Lett. 60 (1988) 389.

/15/ R.J.Baxter, "Exactly solved models in statistical mechanics" Academic Press 1982.

/16/ P.P.Kullish, N.Yu Reshetikhin, E.K.Sklyanin, Lett.Math.Phys. 5 (1981) 393.

/17/ Y.Akutsu, A.Kuniba, M.Wadati, J.Phys.Soc.Japan 55 (1986), 2907.

/18/ L.
H.
31
/19/ I.

/18/ L.Takhtajan, Phys.Lett. 87A (1982) 479.
H.M.Babujian, Nucl.Phys. B215 (1983) 317.

/19/ I.Affleck, Nucl.Phys. B265 (1985) 409.

/20/ O.Foda, Preprint Utrecht 87.

/21/ L.Dixon, P.Ginsparg, J.Harvey, Preprint SLAC 88.