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ON THE STRUCTURE OF QUANTUM PHASE SPACE*

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ABSTRACT

The space of labels characterizing the elements of Schwinger's basis for unitary quantum operators is endowed with a structure of symplectic type.This structure is embodied in a certain algebraic cocycle, whose main features are inherited by the symplectic form of classical phase space. In consequence, the label space may be taken as the Quantum Phase Space: it plays, in the quantum case, the same role played by phase space in classical mechanics, some differences coming inevitably from its non-linear character.

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Abstract

The space of labels characterizing the elements of Schwinger's basis for unitary quantum operators is endowed with a structure of symplectic type. This structure is embodied in a certain algebraic cocycle, whose main features are inherited by the symplectic form of classical phase space. In consequence, the label space may be taken as the Quantum Phase Space: it plays, in the quantum case, the same role played by phase space in classical mechanics, some differences coming inevitably from its non-linear character.

On the Structure of Quantum Phase Space

1. Introduction

The recent extension of Weyl-Wigner transformations to discrete quantum spectra has drawn attention [1] to a certain discrete space which plays the role [2] of a "quantum phase space" (QPS). The extension makes use of Schwinger's complete basis [3] of unitary operators for Weyl's realization of the Heisenberg group. Unlike usual classical phase spaces, QPS is not a linear space: its points, besides being isolated, display themselves on the surface of a torus. The continuum quantum case may be obtained by a standard procedure which corresponds to stretching the torus radii to infinite while bringing the spacing between neighbouring points to zero in a suitable way. This C-number representation of QPS closely parallels the classical picture, its quantum character being signaled by the presence of Planck's constant $\hbar$ in the expressions involved. It is of basic interest to examine the main properties of QPS and their relations to the well known characteristics of the classical phase space. We would of course expect to obtain the classical case as a $\hbar \rightarrow 0$ limit of the quantum case.

The basic geometrical feature of a classical phase space is its symplectic structure, embodied in a differential 2-form Ω which is closed (a cocycle) and non-degenerate The fundamental rôle of this symplectic form is specially visible in the Hamiltonian formulation of mechanics. So strongly does the symplectic structure sticks to the very notion of phase space that QPS will only deserve its name if it includes a structure of similar nature. Although we may not expect the presence of a complete analogue to Ω on QPS, our objective here is to show that a certain structure exists indeed which plays on QPS a role as similar to a symplectic structure as could be expected. Such a "pre-symplectic" structure is actualized in a certain 2-cochain (also a cocycle) acting on the unitary operators, a purely algebraic object which acquires, in the continuous limit, a geometrical nature and tends, in the classical limit, to the symplectic form The 2-cochain marks in reality the projective character of Weyl's realization of the Heisenberg group.

We start in chapter 2 with a sketchy presentation of Hamiltonian mechanics [4], special emphasis being given to the role of the symplectic structure [5] we then address ourselves to quantum kinematics and give a resume on Schwinger's complete basis of unitary operators in chapter 3. A crucial point will be that the basis provides in reality not a linear but a projective representation of the Heisenberg group Preparing to establish that, chapter 4 is a short introduction to the subject of projective representations [6] from the cohomological point of view [7] which, being closer to the formalism of differential forms, is specially convenient to our purposes [8]. The meaning of ray representations becomes specially clear in this language The results are then applied in chapter 5 to the

Schwinger basis for the Weyl representation, emphasis being given to the emergence of the mentioned cocycle and to some of its properties. The continuum limit is examined and comparison is made with another C-number representation of Quantum Mechanics, the Weyl-Wigner-Moyal [9] approach. The meaning of the "pre-symplectic" fundamental cocycle is clarified in terms of well known features of that approach.

2. Classical Phase Space

In the classical description of a system with n degrees of freedom, physical states constitute a differentiable symplectic manifold M of dimension 2n. The fundamental geometrical characteristic of this phase space is the *symplectic 2-form* Ω. In terms of the generalized coordinates $q = (q^1, q^2, \ldots, q^n)$ and momenta $p = (p_1, p_2, \ldots, p_n)$, Ω is written

$$\Omega = dq^i \wedge dp_i. \qquad (2.1)$$

It is clearly a closed form (that is, $d\Omega = 0$), and can be shown to be also nondegenerate. A closed form is also called a *cocycle*. Ω as above is also an exact form (a *coboundary*, or a trivial cocycle) as it is, up to a sign, the differential of the *canonical* form

$$\sigma = p_i\, dq^i \qquad (2.2)$$

The structure defined by a closed nondegenerate 2-form is called a *symplectic structure* and a manifold endowed with such a structure is a *symplectic manifold*. In reality, phase spaces are very particular cases of symplectic manifolds. On general, topologically non-trivial symplectic manifolds there are no global coordinates such as the (q^i, p_i) supposed above and the basic closed nondegenerate 2-form is not necessarily exact. Notice that every coboundary is a cocycle but not vice-versa. A theorem by Darboux ensures the existence of a chart (of "canonical", or "symplectic" coordinates) around any point on a (2n)-dimensional manifold M in which a closed nondegenerate 2-form can be written as in (2.1), so that the equations here written in components hold locally. Notice however that Ω is globally defined and the equations written in the invariant language of forms are valid globally.

The fundamental point about the symplectic structure is that Ω establishes a one-to-one relationship between 1-forms and vector fields on the manifold M. Consider the example of the phase space velocity field,

$$X_H = \frac{dq^i}{dt}\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i} . \qquad (2.3)$$

The time evolution of the state point (q,p) will take place along the integral curves of X_H. Hamilton's equations put this evolution field into the form

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} . \qquad (2.4)$$

The differential operator X_H generates a one-parameter group of transformations, the *Hamiltonian flow*. On the other hand, the Hamiltonian function H(q,p) will have as differential the 1-form

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i . \qquad (2.5)$$

The two equations above show clearly an intimate relation between dH and X_H for which Ω is responsible. The relationship involves the *interior product* of a field by a form. The interior product of a field X by a 1-form σ, denoted $i_X \sigma$, is simply $\sigma(X)$. The interior product of a field X by a 2-form Ω, denoted $i_X \Omega$, is defined as that 1-form satisfying $i_X \Omega(Y) = \Omega(X,Y)$ for any field Y. This is directly generalized to higher-order forms. We find easily that

$$i_{X_H} \Omega = dH . \qquad (2.6)$$

Besides being a particular case of the general one-to-one relationship between fields (vectors) and 1-forms (covectors) on M, this is also an example of relationship between a transformation generator and the corresponding generating function. The Hamiltonian presides over the time evolution of the physical system under consideration: H(q,p) is the *generating function* of the velocity field X_H. Applying X_H to any given differentiable function F(q,p) on M, we find that

$$X_H F = \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}$$

$$= \{F,H\}, \qquad (2.7)$$

the Poisson bracket of F and H, so that its equation of motion is

$$\frac{dF}{dt} = X_H F\,. \qquad (2.8)$$

the Liouville equation X_H is frequently called *Liouvillian operator*. Functions like $F(q,p)$ are the classical observables, or dynamical functions. To each such a function will correspond a field

$$X_F = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i} \qquad (2.9)$$

through the relation

$$i_{X_F}\Omega = dF \qquad (2.10)$$

Given another function $G(q,p)$ and its corresponding field X_G, it is immediate to verify that

$$\Omega(X_F,X_G) = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} =$$

$$= \{F,G\} \qquad (2.11)$$

Each field on M is the local generator of a one-dimensional group of transformations. The response of a tensor to the local transformations generated by a field is measured by the Lie derivative of the tensor with respect to the field. Of course, F (which is a zero order tensor) is an integral of motion if its Lie derivative $L_{X_H} F = X_H F$ vanishes, or $\{F,H\} = 0$. The Lie derivative of Ω with respect to X_H vanishes.

$$L_{X_H}\Omega = 0 , \tag{2.12}$$

because $L_X = d \circ i_X + i_X \circ d$. This means that the 2-form Ω is preserved by the Hamiltonian flow, or by the time evolution. This and the property

$$L_X(\Omega\wedge\Omega) = (L_X\Omega)\wedge\Omega + \Omega\wedge(L_X\Omega)$$

of Lie derivatives establish the invariance of the whole series of Poincare invariants $\Omega\wedge\Omega\ldots\wedge\Omega$, including that with the number n of Ω's, which is proportional to the volume form of M

$$\Omega^n = (-)^n \; dq^1\wedge dq^2\wedge \ldots \wedge dq^n\wedge dp_1\wedge dp_2\wedge \ldots \wedge dp_n$$

The preservation of Ω^n by the Hamiltonian flow is of course Liouville's theorem.

For any field X_F related to a dynamical function F

$$L_{X_F}\Omega = 0 \qquad (2.13)$$

This happens because $L_{X_F}\Omega = d \circ i_{X_F}\Omega + i_{X_F} \circ d\Omega = d^2F = 0$. Such transformations leaving Ω invariant are the *canonical transformations*. X_F is said to be a *hamiltonian field* and F its *generating function*. In a more usual language, F is the generating function of the corresponding canonical transformation. The simplest examples of generating functions are given by $F(q,p) = q^i$, corresponding to the field $X_F = -\partial/\partial p_i$, and $G(q,p) = p_i$, whose field is $X_G = \partial/\partial q^i$. Both lead to $\{q^i, p_j\} = \delta^i_j$. Next in simplicity are the dynamical functions of the type

$$f_{ab} = aq + bp, \qquad (2.14)$$

with a,b real constants. The corresponding fields are $J_{ab} = -a\,\partial/\partial p + b\,\partial/\partial q$. The commutator of two such fields is $[J_{ab},J_{cd}] = 0$ and consequently the corresponding generating function $F_{[J_{ab},J_{cd}]} = F_0$ is a constant. On the other hand, the Poisson brackets are determinants

$$\{f_{ab},f_{cd}\} = \Omega(J_{ab},J_{cd}) = ad - bc \qquad (2.15)$$

We have been using above the holonomic basis $\{\partial/\partial q^i, \partial/\partial p_i\}$ for the vector fields on phase space. In principle, any set of 2n linearly independent fields may be taken as a basis. Such a general basis $\{e_i\}$ will have the dual basis $\{\omega^j\}$ with $\omega^j(e_i) = \delta^j_i$, and its members will have commutators $[e_i,e_j] = C^k{}_{ij}\, e_k$, where the structure coefficients $C^k{}_{ij}$

measure the basis anholonomicity. A general field will be written $X = X^i e_i = \omega^i(X) e_i$; a general 1-form $\sigma = \sigma_i \omega^i = \sigma(e_i)\omega^i$, the differential of a function F will be $dF = e_i(F)\omega^i$, etc. In this basis, the symplectic 2-form will be

$$\Omega = \frac{1}{2}\Omega_{ij}\, \omega^i \wedge \omega^j = \frac{1}{2}\Omega(e_i, e_j)\, \omega^i \wedge \omega^j \tag{2.16}$$

It is interesting to consider $\Omega = (\Omega_{ij})$ as a (antisymmetric) matrix and introduce its inverse $\Omega^{-1} = (\Omega^{ij})$, whose existence is ensured by the nondegeneracy condition:

$$\Omega_{ij}\Omega^{jk} = \Omega^{kj}\Omega_{ji} = \delta^k{}_i \tag{2.17}$$

The Poisson bracket is then

$$\{F,G\} = \Omega(X_F, X_G) = X^i{}_F\, \Omega_{ij}\, X^j{}_G = e_k(G)\Omega^{kj} e_j(F) \tag{2.18}$$

The holonomic vector basis related to the coordinates $\{x^k\} = \{q^i, p_j\}$ will be formed by the fields $e_k = \partial/\partial q^k$ for $k = 1, 2, \ldots, n$ and $e_k = \partial/\partial p_k$ for $k = (n+1), (n+2), \ldots, (2n)$. The matrices Ω and Ω^{-1} will have the forms

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad ; \quad \Omega^{-1} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

where I_n is the n-dimensional unit matrix.

Let us summarize to a contravariant field X

$$X = X_{p_i} \frac{\partial}{\partial p_i} + X_{q^i} \frac{\partial}{\partial q^i} \qquad (2.19)$$

Ω will make to correspond the covariant field

$$i_X \Omega = X_{q^i} dp_i - X_{p_i} dq^i \quad . \qquad (2.20)$$

In reality, although this relationship always holds, most fields do not correspond to a generating function. $i_X \Omega$ is not always exact. When it is, X is more frequently called a *strictly* (or *globally*) hamiltonian field. In general, a generating function exists only locally The 1-form corresponding to any field preserving Ω will be closed, $d(i_X \Omega) = L_X \Omega = 0$. As a closed form is always locally exact, around any point of M there is a neighbourhood where some F(q,p) satisfies $i_X \Omega = dF$

Suppose another field is given, $Y = Y_{p_i}\ \partial/\partial p_i + Y_{q^i}\ \partial/\partial q^i$. The action of the 2-form Ω on X and Y will give

$$\Omega(X,Y) = X_{q^i} Y_{p_i} - X_{p_i} Y_{q^i} \,. \qquad (2.21)$$

This is twice the area of the triangle defined on M by X and Y, as it is still easier to see from (2.14,15)

An n-dimensional subspace of the 2n-dimensional phase space M is a *Lagrange manifold* if $\Omega(X,Y) = 0$ for any two vectors X,Y tangent to it. Examples are the configuration space and the momentum space Canonical transformations preserve such subspaces of M, that is, they take a Lagrange manifold into another Lagrange manifold

Let us finish with a comment on the closedness of the symplectic form. Why have we so much insisted that Ω be a cocycle? The reason is that Ω being a cocycle is equivalent to the Jacobi identity for the Poisson brackets. It is not difficult to find that

$$X(\Omega(Y,Z)) = -\{F_X,\{F_Y,F_Z\}\}$$

and that

$$\Omega(X,[Y,Z]) = [Y,Z]\,F_X = -\{F_X,\{F_Y,F_Z\}\}.$$

Combined with the general expression for the differential of a 2-form,

$$3!\,(d\Omega(X,Y,Z)) = X(\Omega(Y,Z)) + Z(\Omega(X,Y)) + Y(\Omega(Z,X)) + \Omega(X,[Y,Z]) + \Omega(Z,[X,Y]) + \Omega(Y,[Z,X]),$$

this gives

$$3\,d\Omega(X,Y,Z) = -\{F_X,\{F_Y,F_Z\}\} - \{F_Z,\{F_X,F_Y\}\} - \{F_Y,\{F_Z,F_X\}\} = 0 \qquad (2.22)$$

We see in this way the meaning of the closedness of Ω: it is just the Jacobi identity for the Poisson bracket.

The Poisson bracket is antisymmetric and satisfies the Jacobi identity. It is an operation defined on the space $C^\infty(M,\mathbf{R})$ of real differentiable functions on M. Consequently, $C^\infty(M,\mathbf{R})$ is an infinite-dimensional Lie algebra with the operation defined by the Poisson bracket. Actually, $F \longrightarrow X_F$ is a Lie algebra homomorphism (a representation) of $C^\infty(M,\mathbf{R})$ into the algebra of strictly hamiltonian fields on M.

There would be of course much more to be said about phase space: how linear canonical transformations constitute an important group, how a complex structure may be introduced, etc. The brief outline above, however, fixes notation and stresses hopefully enough the basic role of the cocycle Ω. We shall see in paragraph 5 that on quantum phase space also a cocycle is defined which, even in the discrete case, has a comparably fundamental role.

3. Quantum Kinematics

The quantum description of a physical system requires a complete set of observables. Still better, it requires a complete set of operators in terms of which all dynamical operators can be built up. Kinematics is governed by Heisenberg's group [10], whose elements may be represented by real triples (a, b, r) obeying the group product rule [11]

$$(a, b, r) * (c, d, s) = (a + c, b + d, r + s + (1/2)[ad - bc]).$$

The corresponding algebra is formed by triples (a, b, r) satisfying

$$(a, b, r) \oplus (c, d, s) = (0, 0, ad - bc)$$

Weyl introduced a realization in terms of powers of two unitary operators U(a) and V(b) satisfying

$$U(a)\, U(a') = U(a + a'),$$
$$V(b)\, V(b') = V(b + b')$$

and

$$U(a)\, V(b) = V(b)\, U(a)\, e^{i\mu ab}.$$

A particular example is given by $V = e^{ibp}$, $U = e^{iaq}$, which lead to the usual formulation of Heisenberg's algebra using the basic operators **p** and **q**. Schwinger [3] has recognized the fact that the above U and V generate a complete basis for all unitary operators and provided a classificaton of all the possible physical degrees of freedom. We shall here be interested only in some aspects of Schwinger's work. What follows is a short presentation of them.

Consider a space of quantum states of which a basis is given by orthonormalized kets $|v_k\rangle$ with $k = 1,2, \ldots ,N$. A unitary operator U can be defined which shifts these kets through cyclic permutations as

$$U\,|v_k\rangle = |v_{k+1}\rangle, \text{ with } |v_{k+N}\rangle \equiv |v_k\rangle \tag{3.1}$$

Through the repeated action of U, a set of linearly independent unitary operators U^m can be obtained whose action is given by

$$U^m\,|v_k\rangle = |v_{k+m}\rangle. \tag{3.2}$$

As $U^N = 1$, the eigenvalues of U are $u_k = e^{i(2\pi/N)k}$, corresponding to another set of kets fixed by

$$U|u_k\rangle = u_k|u_k\rangle \tag{3.3}$$

Another operator V exists such that

$$V|u_k\rangle = |u_{k-1}\rangle \tag{3.4}$$

and

$$V^n|u_k\rangle = |u_{k-n}\rangle, \text{ with } |u_{k-N}\rangle \equiv |u_k\rangle. \tag{3.5}$$

Here also $V^N = 1$ and the V eigenvalues are $v_k = e^{i(2\pi/N)k}$. The miracle of Schwinger's basis is that the eigenkets $|v_k\rangle$ such that

$$V|v_k\rangle = e^{i(2\pi/N)k}|v_k\rangle \tag{3.6}$$

are just those from which we have started. Of course,

$$V^n|v_k\rangle = e^{i(2\pi/N)kn}|v_k\rangle. \tag{3.7}$$

A direct calculation in any basis shows that

$$V^n U^m = e^{i(2\pi/N)mn} U^m V^n. \tag{3.8}$$

Now, Schwinger's final point: the set of operators

$$S_{mn} = e^{i(\pi/N)mn}\, U^m V^n \qquad (3.9)$$

constitute a complete orthogonal basis in terms of which any dynamical quantity **O** can be constructed as

$$\mathbf{O} = \Sigma_{m,n} O_{mn} S_{mn}\ , \qquad (3.10)$$

the O_{mn}'s being coefficients given by

$$O_{mn} = tr\,[\, S'_{mn}\, \mathbf{O}\,]\ . \qquad (3.11)$$

U and V are each one a generator of the cyclic group Z_N. The operators S_{mn} give a peculiar combination of the two Z_N's, providing a discrete version of Weyl's representation of the Heisenberg group.

The following results are easily obtained:

(i) the action of the basic operators on the kets:

$$S_{mn}\,|\, v_k \rangle = e^{i(\pi/N)(2k+m)n}\,|\, v_{k+m} \rangle\ ; \qquad (3.12)$$

(ii) the group product.

$$S_{rs} S_{mn} = e^{i(\pi/N)(ms-nr)} S_{(m+r)(n+s)} \quad , \tag{3.13}$$

(iii) the group identity.

$$S_{00} \equiv 1; \tag{3.14}$$

(iv) the inverse to a given element:

$$S^{-1}_{mn} = S_{-m,-n} \quad ; \tag{3.15}$$

(v) behaviour under a similarity transformation:

$$S_{mn} S_{rs} S^{-1}_{mn} = e^{-i(2\pi/N)(ms-nr)} S_{rs} . \tag{3.16}$$

(vi) associativity:

$$(S_{mn} S_{rs}) S_{kl} = S_{mn} (S_{rs} S_{kl}) \quad . \tag{3.17}$$

With the periodicity conditions in (3.1) and (3.5), the numbers m, n, etc take values on a torus lattice. It is this lattice who plays the role of a quantum phase space. The points of QPS are so labels of elements of a discrete group. The operators S_{mn}, obeying the product rule (3.13), give a projective representation of the group of transformations on this space, which will be examined in next chapter. Notice that they are themselves only semiperiodical: $S_{Np} = (-)^p S_{0p}$; $S_{pN} = (-)^p S_{p0}$. The quantum continuum limit, which has only been studied in detail in some cases [1,3], is in such cases attained by taking both the torus radii to infinite while making the spacing between neighbouring points go to zero, in such a way that $[\sqrt{2\pi/N}\, m] \longrightarrow$ some real constant a, $[\sqrt{2\pi/N}\, n] \longrightarrow$ another real constant b, etc. In this limit, a particular realization of the above operators is

$$V = e^{i\sqrt{2\pi/N}\, p} \; ; U = ee^{i\sqrt{2\pi/N}\, q} \tag{3.18}$$

$$V^n \longrightarrow e^{ibp} , U^m \longrightarrow e^{iaq} \tag{3.19}$$

where the operators **p** and **q** have eigenvalues $\sqrt{2\pi/N}\, k$. In this case,

$$S_{mn} \longrightarrow S_{ab} = e^{i(aq + bp)} \tag{3.20}$$

The expression (3.13) takes the form

$$S_{cd}\, S_{ab} = e^{(i/2)(ad - cb)}\, S_{(a+c)\,(b+d)} \tag{3.21}$$

The exponent in S_{ab} is the quantum version of the dynamical functions (2.14) and the phase in the group product is just (half) the Poisson bracket (2.15).

To a given degree of freedom corresponds a pair of operators U, V satisfying (3.8) which will provide a basis for a realization of the Heisenberg group. A curious and important example is given by the non-local order and disorder operators determining the confined and unconfined phases in quarkionic matter [12]. The algebra (3.8) appears then because of the crucial role attributed to the centre of the group SU(N), which is precisely Z_N.

The above considerations on the continuum limit suggest that each pair of operators U,V satisfying (3.8) is related to a pair of (exponentiated) canonically conjugate variables and, so, to a degree of freedom. This is true only when N is a prime number [3]. Otherwise, the representation involved is reducible. When N is prime, (3.8) is the only possible combination of powers of U and V leading to such a kind of expression. When N is not prime, however, things are different. N can be written in terms of its prime factors, $N = N_1 N_2 \ldots N_j$ and particular powers of U and V combine to give expressions like (3.8) with N replaced by each one of the factors N_i. The basis can then be redefined to become a direct product [3]. In the continuous limit, N goes to infinity through prime values.

4. Projective Representations

Projective representations [6] are treated, even in the best of older texts, in a rather involved way. The modern, homological approach [7] of which a brief account is given in the following has many advantages, not the least being its assignment of the subjet's correct place in the wider chapter of group extensions [8]. In our case the main advantage is that the evident analogy with the formalism of differential forms allows a clearer view of the connections between Schwinger's basis and classical phase space.

Let us consider, to fix the ideas, a group G of elements g, h, etc, acting through their representative operators U(g), U(h), etc on kets $|\varphi_x\rangle$, $|\psi_y\rangle$, etc. The indices x, y include not only configuration or momentum space coordinates but also spin and/or isospin indices and any other necessary state labels. We shall call them *parameters*. We might alternatively talk of the corresponding wavefunctions $\varphi(r) = \langle r | \varphi \rangle$, etc, but will use kets to keep in pace with previous notation. The space $\{|\varphi_x\rangle\}$ of kets will be the *carrier space* of the representation.

Suppose to begin with that we have

$$U(g)|\varphi_x\rangle = |\varphi_{xg}\rangle , \qquad (4.1)$$

where " xg " is the set of labels as transformed by the action of g. Suppose further that, by compositon,

$$U(h)\,U(g)|\psi_x\rangle = U(gh)|\psi_x\rangle, \qquad (4.2)$$

meaning in particular that the composition by itself is independent of the point x in parameter space. This is what is usually called a representation, but will in the present context be called a *linear representation*. The mapping U: g → U(g) is in this case a homomorphism.

We may next suppose that, instead of (4.1), the action of a transformation is given by

$$U(g)|\psi_x\rangle = e^{i\alpha_1(x;\,g)}|\psi_{xg}\rangle. \qquad (4.3)$$

The wavefunction acquires a phase $\alpha_1(x; g)$ which depends both on the transformation and the point in parameter space. The transformation will operate differently at different x. In Quantum Mechanics, of course, the above phase will not change the state, which is the same for wavefunctions differing only by phases. A state is, in reality, fixed by a *ray* (that is, a function with any phase factor). For this reason, a representation acting according to (4. 3) has been called a *ray representation*. Its more mathematical name is *projective representation*, but these notions will become more precise in the following.

Suppose condition (4.2) holds,

$$U(h)\,U(g)\,|\psi_x\rangle = U(gh)\,|\psi_x\rangle .$$

A direct calculation shows that this implies

$$\alpha_1(xg;h) - \alpha_1(x;gh) + \alpha_1(x;g) = 0, \qquad (4.4)$$

another form of the homomorphic condition. If a function $\alpha_0(x)$ exists such that $\alpha_1(x;g)$ can be written in the form

$$\alpha_1(x;g) = \alpha_0(xg) - \alpha_0(x), \qquad (4.5)$$

then (4.4) holds automatically, (4.3) becomes

$$U(g)\,e^{i\alpha_0(x)}\,|\psi_x\rangle = e^{i\alpha_0(xg)}\,|\psi_{xg}\rangle$$

and phases can be eliminated by redefining

$$|\phi_x\rangle = e^{i\alpha_0(x)}\,|\psi_x\rangle,$$

which brings the group action back to the form (4.1)

In the cohomological theory of group representations, phases such as the above $\alpha_0(x)$ and $\alpha_1(x;g)$ are considered as results of the action of cochains on the group G. Cochains are antisymmetric mappings on the group, purely defined by their action. They have much in common with differential forms (which are in reality special cochains) but it should be kept in mind that here they are not necessarily acting on elements of a linear space. Here they take one, two or more group elements to give numbers. The group elements have the role vectors have in the case of differential forms. Cochains may be defined on any group, even discrete ones - which is just the case of our interest. α_0 is a 0-cochain, a function on parameter space whose value at point x is the phase $\alpha_0(x)$; α_1 is a 1-cochain because it operates on one element g of G at point x of the parameter space to give $\alpha_1(x;g)$; a cochain taking two group elements as arguments will be a 2-cochain, etc. An operation analogous to the exterior differentiation of differential forms is defined [8] on cochains: it is the derivative operation δ taking a p-cochain α_p into a (p+1)-cochain β_{p+1} according to [13]

$$\delta : \alpha_p \rightarrow \text{some } \beta_{p+1} = \delta\alpha_p$$

$$\delta\alpha_p(x, g_1, g_2, \ldots, g_{p+1}) = \alpha_p(x\, g_1; g_2, \ldots, g_{p+1}) - \alpha_p(x; g_1 g_2, \ldots, g_{p+1}) +$$

$$+ \ldots + (-)^{p+1}\alpha_p(x; g_1, g_2, \ldots, g_p). \qquad (46)$$

An important property is the Poincaré lemma $\delta^2 = 0$, which can be verified directly from this expression. The first examples are

$$\delta\alpha_0(x; g) = \alpha_0(x\, g) - \alpha_0(x); \tag{4.7}$$

$$\delta\alpha_1(x; g, h) = \alpha_1(x\, g, h) - \alpha_1(x; g\, h) + \alpha_1(x; g); \tag{4.8}$$

$$\delta\alpha_2(x; g, h, f) = \alpha_2(x\, g; h, f) - \alpha_2(x; g\, h, f) + \alpha_2(x; g, h\, f) - \alpha_2(x; g, h). \tag{4.9}$$

A cochain α_p satisfying $\delta\alpha_p = 0$ is a *closed* p-cochain, or a p-*cocycle*, and a cochain α_p for which a cochain α_{p-1} exists such that $\alpha_p = \delta\alpha_{p-1}$ is *exact*, or a *coboundary* (or trivial cocycle). An exact cochain is automatically closed. We see that condition (4.4) means that α_1 is closed,

$$\delta\alpha_1(x; g, h) = 0, \tag{4.10}$$

still another form of the homomorphic condition. As to (4.5), it says simply that α_1 is exact:

$$\alpha_1(x, g) = \delta\alpha_0(x; g). \tag{4.11}$$

Summing up: the composition rule (4.2) implies the closedness of α_1; if in addition α_1 is a derivative, a redefinition of the functions exists such that it simply disappears. When α_1 is closed but not exact, it cannot be eliminated but the representation is still equivalent to a linear representation. A pure projective representation appears when, instead of (4.2), we only require

$$U(h)\,U(g)\,|\psi_x\rangle = e^{i\alpha_2(x;\,g,h)}\,U(gh)\,|\psi_x\rangle, \qquad (4.12)$$

allowing the composition to depend on the "position" x through a phase factor. The mapping $U: g \to U(g)$ is no more a homomorphism. Applying (4.3) successively, we have

$$U(gh)\;|\psi_x\rangle = e^{i\alpha_1(x;gh)}\;|\psi_{xgh}\rangle\,.$$

$$U(h)\,U(g)\,|\psi_x\rangle = e^{i\alpha_1(x;g)}\,e^{i\alpha_1(xg;h)}\;|\psi_{xgh}\rangle =$$

$$= e^{i[\,\alpha_1(xg;h) - \alpha_1(x;gh) + \alpha_1(x;g)\,]}\,U(gh)\,|\psi_x\rangle. \qquad (4.13)$$

Consequently,

$$\delta\alpha_1(x;g,h) = \alpha_2(x;g,h)\,. \qquad (4.14)$$

In this case α_1 is not closed and the representation is no more equivalent to a linear one. The cochain α_2 is an *obstruction* to homomorphism. On the other hand, ray representations like (4.3) require α_2 to be exact.

Let us see what comes out from the imposition of associativity: equaling

$$U(f)\,[\,U(h)\,U(g)]\ |\psi_x\rangle = e^{i\alpha_2(x;g,h)}\ U(f)\ U(gh)\ |\psi_x\rangle =$$

$$= e^{i\alpha_2(x;g,h)}\ e^{i\alpha_2(x;gh,f)}\ \ U(ghf)\ |\psi_x\rangle \qquad (4.15)$$

and

$$[U(f)\ U(h)]\,[\,U(g)\ \ |\psi_x\rangle\,] = e^{i\alpha_2(xg;h,f)}\ U(hf)\ U(g)\ |\psi_x\rangle =$$

$$= e^{i\alpha_2(xg;h,f)}\ e^{i\alpha_2(x;g,hf)}\ \ U(ghf)\ \ |\psi_x\rangle \qquad (4.16)$$

brings forth, from (4.9), just the closedness of α_2,

$$\delta\alpha_2 = 0. \qquad (4.17)$$

This "associativity condition" is of course coherent with (4.14).

Condition (4.14) has an interesting consequence. Suppose it holds and let us proceed to a redefinition of the operators U: define new operators U^* by

$$U^*(g) = e^{-i\alpha_1(x;g)}\ U(g). \qquad (4.18)$$

They depend, through the phase, on the point x at which they will operate and are, in this sense, "gaugefied" versions of the previous $U(g)$. In terms of such operators, (4.13) becomes

$$U^{*}(h)\, U^{*}(g)\ |\psi_x\rangle = U^{*}(gh)\ |\psi_x\rangle, \qquad (4.19)$$

which is just of the form (4.2).

Concerning only the group operator representatives (and not the particular carrier space), it is expression (4.12) which characterizes a projective representation. Associativity implies that α_2 is a cocycle. If it is also exact, there exists a α_1 satisfying (4.14) which will appear as a ket phase and α_2 can be absorbed by the procedure just described into the "gaugefied" operators, in terms of which the representation reduces (but only locally in parameter space) to a linear one. We will say in this case that the representation is locally linear but globaly projective. The unitary quantum operators to be studied in next chapter will be of this type. If α_2 is closed but not exact, there exists no α_1 as in (4.14) and α_2 cannot be eliminated. The projective representation is not even locally equivalent to a linear representation and is not of the form (4.3). Consequently, it is better to reserve the name "ray representations" to locally linear representations.

If an exact cochain $\delta\beta_1$ is added to α_2, the exact part can be eliminated but the non-exact "core" cannot. Adding an exact cochain is an equivalence relation, the corresponding classes being the elements of the quotient space of the closed by the exact cochains. This quotient space is the additive cohomology group $H^2(G)$. There is a one-to-one relation between the inequivalent projective representations and the elements of $H^2(G)$, which thereby "classifies" them [7,8].

To obtain condition (4.17), we have taken associativity for granted in its usual way. If we are enough of a free thinker to accept that it holds only up to a phase factor,

$$[U(f)\ U(h)]\ U(g)\ |\psi_x\rangle = e^{i\alpha_3(x;g,h,f)} U(f)\,[U(h)\,U(g)]\,|\psi_x\rangle \quad (4.20)$$

then

$$\delta\alpha_2 = \alpha_3 \quad (4.21)$$

instead of (4.17). α_3 is a 3-cochain, as it takes three elements of G to give the number $\alpha_3(x; g, h, f)$. When it is nonvanishing α_2 is no more a cocycle and there is no associativity: α_3 is an obstruction to associativity. In principle, we can proceed with such successive steps of requirements and a corresponding hierarchy of closed and exact cochains. Nevertheless, associativity is part of the definition of a group and so desirable a property for a representation that it is usual to stop at this point. We say then simply that α_3 is an obstruction to the construction of projective representations.

It is also possible to introduce a notion akin to the interior product: given the p-cochain α_p, its "interior product" with $h \in G$ is that (p-1)-cochain $\iota_h \alpha_p$ satisfying

$$[\iota_h \alpha_p](x; g_1, g_2, \ldots, g_{p-1}) = \alpha_p(x; h, g_1, g_2, \ldots, g_{p-1}) \qquad (4.22)$$

for all $g_1, g_2, \ldots, g_{p-1}$. A natural further step is to introduce a formal "Lie derivative" with respect to a $h \in G$ by

$$\lambda_h = \delta \circ \iota_h + \iota_h \circ \delta. \qquad (4.23)$$

Some of its formal properties, again analogous to those of differential forms, are:

$$\delta \circ \lambda_h = \delta \circ \iota_h \circ \delta = \lambda_h \circ \delta; \qquad (4.24a)$$

$$(\lambda_h \alpha_0)(x) = \alpha_0(xh) - \alpha_0(x) = \delta \alpha_0(x;h); \qquad (4.24b)$$

$$(\lambda_h \alpha_1)(x;g) = \alpha_1(xg;h) - \alpha_1(x;hg) + \alpha_1(xh;g); \qquad (4.24c)$$

$$(\lambda_h \alpha_2)(x;g,j) = \alpha_2(xh;g,j) - \alpha_2(x;hg,j) + \alpha_2(xg;h,j). \qquad (4.24d)$$

The limited character of such analogies should however be stressed. Unlike differential forms, the above cochains are not acting on a linear space and consequently share with them only some of their properties. They lack a tensorial character and, as a consequence, all the qualities coming with it. For example, there are no basis in terms of which any p-cochain can be written.

5. The Fundamental Cocycle

As said in paragraph 3, it is the toroidal laticce formed by the labels (m,n) of Schwinger's operators S_{mn} that constitute quantum phase space. Our objective, to which we finally arrive, is to show that indeed a certain cocycle (α_2 below) exists which endows the space of a structure similar to the symplectic structure of classical phase space and tends to the symplectic form in the classical limit. Consider the unitary operators of chapter 3. It comes directly from (3.12) and (4.3) that

$$\alpha_1(k; S_{mn}) = \frac{\pi}{N}\,[(2k+m)\,n]\,, \qquad (5.1)$$

of which two particular cases are

$$\alpha_1(k; V) = \frac{2\pi}{N}\,k \qquad (5.2)$$

and

$$\alpha_1(k; U) = 0\,. \qquad (5.3)$$

We need the two expressions

$$\alpha_1(kS_{mn}, S_{rs}) = \alpha_1(k+m; S_{rs}) = \frac{\pi}{N}\,[2(k+m)+r]\,s \qquad (5.4)$$

and

$$\alpha_1(k; S_{mn}S_{rs}) = \frac{\pi}{N}[(2k+m+r)\,(n+s)] \qquad (5.5)$$

to verify, using (4.8), that

$$\delta\alpha_1(k; S_{mn}, S_{rs}) = \alpha_1(k\,S_{mn}; S_{rs}) - \alpha_1(k; S_{mn}S_{rs}) + \alpha_1(k, S_{mn}) =$$

$$= \frac{\pi}{N}\,[ms-nr]\,. \qquad (5.6)$$

This is nonvanishing in general, hinting, after the discussion of the previous chapter, to a globally projective character. Indeed, from (3.13) we obtain

$$\alpha_2(k; S_{mn}, S_{rs}) = \frac{\pi}{N}\,[ms-nr]\,, \qquad (5.7)$$

so that α_2 is exact:

$$\alpha_2(k; S_{mn}, S_{rs}) = \delta\alpha_1(k; S_{mn}, S_{rs}) \qquad (5.8)$$

for any pair S_{mn}, S_{rs}. This means that the representation only reduces to a linear one if we want to pay the price of "gaugefying" it as in (4.18): it is a ray representation, locally linear although globally projective.

Notice that $\alpha_2(k; S_{mn}, S_{rs})$ is independent of the state label k. A particular value of interest is

$$\alpha_2(k; U, V) = \alpha_2(k; S_{10}, S_{01}) = \frac{\pi}{N}. \tag{5.9}$$

That the cochain α_2 is a cocycle is a consequence of the associativity condition (3.17).

$$\delta\alpha_2(k; S_{mn}, S_{rs}, S_{kl}) = \alpha_3(k; S_{mn}, S_{rs}, S_{kl}) = 0 \tag{5.10}$$

Of course, this was already implied by the triviality (5.8) of α_2 and actually contained in the product rules (3.13). We should call atention to an obvious but important aspect. The cochains act on group elements to produce phases, exponentiated numbers. Unitary operators are not observables, only their hermitian exponents are. The parameters m,n appear always exponentiated also, as in (3.9) and in the continuum limit [as in (3.19)] they do seem to tend to observables with classical counterparts. It is as if the parameters belonged not to the group but to its algebra. We must consequently be prepared to the fact that the relation between α_2 and Ω is exponential and, for facility, compare the results of their respective actions. There is no obvious correlation between associativity and the property related to the closedness of Ω, the Jacobi identity (2.22) for the Poisson bracket. Associativity is a much more general condition, a property of every group while Jacobi identity, typically an integrability condition, appears (exponentiated, as a property of the generators) only for Lie groups. Presumably this general property gets somehow weakened in the limiting process. An analogy may however help to shed some light on this point. There is a strong similarity of the formalism above with the basic structure of gauge theories: α_1 recalls the gauge potential A, δ the covariant derivative D, α_2 the field strength F = DA. Or, it happens that in gauge theories the closedness of F, DF = 0 (the Bianchi identity) is precisely equivalent to the Jacobi identity for the gauge group generators [14]. We might conjecture that the closedness of α_2 is somehow related to that of Ω.

It is instructive to consider on the parameter space of the numbers m, n, r, etc column vectors $X_{mn} = \sqrt{\frac{\pi}{N}}\binom{m}{n}$, $X_{rs} = \sqrt{\frac{\pi}{N}}\binom{r}{s}$ etc, with them as components. The row vectors X^T_{mn}, X^T_{rs} will behave as dual vectors by simple scalar product. Then, with the usual product of rows, matrices and columns,

$$\alpha_2(k; S_{mn}, S_{rs}) = \frac{\pi}{N}\,[ms-nr] = X^T_{mn}\,\Omega\,X_{rs}\,, \qquad (5.11)$$

where Ω is the symplectic matrix of chapter 2. On the toroidal grid formed by the parameters $\alpha_2(k;\, S_{mn}, S_{rs})$ is proportional to the "area" defined by the vectors (m,n) and (r,s), as was the case for Ω in (2.21). We may also check that $\iota_{S_{mn}}\alpha_2$ is closed and takes a column vector X_{mn} into $X_{-n,m}$,

$$[\iota_{S_{mn}}\alpha_2](k;S_{rs}) = \frac{\pi}{N}(-n,m)\binom{r}{s} \qquad (5.12)$$

This duality corresponds to relation (2.10) established by Ω between vectors and forms. Furthermore, putting together the considerations on the continuum limit at the end of paragraph 3 and equations (2.14-15), we see that α_2 plays on the lattice torus a role quite analogous to that of the symplectic form: from (3.21), we see that in the continuum limit α_2 gives (minus) half the value (2.15) of Ω applied to the corresponding vectors:

$$\alpha_2(k;\, S_{ab}, S_{cd}) = -\frac{1}{2}[ad - cb] = -\frac{1}{2}\Omega\,(J_{ab}, J_{cd}).$$

Using (4.23) we find that

$$(\lambda_{S_{mn}}\alpha_2)(k;\, S_{rs}, S_{pq}) = 0 \qquad (5.13)$$

for all S_{mn}, S_{rs} and S_{pq}, stating the invariance of α_2 under all transformations of the Weyl group. In this sense, all of them are "canonical transformations". Another analogy, trivial to obtain but interesting, comes from the very definition of α_2: it vanishes when applied to two commuting elements, just as Ω vanishes when applied to two fields corresponding to dynamical functions whose Poisson bracket vanishes. Such two fields are tangent to the same Lagrange manifold. On QPS, this corresponds to subsets of intercommuting operators. Finally, from (5.8), we see that the role of the canonical form σ is played by the cochain α_1.

Points in QPS can be attained from each other by successive applications of the operators U and V. Operators S_{mn} will meanwhile acquire phases. This is better seen if we start with some state $|v_k\rangle$ and look such successive transformations as forming paths on QPS. Each time U is applied the state is shifted and each time V is applied the ket gains a phase. This phase depends on the state arrived at. In Fig.1(a), operator V acts at "k+1", but its inverse V^{-1} acts at "k". As a consequence of this point dependence, closed loops give a net result phase. Going around the loop in Fig.1(a), for example, will give to $|v_k\rangle$ a phase $\epsilon^2 = (2\pi/N)$. This ϵ^2 is the unit phase: it comes each time a unit cell in QPS is surrounded.

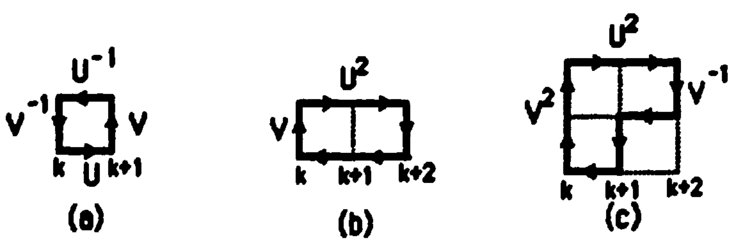

Fig.1 The simplest loops on QPS: (a) an elementary loop; (b) a double loop with negative sense; (c) a triple loop. Each enclosed elementary cell contributes a phase $\frac{2\pi}{N}$ to S_{mn}.

The sum of phases is algebraic: going around the unit loop in the inverse sense changes its sign. In our convention, positive sign is given by counterclockwise motion. So, the path of Fig.1(b) contributes a phase $(-2\varepsilon^2)$, that of Fig.1(c) a phase $(-3\varepsilon^2)$, etc. Closed loops may give vanishing phases. This is trivial for the two

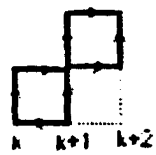

Fig.2 A loop giving a vanishing contribution to the phase of S_{mn}

closed paths generated by U^N and V^N, which simply close around the torus, but there are non-trivial cases: in Fig.2, the contributions from the two unit loops cancel each other. As α_2 measures just (half) the areas in units of ε, there is at work here a version of Gauss theorem: the total (algebraic) area circunvented by a loop is obtained by just following the loop step by step, at each step summing the corresponding α_1, as given by (5.2,3). As exhibited in Fig.3, the product $S_{mn} S_{rs}$ is equivalent to taking S_{mn} then S_{rs} only when $\alpha_2(k; S_{mn}, S_{rs}) = 0$. The closed paths of Figs.1 and 2 are projections of paths in the space of the operators S_{mn}, where the paths are, as a rule, open. A kind of non-integrability appears: starting from a given point, the phase at another point will depend on the path, unless the "flux" of a_2 through the surface defined by any two paths is zero. In this

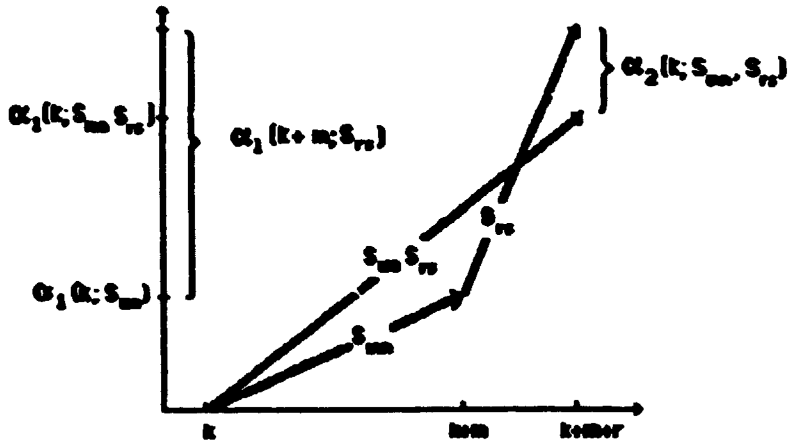

Fig.3 The fundamental cocycle measures the phase difference between $S_{mn} S_{rs}$ and the successive applications of S_{mn} and S_{rs}.

sense α_1 is a non-integrable phase like those of gauge theories [15] and α_2 would act as the corresponding "curvature". As already mentioned, there are many aspects in common with gauge theories in the present formalism, but we shall not discuss them here. Neither shall we consider the possible relation of α_1 to a generalized [16] Berry's phase [17], a subject which deserves further study.

As an example, the commutator $V^{-1}U^{-1}VU$ of Fig.1(a) produces in operator space (see Fig.4) an arc which fails to close precisely by the phase ε^2. Such trajectories in operator space only close when the unit cell is surrounded a multiple of N times, in which case it becomes a closed spiral. The role of α_2, similar to a curvature on QPS, is different here: as it measures such defects in the operator space, it is reminiscent of that of torsion in differential geometry.

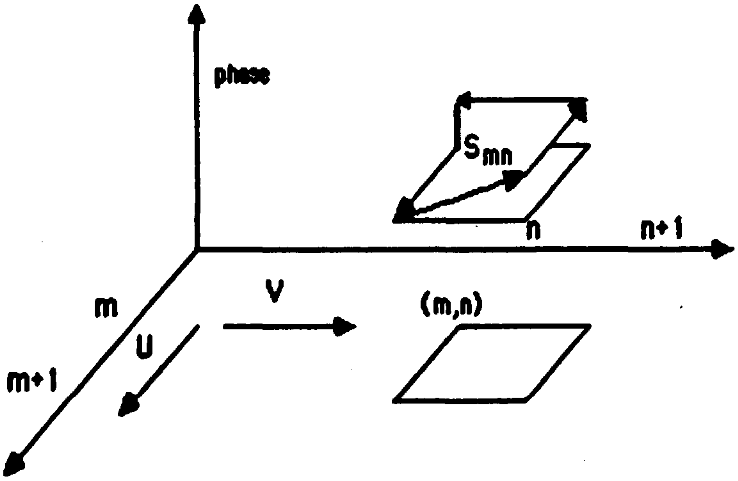

Fig. 4 An elementary loop in parameter space corresponds to an open trajectory in operator space.

In the continuum limit we must consider "large" regions of sizes $m\varepsilon$ and $n\varepsilon$ tending to limits a and b and the operators (now putting $\hbar$ back into our expressions) $U^m \to e^{iap/\hbar}$ and $V^n \to e^{iaq/\hbar}$. The phase $\alpha_2(k; U^m, V^n) = \frac{\pi}{N} mn = \frac{\varepsilon^2}{2} mn$ tends to $\frac{1}{2}ab$, just (half) the value of $\Omega(aX_q, bX_p)$. Actually, to examine the continuum limit, as well as to get some more insight on the role of the cocycle α_2 it is convenient to apply the formula giving the Weyl-Wigner transform $W(\mathbf{A}\,\mathbf{B}) = (\mathbf{A}\,\mathbf{B})_W$ of the product of two operators $\mathbf{A}$ and $\mathbf{B}$ in terms of their transforms $W(\mathbf{A}) = \mathbf{A}_W(q,p)$ and $W(\mathbf{B}) = \mathbf{B}_W(q,p)$, which is

$$W(\mathbf{A}\,\mathbf{B}) = e^{i(\hbar/2)[\partial^A_q \partial^B_p - \partial^A_p \partial^B_q]} \mathbf{A}_W(q,p)\, \mathbf{B}_W(q,p) \qquad (5.14)$$

The upper indices in ∂^A_q, ∂^B_p are reminders: ∂^A_q is the derivative with respect to q but which applies only on $A_W(q,p)$; ∂^B_p derives with respect to p but only acts on $B_W(q,p)$, etc. The Poisson bracket always comes up at first order in $\hbar$:

$$W(A\,B) = A_W(q,p)\,B_W(q,p) - (\hbar/2i)\{A_W(q,p), B_W(q,p)\} + \ldots \qquad (5.15)$$

but the Weyl-Wigner transformed functions $A_W(q,p)$ and $B_W(q,p)$ may still exhibit additional powers of $\hbar$, depending on their explicit form in terms of q and p. In fact, only in the strict classical ($\hbar \to 0$) limit will such functions reduce to their classical counterparts. Getting the Poisson bracket from a quantum commutator is only achieved when we pass from a non-commutative algebra to a commutative one at the price of ignoring the cell structure of quantum phase space. Only then $(i/\hbar)[A, B] \to \{A_{clas.}, B_{clas}\}$ [18]. To clarify this point, let us consider the operators $A = S_{a0} = e^{(i/\hbar)\,a\,q}$ and $B = S_{0b} = e^{(i/\hbar)\,b\,p}$. From the previous formalism, their product will be

$$A\,B = e^{i\alpha_2(A,B)}\,S_{ab} = e^{i\alpha_2(A,B)}\,e^{(i/\hbar)(a\,q + b\,p)} \qquad (5.16)$$

The Weyl-Wigner transform of the right-hand side is

$$W(A\,B) = e^{i\alpha_2(A,B)}\,e^{(i/\hbar)(a\,q + b\,p)}, \qquad (5.17)$$

where now q and p behave like classical variables. On the other hand, (5.14) will say that

$$W(A\,B) = e^{i(\hbar/2)[\partial A_q \partial B_p - \partial A_p \partial B_q]} \left[e^{(i/\hbar)(aq+bp)} \right]. \qquad (5.18)$$

we see that in some way α_2 sums up all the intricate action of the exponentiated operator. The present example is specially simple but reflects much of the fundamental structure of the continuum quantum phase space, as in this case S_{ab} is a typical base element. The Poisson bracket is constant and it is possible to write down the exact result,

$$W(A\,B) = e^{-(i/2\hbar)ab}\, e^{(i/\hbar)(aq+bp)}, \qquad (5.19)$$

so that

$$\alpha_2(A, B) = -(i/2\hbar)\{A, B\}. \qquad (5.20)$$

An analogous result would come if we took operators of type (3.21). In such cases related to the harmonic oscillator, whose semiclassical aproximation is exact, α_2 gives the classical result up to a factor $\hbar^{-1}$. This is indeed the hallmark of the quantum structure of phase space embodied in α_2, which is not at all a classical object. It is expressed above in terms of the Poisson bracket, but of Weyl-Wigner representatives of quantum objects. In this continuum case, α_2 heralds the non-commutativity of the basic pair **q** - **p**. In the general case, it highlights the fundamental cellular structure of QPS.

6. Summing up

Every feature of Classical Mechanics stems from some quantum mechanical feature. Let us try to review the analogies and differences between the cocycle α_2 and the symplectic form. To begin with, Ω is globally defined on the classical state space and $\alpha_2(k; S_{mn}, S_{rs})$ is independent of the state label k. The first is invariant under canonical transformations, the second under all unitary transformations. Both measure areas defined by vectors in the corresponding spaces. The closedness of Ω guarantees the Jacobi identity for the Poisson brackets, that of α_2 the projective character of the Weyl representation. Classical Lagrange manifolds are on QPS replaced by subsets of intercommuting unitary operators. The symplectic form is a linear operator, which we could not expect of α_2. Finally, α_2 tends to the symplectic form Ω when, in the continuum limit, the non-commutativity of dynamical variables is relaxed. The cocycle α_2 is that feature of quantum mechanics on which the symplectic structure of classical mechanics casts its roots.

References

[1] D. Galetti and A.F.R. de Toledo Piza, Physica **149A** (1988) 267.

[2] W.K. Wooters, Ann. of Phys. **176** (1987) 1

[3] J. Schwinger, Proc.Nat.Acad.Sci. **46** (1960) 570,893, **47** (1961) 1075. See also J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970)

[4] V.Arnold, *Mathematical Methods in Classical Mechanics* (Springer, New York, 1978)

[5] R. Abraham and J.E. Marsden, *Foundations of Mechanics* ,2nd ed. (Benjamin-Cummings,Reading, Mass. 1978)

[6] M. Hamermesh, *Group Theory and its Applications to Physical Problems* (Addison-Wesley, Reading, Mass. 1962)

[7] L. Michel, in F.Gursey (ed.), *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Istambul Summer School of Theoretical Physics 1963 (Gordon and Breach, New York, 1964).

[8] A.A. Kirillov, *Elements of the Theory of Representations* (Springer, Berlin,1976)

[9] J.E. Moyal, Proc. Cambridge Phys. Soc. **41** (1949) 99

[10] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950), chap.4, S 14.

[11] C. Isham, in B.deWitt and R.Stora (eds.), *Relativity, Groups and Topology II*, Les Houches Summer School 1983 (North Holland, Amsterdam, 1984)

[12] G. 't Hooft, Nucl.Phys. **B138** (1978) 1

[13] L.D. Faddeev and S.L. Shatashvilli, Theor. Math.Phys. **60** (1984) 770

[14] Y. M. Cho, J. Math.Phys. 16 (1975) 2029

[15] T.T. Wu and C.N.Yang, Phys.Rev. **D12** (1975) 3845

[16] Y. Aharonov and J. Ananden, Phys.Rev.Lett. **58** (1987) 1593

[17] M.V. Berry, Proc.Roy.Soc. **A392** (1984) 45

[18] S.R. de Groot & L.G. Suttorp, *Foundations of Electrodynamics* (North-Holland, Amsterdam, 1972), chap. VI and appendix.