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HARMONIC MAP HEAT FLOW WITH FREE BOUNDARY

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HARMONIC MAP HEAT FLOW WITH FREE BOUNDARY *

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ABSTRACT

In this paper, we study the harmonic map heat flow with free boundary from a Riemannian surface with smooth boundary into a compact Riemannian manifold. As a consequence, we get at least one disk–type minimal surface in a compact Riemannian manifold without minimal 2–sphere.

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1. THE MOTIVATION

Let (M,g) be a normalized compact Riemannian surface, i.e. a Riemannian surface whose boundary consists of a finite number of closed geodesics of (M,g). Let (N,h) be a compact Riemannian manifold of dimension n. For any smooth map $u : M \to N$, we define an energy of u by $E(u) = \int_M |du|^2 dM$ where $|du|$ is the Hilbert–Schmidt norm of the operator du.

Let K be a k–dimensional closed submanifold of N and we define

$$C(K) = \{u \in C^2(M,N);\ u(\partial M) \subset K\}$$

Then the critical point of $E(\ )$ over $C(K)$ is a harmonic map with free boundary. The motivation of this paper comes from following basic problem: WHEN DOES A GIVEN COMPONENT OF $C(K)$ CONTAIN A HARMONIC MAP WITH FREE–BOUNDARY.

A natural way to attack this problem is to flow u_0 in $C(K)$ along the negative L^2–gradientline of E. That is, we want to find $u : M \times (0,+\infty) \to N$ satisfying following equations:

$$\partial_t u(x,t) = \tau(u(x,t)), \forall (x,t) \in M \times (0,\infty) \tag{1}$$

$$u(\,,0) = u_0 \tag{2}$$

$$u(x,t) \in K, \quad a.a.x \in \partial M \quad \text{for} \quad t \geq 0, \tag{3}$$

$$du_{(x,t)}(n) \perp T_{u(x,t)}K, \quad \forall (x,t) \in \partial M \times (0,\infty) \tag{4}$$

where $\tau(u)$ is the tension field of u in the sense of Eells–Sampson [ES], n is the inward unit normal on M, and "$\perp$" means orthogonal.

A global way to write (1) in one chart in the range was found in [ES]. We adapt a statement from [H]. We imbed N in a Euclidean space E^d by Nash's isometric imbedding theorem, but we prefer to change the ordinary metric on E^d. Let T be tubular neighbourhood of N in E^d, define $\iota : T \to T$ be a involution having precisely N for its fixed point set, take an extension of the metric h to T, and average it under the action of ι, then ι is an isometry. Finally, let B be a large ball (in E^d) containing T, and extend the metric on T smoothly to all of E^d so as to equal the Euclidean metric outside of B, then we obtain a new metric vector space R^d. In R^d, B is also convex. If we consider u as a map from M to R^d, then (1) can be written as:

$$\partial_t u^i = \Delta_M u^i + A^i_u(\nabla u, \nabla u),\ 1 \leq i \leq d \tag{5}$$

where $A_u : T_u N \times T_u N \to R^d$ be the second fundamental form of N in R^d.

Since the natural space for E making sense is $H^1(M,N)$, we shall enlarge the initial value class of (1) to

$$H(K) = \{u \in H^1(M,N);\ u(x) \in K\ a.a.x \in \partial M\}$$

By the argument of [SU], we know that $C(K)$ is dense in $H(K)$. Our main result is the following:

Theorem 1 Let $u_0 \in H(K)$, then there exists a (distribution) global solution u of (1–4), which is regular except a finite point set $\{(x^l, T^l)\}_{1\leq l\leq L} \subset M \times (0,\infty)$ such that at each point (x^l, T^l), there exist sequences $x^l_j \to x^l, t^l_j \nearrow T^l$, and $R^l_j \searrow 0$ such that (a) if $x^l \in intM$, then $u(x^l + R^l_j \times x, t^l_j) \to u_l(x)$ in $H^2_{loc}(R^2, N)$ where $u_l : R^2 \to N$ can be extended to a minimal 2–sphere in N.

(b) if $x^l \in \partial M$, then

$$u(x^l + R^l_j \times x, t^l_j) \to u_l(x) \quad \text{in} \quad H^2_{loc}(R^2_+, N)$$

where $u_l : R^2_+ \to N$ can be extended to minimal 2–disk in N with free boundary in K.

The solution u is unique in this class and the singularities are characterized by the condition that $\lim_{t^l_j \to T^l} E(u(t), B_R(x^l)) \geq \epsilon_0$, $\forall R > 0$ where ϵ_0 is a uniform constant.

If $u_0 \in C(K)$, then the solution u is regular on $M \times [o, \bar{T})$ where $\bar{T} = \min\{T^l\}$.

As a direct consequence of Theorem 1, we obtain

Theorem 2 Let N be a compact Riemannian manifold within no minimal 2–sphere. Let K be a connected and simply connected closed submanifold of N. There exists at least one minimal 2–disk with free boundary in K.

So our Theorem 2 is a generalization of the reuslt of [St1]. The main difficulty in proving Theorem 1 is as follows. We cannot directly use implicity function theorem to obtain the small time existence of a solution of (1)–(4), but we work through by proving a uniqueness theorem and using fixed point theorem after a subtle geometrical construction. The large time behaviour of the solution was studied as in [St]. Our argument is similar to [St2]. I should mention the work of Kungching Chang [C] which is very important in relating harmonic map heat flow with minimax principle and Morse theory. In the following, without loss of generality, we will assume that the injectivity radius of (M, g) is one and the sect$(N) \leq 1$.

NOTATIONS

For a domain $\Omega \subset M$, $-\infty \leq s < t \leq +\infty$ let

$$\Omega^t_s = \Omega \times (s,t), \Omega^0_s = \Omega_s, \Omega^t_0 = \Omega^t$$

and the space

$$V(\Omega^t_s) = \{u \in C^0([s,t]; H^1(\Omega, N));\ |\nabla^2 u|,\ \partial_t u| \in L^2(\Omega^t_s)\}$$

where the derivatives are taken in the distribution sense.

∇ denotes the covariant derivative on (M,g). In particular, for a local frame $\{e_1, e_2\}$ we write $\nabla_i = \nabla_{e_i}$ for $i = 1,2$.

At a local frame of x_0, we denote $x_0 + x = \exp_{x_0} x$ for $x \in R^2$. $\zeta(x) \in C_0^3(B_{2R}^M(x_0))$ denotes a non–increasing function of the distance $dist_M(x, x_0)$ such that $\varsigma = 1$ on $B_R^M(x_0)$ and $= 0$ outside $B_{2R}^M(x_0)$ and $|\nabla \varsigma| \leq 3R^{-1}$

c denotes a generic constant depending only on M, N and K sometimes numbered for clarity.

2. A–PRIORI BOUNDS

The following Sobolev–type equality is crucial in our estimates.

Lemma 1 There exist two constants c and R_0 for any $R \in (0, R_0)$, $T \leq \infty$, and any $u \in V(M^T)$, there holds the estimate

$$\int_{M^T} |\nabla u|^4 \, dM\,dt \leq c \operatorname*{essup}_{(x,t)\in M^T} \int_{B_R(x)} |\nabla u|^2 \, dM \Big(\int_{M^T} |\nabla^2 u|^2 \, dM\,dt + R^{-2} \int_{M^T} |\nabla u|^2 \, dM\,dt \Big) .$$

Moreover, for any $x_0 \in M$, ς as before, we have the estimate:

$$\int_{M^T} |\nabla u|^4 \varsigma dM\,dt \leq c \operatorname*{essup}_{0\leq t\leq T} \int_{B_R(x_0)} |\nabla u|^2 \, dM \Big(\int_{M^T} |\nabla^2 u|^2 \varsigma dM\,dt + R^{-2} \int_{M^T} |\nabla u|^2 \varsigma dM\,dt \Big) .$$

Proof The same argument as that of Lemma 3.2 in [ST].

Q.E.D

Now let us introduce a key quantity

$$\epsilon(R) = \sup_{(x,t)\in M^T} R(u(\cdot, t); B_R(x)) ,$$

which describes the energy distribution of the map u over all balls of radius R in M and plays an important role in following estimates.

Lemma 2 If $u \in V(M^T)$ be a solution of (1)–(4), then $E(u(t))$ is absolutely continuous in $t \in [0, T]$ and there holds the estimate

$$E(u(t)) + \int_{M^t} |\partial_t u|^2 \leq E(u_0) \tag{6}$$

Proof By Lemma 1 we may multiply (5) by u_t and integrate. On account of (3)–(4) this gives for any $s, t \in [0, T]$:

$$\int_s^t \frac{d}{dt} E(u(t))\, dt + \int_{M_s^t} |\partial_t u|^2 \, dM\,dt = 0$$

and the claim follows.

Q.E.D

Lemma 3 There exist constants $c_1, \epsilon_1 > 0$ depending only on M, K and N such that for any solution $u \in V(M^T)$ of (1)–(4) and any $R \in (0, 0.5]$ there holds the estimate

$$\int_{B_R(x)} |\nabla u(T)|^2 + \int_{B_R(x)^T} |\nabla^2 u|^2 \leq c_1 E(u_0) + c_1 T R^{-2} \sup_{0 \leq t \leq T} E(u(t))$$

provided $\epsilon(2R) \leq \epsilon_1$.

Proof Testing (5) with $-\Delta_M u \varsigma^2$ we obtain

$$\int_{M^T} |\Delta u|^2 \varsigma^2 \, dM \, dt + \int_M |\nabla u(T)|^2 \varsigma^2 \, dM - \int_M |\nabla u_0|^2 \, dM$$

$$\leq c\varepsilon_1 \int_{M^T} |\nabla^2 u|^2 \varsigma^2 \, dM \, dt + cTR^{-2} \sup_{0 \leq t \leq T} E(u(t)) \, . \tag{7}$$

Without loss of generality, we assume $x_0 \in \partial M$. Let $\{e_1, e_2\}$ be a local frame on $B_R(x_0)$ such that e_1 is the outward normal on ∂M. Testing (5) with $-\nabla_2^2 u \varsigma^2$ we find

$$\int_{M^T} \partial_t \left(\frac{1}{2} |\nabla_2 u|^2\right) \varsigma^2 + \int_{M^T} |\nabla \nabla_2 u|^2 \varsigma^2 - CTR^{-2} \sup_{0 \leq t \leq T} E(u(t))$$

$$\leq c\Big(\int_{M^T} |\nabla_2 u|^2 \varsigma^2 + \int_{M^T} |\nabla u|^2 |\nabla_2^2 u|\Big) + \Big|\int_{(\partial M)^T} \langle \nabla_1 u, \nabla_2^2 u \rangle \varsigma^2\Big| + \int_{M^T} |\nabla u|^2 \varsigma^2 \, . \tag{8}$$

For $u \in K$ now let $\{G_1(u), \ldots, G_{n-k}(u)\} \subseteq TN$ be the outward normal frame to K. By (4),

$$\nabla_1 u = \sum_{i=1}^{n-k} \langle \nabla_1 u, G_i(u) \rangle G_i(u) = \sum_{i=1}^{n-k} (\nabla_1 u \cdot G_i(u)) G_i(u) \quad \text{on} \quad \partial M \, .$$

By (3)

$$\langle \nabla_2 u, G_i(u) \rangle = 0 \quad a.e. \quad \text{on} \quad \partial M \, .$$

Differentiating this equality w.r.t e_2 on ∂M yields that

$$\langle G_i(u), \nabla_2^2 u \rangle = -\langle \nabla G_i(u) \cdot \nabla_2 u, \nabla_2 u \rangle \, ,$$

hence on ∂M,

$$\langle \nabla_1 u, \nabla_2^2 u \rangle = -\sum_{i=1}^{n-k} (\nabla_1 u \cdot G_i(u))(\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u) \, .$$

Smoothly extended G_i to $B \subset R^d$ and by Stokes' theorem,

$$\begin{aligned}\int_{\partial M} \nabla_1 u \cdot \nabla_2^2 u \varsigma^2 &= \int_M \Delta u \cdot [G_i(u)(\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u)]\varsigma^2 \\ &+ \int_M \nabla u \cdot \nabla[G_i(u)(\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u)]\varsigma^2 \\ &+ \int_M \nabla u \cdot G_i(u)(\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u)\nabla \varsigma^2 \\ &\leq c\int_M (|\nabla^2 u||\nabla u|^2 + |\nabla u|^4)\varsigma^2 + |\nabla\varsigma|^2 |\nabla u|^2\end{aligned}$$

Hence for any $\delta \in (0,1)$ we know from (8):

$$\begin{aligned}&\frac{1}{2}\int_M |\nabla_2 u(T)|^2\varsigma^2 - \frac{1}{2}\int_M |\nabla_2 u_2|^2\varsigma^2 + \int_{M^T} |\nabla\nabla_2 u|^2 \varsigma^2 \\ &\leq (c\varepsilon_1 + \delta)\int_{M^T} |\nabla^2 u|^2\varsigma^2 + c(\delta)TR^{-2} \sup_{0\leq t\leq T} E_{(u(t))} .\end{aligned}$$

Since

$$\begin{aligned}|\nabla^2 u|^2 &= |\nabla_1^2 u|^2 + |\nabla_2^2 u|^2 + 2|\nabla_1\nabla_2 u|^2 \\ &= (\Delta u - \nabla_2^2 u)^2 + |\nabla_2^2 u|^2 + 2|\nabla_1 \nabla_2 u|^2 \\ &\leq 2(|\Delta u|^2 + 2|\nabla\nabla_2 u|^2) ,\end{aligned} \tag{9}$$

we obtain from (9) and (7):

$$\int_{M^T} |\nabla^2|^2\varsigma^2 \leq (c\varepsilon_1 + 4\delta)\int_{M^T} |\nabla^2 u|^2 + cE(u_0) + c(\delta)TR^{-2} \sup_{0\leq t\leq T} E(u(t)) ,$$

and the claim follows for $\epsilon_1, \delta > 0$ small enough.

Q.E.D

Using

$$\int_{M^T} |\partial_t u|^2\varsigma^2\, dM\, dt \leq c\int_{M^T} |\nabla^2 u|^2 \varsigma^2\, dM\, dt + c\int_{M^T} |\nabla u|^4 \varsigma^2\, dM\, dt ,$$

in the above argument we obtain the following:

Lemma 4

There exist constants $c_2, \epsilon_2 > 0$ depending only on M, K and N such that for any solution $u \in V(M^T)$ of (1)–(4), any $R \in (0, 0.5)$, any $x \in M$ there holds:

$$E(u(T), B_R(x_0)) \leq 2E(u_0; B_{2R}(x_0)) + c_2 TR^{-2}$$

and

$$\int_{B_R(x_0)} |\nabla^2 u|^2\, dM\, dt \leq 4c_2(1 + TR^{-2})$$

provided $\varepsilon(2R) \leq \epsilon_2$.

3. UNIQUENESS

Let $\hat{M}$ be the double of M, it is well–known that $\hat{M}$ is a closed Riemannian surface. We will denote M_a the a–tubular neighbourhood of M in $\hat{M}$. Then for a small enough and for any $x \in M \cap M_a$, there exists a unique projection point $x' \in \partial M$ and a unique point $\bar{x} \in M_a/M$ that has also x' as the unique projection onto ∂M along the geodesic $c(t) = \exp_{x'}\{-\exp_{x'}^{-1} x\}$. So we may define an involution Φ (that is $\Phi^2 = Id$) by

$$\Phi(x) = \bar{x} \quad \text{if} \quad x \in M - M_a \quad \text{and} \quad \Phi(\bar{x}) = x \quad \text{for} \quad \bar{x} \in M_a/M\,.$$

Similarly, we can define an involution I in a b–neighbourhood K_b of K in N. In what follows, we will write $M_{-a} = \{x \in M;\ dist(x, \partial M) \geq a\}$. For a solution $u \in V(M^T)$ to (1)–(4)now we consider the set

$$\tilde{\Omega} = \{(x,t) \in (M_a/M))^T,\ u(\Phi(x),t) \in K_b\}$$

and define an extension of u to $\hat{\Omega} = M^T \cup \tilde{\Omega}$

$$\hat{u}(x,t) = \begin{cases} u(x,t), & \text{if } x \in M; \\ I(u(\Phi(x),t)), & \text{if } (x,t) \in \tilde{\Omega} \end{cases}$$

By the interior regularity results of (1) (see [C]), $\tilde{\Omega}$ is open in $M_a \times (0,T)$ and it is meaningful to consider

$$I(\hat{u}(\Phi(x),t) = u(x,t)$$

for $(x,t) \in M^T$ with $(\Phi(x),t) \in \tilde{\Omega}$. It can be verified that:

Lemma 5 Suppose $u \in V(M^T)$ solves (1)–(4), then $\hat{u}$ satisfies

$$\nabla^2 \hat{u}, \partial_t \hat{u} \in L^2_{loc}(\hat{\Omega})$$

and $\hat{u}$ is a weak solution to the system

$$\partial_t \hat{u} - \Delta_{\hat{M}} \hat{u} + \Gamma_{\hat{u}}(\nabla \hat{u}, \nabla \hat{u}) = 0 \tag{10}$$

on $\hat{\Omega}$, where $\Gamma_{\hat{u}}(\cdot,\cdot)$ be a bounded bilinear form.

Proof To obtain (10) note the pointwise estimates for all $(x,t) \in \tilde{\Omega}$.

$$|\partial_t \hat{u}(x,t)| \leq |DI(u) \cdot \partial_t u(\Phi(x),t)|$$

$$\leq c|\partial_t u(\Phi(x),t)|,\ etc. \tag{$*$}$$

Moreover, it is easy to see that $\hat{u}(t)$ is of class H^2 for a.e. $t \in [0,T]$ on its domain $\hat{\Omega}(t)$. Indeed $\hat{u}(t) \in H^2$ separately on M and $\hat{\Omega}(t)/M =: \tilde{\Omega}(t)$ for a.e.t. By properties (3)–(4) we have that for any

$$\int_{\hat{\Omega}(t)} \hat{u}\nabla^2 \varphi dM = \int_{\tilde{\Omega}(t)} \hat{u}\nabla^2 \varphi dM + \int_M \hat{u}\nabla^2 \varphi dM$$
$$= \int_{\tilde{\Omega}(t)} \nabla^2 u\varphi dM + \int_M \nabla^2 u\varphi dM \,,$$

while the boundary terms cancel. Hence the L^2–function $\nabla^2 u(t)$ (defined on $M \cup \tilde{\Omega}(t)$) is the second distributional derivative of $\hat{u}(t)$, and $\hat{u}(t) \in H^2(\hat{\Omega}(t), R^d)$ for a.e.t.

Since $I^2 = Id$ on K_b and for $v \in TK_b$, we obtain that

$$DI_{I(y)} DI_y v = v$$

for $y \in K_b$. Let $\bar{v} = DI(v)$, then by differentiate the above equation we have that

$$D^2 I_{I(y)}(DI_y(v), DI_y(v)) + DI_{I(y)} D^2 I_y(v,v) = 0$$

here we assume $D_v v = 0$ at the point y. Then

$$D^2 I_{I(y)}(\bar{v},\bar{v}) = -DI_{\bar{y}} D^2 I_y(v,v)$$

where $\bar{y} = I(y)$. Let e_i be a moving frame on M_a, by the relation

$$u(x,t) = I(\hat{u}(\Phi(x),t))$$

we obtain further that

$$\begin{aligned}
\partial_t u(x,t) &= DI_{\hat{u}} \cdot \partial_t \hat{u}(x,t) \,, \\
\nabla_i u(x,t) &= DI_{\hat{u}} \cdot \nabla \hat{u}_{(\bar{x},t)}(\nabla_i \Phi(x), t) \,, \\
\Delta_M u(x,t) &= \nabla_i \nabla_i u(x,t) \\
&= D^2 I_{\hat{u}}(\nabla_{\bar{e}_i}\hat{u}, \nabla_{\bar{e}_i}\hat{u}) + DI_{\hat{u}} \cdot \nabla^2 \hat{u}_{(x,t)}(\bar{e}_i, \bar{e}_i) \\
&= DI_{\hat{u}} \cdot \Delta_{M_a} \hat{u}_{(\bar{x},t)} + \\
&\quad + DI_{\hat{u}} \cdot D^2 I_{\Phi(\hat{u})}(\nabla\Phi(\nabla_{\bar{e}_i}\hat{u}), \nabla\Phi(\nabla_{\bar{e}_i}\hat{u}))
\end{aligned}$$

where $\bar{e}_i = \nabla\Phi(e_i)$. By (1) we have (10).

Q.E.D

Later we will use another implication of (3)–(4). Define $\phi \in C_0^6(R)$ satisfying $0 \le \phi \le 1$, $\phi(s) = 1$ if $|s| < \frac{a}{2}$, while $\phi(s) = 0$, if $|s| > a$. Then for any solution $u \in V(M^T)$ with extension $\hat{u}$ the function

$$\hat{\varphi}(x,t) = \begin{cases} 1 & \text{if } x \in M \\ \phi(dist_N(\hat{u}(x,t), \Phi(\hat{u}(x,t)))), & \text{if } (x,t) \in \tilde{\Omega} \\ 0 & \text{if } (x,t) \notin \tilde{\Omega} \end{cases} \qquad (11)$$

belongs to $H^2_{loc}(\hat{\Omega}(t))$ for a.e.t. and satisfies a.e.

$$|\nabla\hat{\varphi}| \leq c|\nabla\hat{u}|, \quad |\nabla^2\hat{\varphi}| \leq c|\nabla^2\hat{u}|, \tag{11}'$$

Moreover, the distributional derivative $\partial_t\hat{\phi} \in L^2_{loc}$ and

$$|\partial_t\hat{\phi}| \leq c|\partial_t\hat{u}| \quad a.e. \tag{12}$$

Lemma 6 Suppose $u_1, u_2 \in V(M^T)$ are two weak solutions to (1)–(4) with $u_1(0) = = u_2(0) = u_0$, then $u_1 = u_2$.

Proof Let $\hat{u}_j$ be the extension of u_j defined in the above of Lemma 5 and $\hat{\phi}_j$ the associated truncation function $j = 1, 2$. Define $\hat{\phi} = \min\{\hat{\phi}_1, \hat{\phi}_2\}$ and $|\nabla\hat{u}| = |\nabla\hat{u}_1| + |\nabla\hat{u}_2|$, etc. Subtracting equations (5) for u_1, u_2 and testing with $(u_1 - u_2)\hat{\phi}^2$ we obtain the following estimates for $\hat{v} = \hat{u}_1 - \hat{u}_2$:

$$|\partial_t\hat{v} - \Delta_{\hat{M}}\hat{v}| \leq c(|\nabla\hat{u}||\nabla\hat{u}| + |\nabla\hat{u}|^2|\hat{v}|)$$

$$\begin{aligned}\int_{M^T}(\frac{1}{2}\partial_t(|\hat{v}|^2\varphi^2) + |\nabla\hat{u}|^2\hat{\varphi}^2) \leq c\int_{\hat{M}^T}\{&|\hat{v}|^2|\partial_t\hat{\varphi}|\hat{\varphi} + \\ &+ |\nabla\hat{v}||\hat{v}||\nabla\hat{\varphi}|\hat{\varphi} + |\nabla\hat{v}||\hat{v}|\hat{\varphi}^2 + \\ &+ |\nabla\hat{u}||\nabla\hat{v}||\hat{v}|\hat{\varphi}^2 + |\nabla\hat{u}|^2|\hat{v}|^2\hat{\varphi}^2\}dM\,dt\end{aligned}$$

By (11)–(12) we may bound the above by

$$c\int_{M^T}[|v|^2(|\partial_t u| + |\nabla u|^2) + |\nabla v||v|(1 + |\nabla u|)]dM\,dt$$

here we have used the mean value theorem:

$$\begin{aligned}\hat{v}(x,t) = (\hat{u}_1 - \hat{u}_2)(x,t) &= (I \circ u_1 - I \circ u_2)(\Phi(x),t) \\ &= \int_0^1 DI(u_2 + \theta(u_1 - u_2)) \cdot (u_1 - u_2)\,d\theta|_{(\Phi(x),t)}\end{aligned}$$

and the bounds

$$|\hat{v}(x,t)| \leq c|v(\Phi(x),t)|, \quad |\nabla\hat{v}(x,t)| \leq c|\nabla v(\Phi(x),t))|$$

for all $x \notin M$, and integral estimates for $\hat{v}$ and $\nabla\hat{v}$ can be obtained from estimates for v and ∇v on M^T respectively.

From Holder's inequality we now obtain

$$\begin{aligned}&\int_M |v(T)|^2 dM + \int_{M^T}|\nabla v|^2 dM\,dt \\ &\quad \leq c(\int_{M^T}|\partial_t u|^2 + |\nabla u|^4)^{1/2} \cdot (\int_{M^T}|v|^4 dM\,dt)^{1/2} \\ &\quad\quad + \frac{1}{2}{}_{M^T}|\nabla v|^2 dM\,dt + c\int_{M^T}|v|^2 dM\,dt.\end{aligned}$$

Without loss of generality, we may assume that T is chosen such that

$$\int_M |v(T)|^2\, dM = \sup_{0\le t\le T}\int_M |v(t)|^2\, dM\ .$$

Using Lemma 1 for the estimate of L^4–norm of v, we conclude that

$$\sup_{0\le t\le T}\int_M |v(t)|^2\, dM + \int_{M^T} |\nabla v|^2\, dM\, dt \le cT \sup_{0\le t\le T}\int_M |v(t)|^2\, dM\, dt+$$

$$+c\Big(\int_{M^T} |\partial_t u|^2 + |\nabla u|^4\Big)^{1/2}\Big(\sup_{0\le t\le T}\int_M v(t)^2 + \int_{M^T}|\nabla v|^2\Big)$$

and by absolute continuity of the Lebesgue integral the above can be bounded by

$$\frac{(\sup_{0\le t\le T}\int_M |v(t)|^2 + |\nabla v|^2)}{2}$$

for $T > 0$ sufficiently small. Hence $v = 0$ on M^T for this T. By iteration $v = 0$ on M^T for any $T > 0$ for u_1, u_2 both making sense.

4. LOCAL EXISTENCE

Theorem A For $u_0 \in C(K)$, there exists a number $T > 0$ depending only on M, K and N such that (1)–(4) admits a solution in $V(M^T)$.

Proof We want to prove this theorem by Schauder's fixed point principle. Let

$$\hat{u}_0(x) = \begin{cases} u_0(x,t) & \text{if } x \in M; \\ I(u_0(\Phi(x))), & \text{if } x \in M_a/M \end{cases}\ .$$

and for any $T > 0$ and $\sigma > 0$ sufficiently small let

$$\mathcal{M} = \{\hat{u} \in V(M_a^T; R^d)\,|\,\hat{u}(0) = \hat{u}_0,\ \operatorname*{esssup}_{0\le t\le T}\int_{M_a^T}|\nabla\hat{u}(t) - \nabla\hat{u}_0|^q \le \sigma\}$$

where q be a number ≥ 4 to be determined later. $\mathcal{M}$ be a closed convex set in $V(M^T)$. To $\hat{u} \in \mathcal{M}$ we now associate the unique solution $\hat{v} = f(\hat{u}) \in V(M_a^T)$ of the Cauchy–Dirichlet problem

$$\partial_t\hat{v} - \Delta_{\hat{M}}\hat{v} + \Gamma_{\hat{u}}(\nabla\hat{u}, \nabla\hat{u}) = 0 \quad \text{in} \quad M_a^T \tag{13}$$

$$\hat{v}(0) = \hat{u}_0 \tag{14}$$

$$\hat{v} = \hat{u} \quad \text{on} \quad \partial M_a \times [0, T] \tag{15}$$

It is a standard fact that we have the following uniform estimate

$$\int_{M_{-a}^T}(|\partial_t\hat{v}|^{q/2} + |\nabla^2\hat{v}|^{q/2})\, dM\, dt \le c(u_0)$$

(see [LSU], Theorem 4.9.1). By Lemma 4.2 of [C] we may bound any derivative of $\hat{v}$ on $M_{-a} \times [O,T]$ if u_0 is sufficiently smooth on M. This is a parabolic–type Bernstein's estimate. Hence for $a > 0, T > 0, \sigma > 0$ small enough, the point set $\{\hat{v}(\Phi(x),t);\ x \in \partial M_a\}$ will lie in a b–neighbourhood of K in N and their reflection in K will be defined. We may therefore define a map $F : \mathcal{M} \to \mathcal{M}$ by letting $\hat{w} = F(\hat{u})$ be the unique solution to the problem

$$\partial_t \hat{w} \Delta_{\hat{M}} \hat{w} + \Gamma_{\hat{u}}(\nabla \hat{u}, \nabla \hat{u}) = 0 \quad \text{in} \quad M_a^T\,, \tag{16}$$

$$\hat{w}(0) = \hat{u}_0 \tag{17}$$

$$\hat{w}(x,t) = I(\hat{v}(\Phi(x),t)) \quad \text{on} \quad \partial M_a \times [0,T] \tag{18}$$

Note that if $q > 4$ is chosen sufficiently large from Lemma 4.2 of [C] and the regularity of v on ∂M_{-a} we obtain uniform Holder estimates for $\nabla \hat{w}$ in (x,t). Therefore, for this small T we have $\hat{w} \in \mathcal{M}$ and $F : \mathcal{M} \to \mathcal{M}$ and $F(\mathcal{M})$ is bounded in $V(M^T)$.

In fact, F is a compact operator. Let $\mathcal{B}$ be a bounded subset of $\mathcal{M}$, then this $\mathcal{B}$ is compact in $L^2([0,T];\ L_1^4(M_a))$ by the weak compactness of $V(M_a^T)$ and uniform boundedness of $\nabla \hat{u}(t)$ in $L^q(M_a)$ for any $t \in [0,T]$ and any $u \in \mathcal{M}$. Moreover, the associated set of traces

$$\{\hat{v}|_{\partial M_a^T};\, v \in \mathcal{B}\}$$

is compact in $W_2^{\frac{3}{2},\frac{3}{4}}$. From (16)–(18) and Lemma 4.2 of [C] it follows that F is compact in $V(M^T)$. By Schauder's fixed point theorem F has a fixed point $\hat{u} = \hat{w}$. Necessarily

$$\hat{v}(x,t) = \hat{u}(x,t) = \hat{w}(x,t) = I(\hat{v}(\Phi(x),t)) \quad \text{on} \quad \partial M_a \times [0,T]$$

i.e. $\hat{v}$ is also a solution to (16)–(18). Hence $\hat{u} = \hat{v} = \hat{w}$ and $\hat{u}$ is a solution to (10). But by our construction also $I(u(\Phi(x),t))$ is a solution to (10) in $(M_a/M_{-a})^T$ with the same initial and boundary values as $\hat{u}$. By an argument like the proof of uniqueness for (1)–(4) we get that $\hat{u}(x,t) = I(\hat{u}(\Phi(x),t))$. So (3) is satisfied. (4) should hold–otherwise $\nabla^2 u$ would not be in $L^2(M_a^T)$.

5. THE LARGE TIME BEHAVIOUR

The following Lemma was obtained by M. Struwe [St] for harmonic map heat flow in closed Riemannian surface.

Lemma 7 There exists a constant $\varepsilon_3 > 0$ depending only on M, K and N such that the following is true:

Any weak solution $u \in V(M^T)$ to (1)–(4) with initial data $u_0 \in H^1$ is Holder continuous on $M \times (0,T)$, and any subinterval $[s,T]$, $s > 0$, the Holder norm of u is uniformly bounded in terms of T, S and the number

$$R = \sup\{R \in (0, \frac{1}{2}];\ \in (2R) \leq \epsilon_3\}$$

If $u_0 \in C(K)$, the solution u is Holder continuous on $M \times [0,T]$ and its Holder norm is bounded in terms of T, R, and the H^2–norm of u_0.

Proof First we derive uniform bounds for smooth solutions for the L^2–norm of $\partial_t u(t)$ for a.e.t> 0.

Differentiate (10) with respect to t to obtain the equality

$$|\partial_t^2 \hat{u} - \Delta_{\hat{M}} \partial_t \hat{u}| \leq c|\nabla \partial_t u||\nabla \hat{u} + c|\partial_t \hat{u}||\nabla \hat{u}|^2$$

on $\hat{\Omega}$. Testing this inequality with the function $\partial_t \hat{u} \hat{\phi}^2 \varsigma^2$ and integrating over M_a and $[t_1, t_2] \subset [0,T]$ we obtain that

$$\frac{1}{2} \int_{M_a} |\partial_t \hat{u}(t)|^2 \hat{\varphi} \zeta^2 |_{t=t_1}^{t_2} + \int_{(M_a)_{t_1}^{t_2}} |\nabla \partial_t \hat{u}| \hat{\varphi} \zeta^2 =$$

$$= \int_{(M_a)_{t_1}^{t_2}} (\frac{1}{2} \partial_t (|\partial_t \hat{u}|^2 \zeta^2) + |\nabla \partial_t \hat{u}| \hat{\varphi}^2 \zeta^2)$$

$$\leq c \int_{(M_a)_{t_1}^{t_2}} \{|\partial_t \hat{u}| \hat{\varphi} \zeta^2 + |\nabla \partial_t \hat{u}| \partial_t \hat{u}| \hat{\varphi}^2 \zeta^2 + |\nabla \partial_t \hat{u}||\partial_t \hat{u}||\nabla \hat{u}| \hat{\varphi}^2 \zeta^2 +$$

$$+ |\nabla \partial_t \hat{u}||\partial_t \hat{u}| \hat{\varphi}^2 \zeta |\nabla \zeta| + |\partial_t \hat{u}|^2 |\nabla \hat{u}|^2 \hat{\varphi} \zeta^2\} dM dt .$$

Note that we have used the bounds of $\hat{\phi}$.

Next recall that integrals and derivatives of $\hat{u}$ over $\hat{\Omega}$ can be estimated by corresponding integrals of u over M; see Lemma 5. Hence we may replace the domain M_a by M and omit $\hat{\phi}$ in all integrals at the expense of enlarge our constants c. Applying Holder's inequality we obtain

$$\int_M |\partial_t u(t_2)|^2 \zeta^2 + \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 \leq c \int_M |\partial_t u(t_1)|^2 \zeta^2 + \frac{1}{2} \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 +$$

$$+ c(\int_{M_{t_1}^{t_2}} |\partial_t u|^2 \zeta^2 + |\nabla u|^4 \zeta^2))^{1/2} [(\int_{M_{t_1}^{t_2}} |\partial_t u|^4 \zeta^2)^{1/2}$$

$$+ c \int_{M_{t_1}^{t_2}} |\partial_t u|^2 (\zeta^2 + |\nabla \zeta|^2) . \tag{19}$$

From (5) there holds

$$|\partial_t u| \leq |\nabla^2 u| + c|\nabla u|^2 , \tag{20}$$

we now have the estimate

$$\int_M |\partial_t u|^2\zeta^2 + |\nabla u|^2\zeta^2 \le c\int_{M_{t_1}^{t_2}} |\nabla^2 u|^2\zeta^2 + |\nabla u|^4\zeta^2$$

$$\le c(\int_{M_{t_1}^{t_2}} |\nabla^2 u|^2\zeta^2 + |t_2 - t_1|R^{-2} \sup_{t_1\le t\le t_2} E(u(t), B_{2R}^M(x_0))$$

$$\le c(t_2 - t_1)R^{-2} + \varepsilon_3 \ .$$

For the last inequality we have also used Lemma 1, Lemma 4, and our assumption that $\epsilon(2R) \le \epsilon_3$. Moreover, note that we may also apply Lemma 1 to the term $\int_{M_{t_1}^{t_2}} |\partial_t u|^4 \varsigma^2$ appearing on the right of (19). Going back to (19) we now simply write

$$\int_M |\partial_t u(t_2)|^2\zeta^2 + \int_{M_{t_1}^{t_2}} |\nabla\partial_t u|^2\zeta^2 \le c\int_M |\partial_t u(t_1)|^2\zeta^2 +$$

$$+c[(1 + (t_2 - t_1)R^{-2})\varepsilon_3]^{1/2}[\sup_{t_1\le t\le t_2} \int_M |\partial_t u(t)|^2\zeta^2(\int_{M_{t_1}^{t_2}} |\nabla\partial_t u|^2\zeta^2 + R^{-2}|\partial_t u|^2\zeta^2)]^{1/2} +$$

$$+c[(t_2 - t_1)R^{-2} + \varepsilon_3] \sup_{t\le t\le t_2} \int_{B_{2R}(x_0)} |\partial_t u(t)|^2$$

$$\le c\int_M |\partial_t u(t_1)|^2\zeta^2 + c[1 + (t_2 - t_1)R^{-2}\varepsilon_3]^{1/2} \int_{M_{t_1}^{t_2}} |\nabla\partial_t u|^2\zeta^2$$

$$+c(\varepsilon_3 + (t_2 - t_1)R^{-2}) \sup_{t_1\le t\le t_2} \int_{B_{2R}(x_0)} |\partial_t u(t)|^2 \ .$$

i.e. for sufficiently small $\epsilon_3 > 0$, $t_2 - t_1 \le \epsilon_3 R^2$ there holds

$$\int_{B_R(x_0)} |\partial_t u(t_2)|^2 \le c\int_{B_{2R}(x_0)} |\partial_t u(t_1)|^2 + c\varepsilon_3 \sup_{t_1\le t\le t_2} \int_{B_{2R}(x_0)} |\partial_t u(t)|^2 \ .$$

This inequality will hold for any $x_0 \in M$ and any $t_1, t_2 \in [0, T]$ such that $t_2 - t_1 \le \epsilon_3 R^2$. Fix

$$0 \le \bar t_1 \le \bar t_3 \le \bar t_1 + \varepsilon_3 R^2, \quad \bar t_2 = \frac{1}{2}(\bar t_1 + \bar t_3) \ ,$$

and for $t_1 \in [\bar t_1, \bar t_2]$ let $x_0 \in M$, $t_2 \in [t_1, \bar t_3]$ be defined such that

$$2\int_{B_R(x_0)} |\partial_t u(t_2)|^2 \ge \operatorname*{essup}_{(\bar x,\bar t)\in M_{\bar t_1}^{\bar t_3}} \int_{B_R(\bar x)} |\partial_t u(\bar t)|^2 \ .$$

Covering M with balls of radius R, for sufficiently small $\epsilon_3 > 0$ and suitably chosen $t_1 \epsilon [\bar t_1, \bar t_2]$ we obtain that

$$c^{-1}R^2\int_M |\partial_t u(\bar t_2)|^2 \le \int_{B_R(x_0)} |\partial_t u(t_2)|^2 \le c\int_{B_{2R}(x_0)} |\partial_t u(t_1)|^2$$

$$\leq c \inf_{\bar{t}_1 \leq t \leq \bar{t}_2} \int_M |\partial_t u(t)|^2 \leq \frac{c}{\bar{t}_2 - \bar{t}_1} \int_{M_{\bar{t}_1}^{\bar{t}_2}} |\partial_t u|^2 \tag{21}$$

By (20) and Lemma 3 we finally get

$$\int_M |\partial_t u(t_2)|^2 \leq \frac{c}{t_2 - t_1} \tag{22}$$

for all $t_2, t_1 \in [0, T]$ such that $0 < t_2 - t_1 \leq \epsilon_3 R^2$, with constant $c = c(T, R)$. i.e. for all $t \in (O, T]$ there holds

$$\int_M (\partial_t u(t)|^2 \leq c(1 + t^{-1}) . \tag{23}$$

If $u_0 \in C(K)$ from (23)–(24) we obtain

$$\int_M |\partial_t u(t)|^2 \leq c \tag{24}$$

uniformly, with c depending in addition on $H^2(M)$–norm of u_0. Now we derive pointwise estimates for

$$\int_M |\nabla^2 u(t)|^2$$

by using (10). Note that (10) implies that

$$|\Delta \hat{u}(t)| \leq c(|\partial_t \hat{u}(t)| + |\nabla \hat{u}(t)|^2) .$$

Testing it with $\Delta \hat{u}(t) \hat{\varphi}^2$ we find that

$$\int_{\hat{M}} |\nabla^2 \hat{u}|^2 \hat{\varphi}^2 \leq c \int_{\hat{M}} (|\partial_t \hat{u}|^2 + |\nabla \hat{u}|^2) \hat{\varphi}^2 + c \int_{\hat{M}} |\nabla^2 \hat{u}| \nabla \hat{u}|^2 \hat{\varphi}^2$$

i.e. by (*) again

$$\int_M |\nabla^2 u(t)|^2 \leq c \int_M (|\partial_t u(t)|^2 + |\nabla u(t)|)^4),$$

Lemma 1 and our assumption $\epsilon(2R) \leq \epsilon_3$ now imply that

$$\int_M |\nabla^2 u(t)|^2 \leq c \int_M |\partial_t u(t)|^2 + cR^{-2},$$

and (24) yields the estimate

$$\int_M |\nabla^2 u(t)|^2 \leq c(T, s, R)$$

for all $t \in [s, T], s > 0$ and the global bound

$$\int_M |\nabla^2 u(t)|^2 \leq c(T, R, ||u_0||_{H^2(M)})$$

for all $u_0 \in C(K)$. By Sobolev's imbedding theorem $H^2(M) \to C^0(M)$ and $u(t)$ is locally uniformaly continuous on $(0, T]$ resp. on $[0, T]$ for regular u_0. In particular, $\hat{\Omega}$ contains a uniform neighbourhood of $M \times (0,T]$, resp. of $M \times [0,T]$, and we can use Lemma 4.2 of [C] to derive all required estimates for u.

For weak solution $u \in V(M^T)$ the time derivative has to be replaced by difference quotients. One can proceed as above.

Q.E.D.

Remark Now we can prove the small time existence of $V(M^T)$–weak solution to (1)–(4) for $u_0 \in H^1$. To see this we approximate u_0 by smooth data $u_0^m \in C(K)$ and let $u^m \in V(M^{T_m})$ be the corresponding solution of (1)–(4). By Lemma 3, 4 and 7, each u_m persists as a regular solution to (1)–(4) for at least $\bar{T} = \frac{\bar{\epsilon}R^2}{2c_2} > 0$. In fact, Lemma 3 guarantees the estimate $\epsilon(2\bar{R}) \leq \bar{\epsilon}$ for all u^m on $[0,\bar{T}]$, and Lemmas 4 and 7 apply. Moreover, we have a uniform bound

$$\int_{M^{\bar{T}}} (|\partial_t u^m|^2 + |\nabla^2 u^m|^2 + |\nabla u^m|^4)\, dM dt + \sup_{0\leq t\leq \bar{T}} E(u^m(t)) \leq c(\bar{R}) ,$$

and we may extract a subsequence that convergences weakly in $V(M^T)$ to a solution of (1)–(4).

Now we are ready to following

Lemma 8 Suppose $u \in \sup_{T<\infty} V(M^T)$ is a solution to (1)–(4) and for some $R > 0$ there holds

$$\sup_{(x,t)\in M^\infty} E(u(t), B_R(x)) \leq \bar{\varepsilon} .$$

Then u is globally regular and there exists a sequence $t_m \to \infty$ such that $u(t_m) \in C^2$ and

$$u(t_m) \to \bar{u} \quad \text{in} \quad H^2(M)$$

where $\bar{u}$ is a harmonic map from M into N with free boundary in K.

Proof Our assumption and Lemma 3 imply that for $m = 1, 2$.

$$\int_{m_m^{m+1}} |\partial_t u|^2 \, dM dt \to o \quad (m \to \infty)$$

$$\int_{M_m^{m+1}} |\nabla^2 u|^2 \, dM dt \leq c \quad \text{uniformly in} \quad m .$$

By Fubini's theorem we may choose a sequence $t_m \to \infty$ such that $u_m = u(t_m) \in C^2$ satisfies

$$\partial_t u(t_m) = \Delta_M u_m + A(u_m)(\nabla u_m, \nabla u_m) \to 0 \quad \text{in} \quad L^2(M) ,$$

$$\partial_t u_m(x) \perp T_{u_m(x)} K \quad \text{a.e. on} \quad \partial M ,$$

$$\sup_{x\in M} E(u_m, B_R(x)) \leq \bar{\varepsilon}, \quad \text{for all } m .$$

Now the claim follows from Theorem B below.

Q.E.D

As in [St] we study singularities created by concentration of energy near some points.

Lemma 9 Suppose that for some $\bar{T} \leq \infty$, $u \in \cap V(M^T)$ is a solution of (1)–(4). If for all $R > 0$,

$$\lim_{T \uparrow \bar{T}} \sup_{(x,t) \in M^T} E(u(t), B_R(x)) \geq \bar{\varepsilon},$$

then there exist sequence $t_m \nearrow \bar{T}, x_m \in M, R_m \searrow 0$ and a map $\bar{u} \in H^2_{loc}$ such that $u(t_m) \in C^2$ and the rescaled functions

$$u_m(t) \equiv u(x_m + R_m x, t_m) \to \bar{u} \quad \text{in} \quad H^2_{loc},$$

after a possible rotation of coordinates. Moreover, $E(\bar{u}) < \infty$, and $\bar{u}$ is conformal to either some minimal 2–disk with free boundary in K or some minimal 2–sphere in N.

Proof For any sequence $R_m \searrow 0$ let $t_m \leq \bar{T}$ be maximal with the property that for some $x_m \in M$

$$E(u(t_m),\, B_{Rm}(x_m)) = \sup_{(x,t) \in M^{t_m}} E(u(t), B_{Rm}(x)) = \bar{\varepsilon}\,.$$

Clearly $t_m \nearrow \bar{T}$ as $m \to \infty$. After possibly take a subsequence we assume that $x_m \to \bar{x}$ in M. Let $\mathcal{O}$ be a local chart of $\bar{x}$. By Lemma 4 there exists a constant $c_3 = \bar{\varepsilon}(2c_2)^{-1}$ such that for all $t \in [t_m - c_3 R_m^2, t_m]$

$$E(u(t), B_{2Rm}(x_m)) \geq \frac{\bar{\varepsilon}}{4} > 0\,.$$

Moreover, by Lemma 3 and Lemma 1

$$\int_{m_{t_m - c_3 R_m^2}^{t_m}} |\nabla^2 u|^2 \leq c$$

uniformly for all m, while again by Lemma 3 and absolute continuity of the Legesgue integral

$$\int_{m_{t_m - c_3 R_m^2}^{t_m}} |\nabla^2 u| dM\, dt \to 0 \quad (m \to +\infty)\,.$$

Finally, by Lemma 3 also

$$\operatorname*{essup}_{t_m - c_3 R_m^2 \leq t \leq t_m} E(u(t)) \leq c < \infty$$

uniformly for all m. Hence if we let

$$u^m(x,t) = u(x_m + R_m x, t_m^+ R_m^2 t)$$

and

$$M^m = \{x \in R^2 \,|\, x_m + R_m x \in \mathcal{O}\}$$

then $u^m \in V(M^m)_{-c}$ will satisfy (1) on $(M^m)_{-c_3}$ and following estimates

$$\int_{(M^m)_{-c_3}} |\partial_t u^m|^2 dM\,dt \to 0 \quad (m \to \infty)$$

$$\int_{(M_m)_{-c_3}} |\nabla^2 u^m| dM\,dt \le c\,,$$

$$\operatorname*{essup}_{-c_3 \le t \le 0} E(u^m(t), B_2(0) \cap M^m) > 0\,,$$

$$\operatorname*{essup}_{(x,t)\in M^m)_{-c_3}} E(u^m(t)) \le c < \infty\,,$$

uniformly in m. As in [C], we can choose $s_m \in [-c_3, 0]$ such that $u_m = u^m(s_m)$ satisfies

$$u_m \in M^2(M^m, R^d)\,,$$

$$\partial_t u^m(s_m) = \Delta_M u_m + A_{u_m}(\nabla u_m, \nabla u_m)\,,$$

$$\partial_{\vec{n}} u_m(x) \perp T_{u_m(r)} K\,, a.e. \text{on} \quad \partial M^m \cap \partial R^2_+ \equiv \partial M^m_+$$

$$0 < c \le E(u_m, B_2(0) \cap M^m) \le E(u_m) \le c' < \infty\,,$$

$$\sup_{x \in M^m} E(u_m,\ B_1(x) \cap M^m) \le \bar{\varepsilon}\,,$$

uniformly in m. After shifting time we assume $s_m = 0$ and invoke theorem B below to conclude our lemma.

Q.E.D

Theorem B Suppose $M^m \subset R^2$, is a sequence of bounded domains that converges to a limiting domain $M^\infty \subset R^2$, $u^m \in H^2(M^m)$ satisfies that

$$u_m(\partial M^m_+) \subset K \tag{26}$$

$$\partial_{\vec{n}} u_m(x) \perp T_{u_m(x)} K \quad a.e. \text{on}\, \partial M^m_+ \tag{27}$$

$$E(u_m) \le c \le \infty \quad \text{uniformly in} \quad m\,, \tag{28}$$

$$\sup_{r \in M^m} E(u_m, B_1(0) \cap M^m) \le \bar{\varepsilon} \quad \text{uniformly in} \quad m\,. \tag{29}$$

$$\int_{M^m} |\Delta_{M^m} u_m + A_{u_m}(\nabla u_m, \nabla u_m)|^2 \to 0 \quad (m \to \infty)\,. \tag{30}$$

Then there exists a map $\bar{u} \in H^2_{loc}(M^\infty, R^d)$ and a subsequence (we still denote) $m \to \infty$ such that

$$u_m \to \bar{u} \quad \text{in} \quad H^2_{loc}(M^\infty, R^d)$$

where $\bar{u} \in H^1(M^\infty)$ solves

$$\Delta \bar{u} + A_{\bar{u}}(\nabla \bar{u}, \nabla \bar{u}) = 0 \quad \text{in} \quad M^\infty\,, \tag{31}$$

$$|\bar{u}_{x_1}|^2 - |\bar{u}_{x_2}|^2 = 0 = \bar{u}_{x_1} \cdot \bar{u}_{x_2} \quad \text{in} \quad M^\infty \,, \tag{32}$$

$$\bar{u}(\partial M_+^\infty) \subset K \,, \tag{33}$$

$$\partial_{\bar{n}} \bar{u}(x) \perp T_{\bar{u}(x)} K, \quad \text{on} \quad \partial M_+^\infty \tag{34}$$

and

$$E(\bar{u}) < \infty \,. \tag{35}$$

If $M^\infty = R_+^2$, then $\bar{u}$ is conformal to a minimal 2–disk.

If $M^\infty = R^2$, then $\bar{u}$ is conformal to minimal 2–sphere.

Proof By (30) we obtain

$$\int_{M^m} |\Delta u_m|^2 \le c \int_{M^m} |\nabla u_m|^2 |\Delta u_m| + 0(1) (\int_{M^m} |\Delta u_m|^2)^{1/2} \,.$$

By Lemma 1 this gives

$$\int_{M^m} |\Delta u_m|^2 \le c \int_{M^m} |\nabla u_m|^4 + 0(1)$$

$$\le c\bar{\varepsilon} \int_{M^m} |\nabla^2 u_m|^4 + c + 0(1) \,.$$

As in the proof of Lemma 3, we have

$$\int_{M^m} |\nabla^2 u_m| \le c \quad \text{uniformly in} \, m \,, \tag{36}$$

By Sobolev's imbedding theorem, u_m are equicontinuous and hence may be extended to a r–neighbourhood $\hat{M}^m$ of M^m by reflection in K as before, here $r > 0$ independent of m (26)–(27) guarantee that these extensions $\hat{u}_m \in H^2(\hat{M}^m; R^d)$ and satisfy (20) on $\hat{M}^m$, as we verified in Lemma 5. Moreover, as in the proof of Lemma 5, for the extension $\hat{u}_m$ of u_m we derive that

$$\int_{\hat{M}^m} |\Delta \hat{u}_m + A_{\hat{u}_m}(\nabla \hat{u}_m, \nabla \hat{u}_m)|^2 \to 0$$

for all m.

By (31) we may select a subsequence $\hat{u}_m$ such that for any bounded domain $\Omega \subset R^2$

$$\hat{u}_m \to \bar{u} \quad \text{weakly in} \quad H^2(M^\infty \cap \Omega; R^d)$$

By Rellich's theorem we have

$$\hat{u}_m \to \bar{u} \quad \text{strongly in} \quad L^{1,q}(M^\infty \cap \Omega;\ R^d)$$

and

$$\hat{u}_m \to \bar{u} \quad \text{strongly in} \quad L^{1,q}(M^\infty \cap \Omega;\ R^d)$$

for any $q < \infty$. Hence we may pass to the limit $m \to \infty$ in (30), (26), (27), and find that $\bar{u}$ be a harmonic map as claimed.

Moreover, letting $\hat{\phi}_m$ be the cutoff function associated with $\hat{u}_m$ by (11). Given j, k let $\hat{\phi} = \min\{\hat{\phi}_j, \hat{\phi}_k\}$, $\xi \in C_0^3(\Omega)$ upon testing the difference of Eq.(5) for u_j, u_k with the function $\Delta(u_j - u_k)\hat{\phi}\xi$ we obtain

$$\int_{R^2} |\nabla^2(\hat{u}_j - \hat{u}_k)|^2 \hat{\varphi}\xi = \int_{R^2} |\Delta(\hat{u}_j - \hat{u}_k)|^2 \hat{\varphi}\xi + 0(1)$$

$$\leq c \int_{R^2} [A_{\hat{u}_j}(\nabla\hat{u}_j, \nabla\hat{u}_j) - A_{\hat{u}_k}(\nabla\hat{u}_k, \nabla\hat{u}_k)] \cdot$$

$$\cdot\Delta(\hat{u}_j - \hat{u}_k)\hat{\varphi}\xi + 0(1)$$

as $m \to \infty$,

and $\hat{u}_m \to \hat{u}$ strongly in $H^2(M^\infty \cap \Omega, R^d)$ for any bounded $\Omega \subset R^2$.

The remaining part of our claim are now easily verified. By (31) the complex valued function of $z = x + iy \in M^\infty \subset R^2 = C$ is holomorphic and

$$H(z) = |\bar{u}_x|^2 - |\bar{u}_y|^2 - 2i\bar{u}_x \cdot \bar{u}_y$$

by (35) is integrable over M^∞. In case $M^\infty = R^2$ from the mean value theorem for harmonic functions

$$H(z) = \frac{1}{2\pi R}\int_{\partial B_R(z)} H(z')dz'$$

upon letting $R \to \infty$ we obtain that $H(z) = 0$, i.e. that $\bar{u}$ is weak conformal. By [S],$\bar{u}$ is smooth and hence a minimal 2–sphere.

Similarly, if $M^\infty = R^2_+$, by (33)–(34) H is real on ∂M^∞. By reflection and (35) the imaginary part of H may be extended to a harmonic function in $L^1(R^2)$, hence it must vanish identically by the above argument. The Cauchy–Riemann equations now imply that H=constant. But $H \in L^1(R^2_+)$, thus $H = 0$, and $\bar{u}$ is weak conformal. By the conformal equivalence $D^2 = R^2_+$, the map $\bar{u}$ will be conformal to a weak harmonic map. By the regularity result of [S] again $\bar{u}$ is smooth and hence a minimal 2–sphere.

Q.E.D.

6. THE END

Now, the proof of Theorem 1 is a direct consequence of Theorem A, the remark, and Theorem B.

Proof of Theorem B First, suppose K is diffeomorphic to S^2. Let (τ,ϕ) be a polar coordinate on S^2 and denote $C = \partial D^2$, then $S^2 = C \times [0,\pi]$ with $C \times \{0\}$, $C \times \pi$ collapsed to points. Let $D : K \to S^2$ be the diffeomorphism in our assumption. Then any continuous mapping $p : [0,\pi] \to H(K)$ such that $p(0), p(\pi)$ are constant maps induces a mapping $\delta p : S^2 \to S^2$ by letting

$$\delta p(\gamma,\phi) = D(p(\phi)(e^{i\tau})) \ . \tag{37}$$

Endowing the space of mappings $S^2 \to S^2$ iwth C^0–topology set

$$p = \{p \in C^0([0,\pi]; H(K)) | p(0) \equiv const,$$

$$\delta p \in c^0(S^2, S^2) \quad \text{is homotopic to the identity on } S^2\} \ .$$

Since (36) for $\delta p = id|_{S^2}$ defines a path $p \in C^1$. Clearly $p \neq \phi$.

Suppose the conclusion of Theorem 2 is false, then for any $u_0 \in H(K)$, there is a global regular solution of (1)–(4) which convergences to constant at infinity. Hence $id|_{S^2}$ is homotopic to constant mapping in $C^0(S^2, S^2)$, a contradiction.

The general case can be proved in the same spirit. More precisely, it goes as the same as the proof of Theorem 8.50 in the book of J.T. Schwartz: Nonlinear Functional Analysis, Gordon and Breach, New York, 1969.

Q.E.D.

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