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NOTES ON SOME ALGEBRAIC STRUCTURES COMMON TO INTEGRABLE LATTICE SYSTEMS AND CONFORMAL FIELD THEORIES

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Introduction

The relations between integrable systems and conformal field theories have first been worked out in the continuum limit[1]. Using Coulomb gas[2] and Bethe ansatz[3] techniques, critical exponents and partition functions were obtained for various lattice models, giving exact results of practical interest[4] as well as a precise correspondence with conformal theories[5]. It was later realized that the two subjects exhibit numerous similarities, and although this is maybe unexpected a priori, that some prints of the conformal invariant structure are present on the lattice before the continuum limit[6,7]. We review here recent progress on this question following refs.[8–10].

One of the key ingredients in conformal field theories in the study of the Virasoro algebra which has mainly been accomplished via the Feigin Fuchs construction[11]. On the lattice it turns out that the Temperley Lieb algebra[12] and the quantum group[13] play a similar role. Several aspects of this correspondence are described in parts 1, 3, 5.

Besides this algebraic point of view we show how critical exponents may be calculated using a lattice version of braiding[14] before the continuum limit (part 2). We also discuss the modular invariance constraint for finite systems (part 4).

Possible extensions are considered in the conclusion.

The following are notes of lectures given at the 8^{th} Symposium of Theoretical Physics on "Conformal Field Theory and Statistical Mechanics". Sokcho (Korea) July 3-July 8 (1989). I want to thank the organizers D. Kim and D. Song who invited me to this conference and kindly introduced me to Korean civilization. I am also grateful to the students and other lecturers, especially T. Eguchi, D. Kim, I.G. Koh, for interesting discussions.

I. $U_q sl(2)$as lattice screening operators algebra

A standard way of studying conformal field theories is the Feigin Fuchs construction[11]. Restricting for simplicity to minimal $c < 1$ theories it involves as a first ingredient a free bosonic field[15] with charge α_0 at infinity ($c = 1-24\alpha_0^2$). Screening operators $Q_\pm$ are then introduced as contour integrals of vertex operators $V_{\alpha\pm}$ of dimension $h = 1$, $\bar{h} = 0$ ($\alpha_\pm = \alpha_0 \pm \sqrt{1+\alpha_0^2}$). For $\alpha_0^2 = \frac{(p-p')^2}{4pp'}$, $Q_\pm$ are nilpotent operators and they can be used in a BRS way to restrict the Fock space of the bosonic theory to the one of the minimal theory[16].

From the lattice point of view it is known that vertex models are described by free fields in the continuum limit[17]. These models have degrees of freedom associated to bonds of the square lattice, and taking values in some representation of a Lie algebra. For the minimal series we have to consider the 6-vertex model associated to spin $j = \frac{1}{2}$ $sl(2)$ representation[2]. Its Boltzmann weights are encoded in a $\check{R}$ matrix satisfying the Yang Baxter equation (u is the spectral parameter)

$$\check{R} = \begin{bmatrix} \sin(\gamma-u) & 0 & 0 & 0 \\ 0 & \sin\gamma & \sin u & 0 \\ 0 & \sin u & \sin\gamma & 0 \\ 0 & 0 & 0 & \sin(\gamma-u) \end{bmatrix} \tag{1.1}$$

The equivalent of adding a charge α_0 at infinity consists in performing a gauge transformation that breaks Z_2 symmetry, giving

$$\check{R} = \begin{bmatrix} \sin(\gamma-u) & 0 & 0 & 0 \\ 0 & \sin\gamma e^{iu} & -\sin u & 0 \\ 0 & -\sin u & \sin\gamma e^{-iu} & 0 \\ 0 & 0 & 0 & \sin(\gamma-u) \end{bmatrix} \tag{1.2}$$

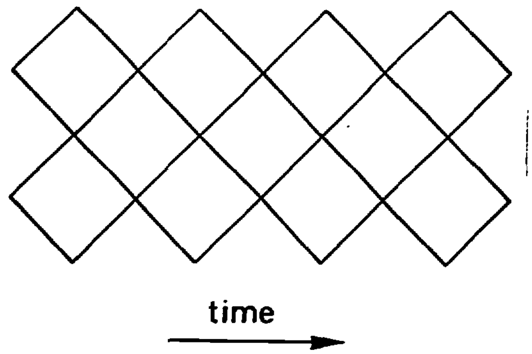
i.e.

$$\check{R} = \sin(\gamma-u) + 2\sin u \, \cos\gamma \, \mathrm{e} \tag{1.3}$$

where

$$\mathrm{e} = \frac{1}{q+q^{-1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad q = e^{i\gamma} \tag{1.4}$$

Conformal theories are usually studied by separating z and $\bar{z}$components. It is not clear how to do so in general on the lattice. In our case the most convenient is to consider a system with boundaries, whose continuum limit will be described by a single Virasoro algebra[18]. Hence we consider the vertex model (1.2) in the geometry



(1.5)

with transfer matrix

$$\mathcal{T}(u) = \prod_i \check{R}_{2i} \prod_i \check{R}_{2i-1} \tag{1.6}$$

In the following it is simpler to deal with (1.6) in the very anisotropic limit with $\mathcal{T} \sim e^{-u\ H}$

$$H = \sum_{i=1}^{L-1} e_i = \sum_{i=1}^{L-1} S_i^X S_{i+1}^X + S_i^Y S_{i+1}^Y + \frac{q+q^{-1}}{2} S_i^Z S_{i+1}^Z + \frac{(q-q^{-1})}{4} (S_1^Z - S_L^Z) \tag{1.7}$$

The critical region corresponds to γ real. Then we get the XXZ hamiltonian plus an imaginary boundary term. Eigenvalues can nevertheless be proven to be real.

The interesting property[9] of (1.7) is that it commutes with $U_q sl(2)$. This is obvious when $q = 1$. In general defining

$$\begin{aligned} S^{\pm} &= \sum_i S_i^{\pm} = \sum_i q^{S_1^Z} \otimes ... \otimes q^{S_{i-1}^Z} \otimes S_i^{\pm} \otimes q^{-S_{i+1}^Z} \otimes ... \otimes q^{-S_L^Z} \\ S^Z &= \sum_i S_i^Z \end{aligned} \tag{1.8}$$

one checks that

$$[H, S^{\pm}] = [H, S^Z] = 0 \tag{1.9}$$

The operators (1.8) satisfy the commutation relations

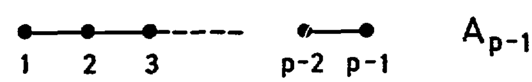
$$\begin{aligned} [S^+, S^-] &= \frac{q^{2S^Z} - q^{-2S^Z}}{q - q^{-1}} \equiv (2S^Z)_q \\ [S^Z, S^{\pm}] &= \pm S^{\pm} \end{aligned} \tag{1.10}$$

and (1.8) derives in fact from multiple applications of the coproduct formula[13,19]

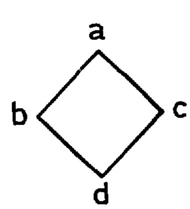
$$\begin{aligned} \Delta : U_q sl(2) &\longrightarrow U_q \otimes U_q \\ S^{\pm} &\longrightarrow q^{S^Z} \otimes S^{\pm} + S^{\pm} \otimes q^{-S^Z} \\ S^Z &\longrightarrow 1 \otimes S^Z + S^Z \otimes 1 \end{aligned} \tag{1.11}$$

The generators $(S^{\pm}, S^Z)$ have qualitatively the same properties[15] than $(Q_{\pm}, Q_3)$ (where $Q_3 = \int \partial\varphi$). Indeed applied to a given eigenstate $S^{\pm}$ do not change its eigenvalue (analogous to L_0) but increase or lower its $U(1)$ charge S^Z.

For q a root of unity $q = \exp i\pi(p-p')/p$, it is also known that lattice models whose continuum limit is in the minimal series are of Restricted Solid on Solid (RSOS) type[5,20]. In the simplest (diagonal) case such models have variables associated to sites of the square lattice, and taking values in the A_{p-1} Dynkin diagram



Face Boltzmann weights are



$$w(abcd) = \sin(\gamma - u)\delta_{bc} + 2\sin u \,\cos\gamma_{acd}e_{abd} \tag{1.12}$$

where

$$_{acd}e_{abd} = \frac{1}{(2)_q}\delta_{ad}\frac{[(b)_q(c)_q]^{1/2}}{(a)_q} \tag{1.13}$$

The essential result shown in[9] is that the vertex space $\rho_{1/2}^{\otimes L}$ can be reduced in this case to the RSOS space by a BRS construction using quantum groups.

The proof of this relies on the analysis of $U_q sl(2)$ representation theory when q is a root of unity . In the generic case, it is known that results are in one to one correspondence[21] with $q = 1$ ones. The Casimir operator being given by

$$C = S^- S^+ + \left(S^Z + \frac{1}{2}\right)_q^2 - \left(\frac{1}{2}\right)_q^2 \tag{1.14}$$

with $C|\alpha\rangle = C_j|\alpha\rangle$ if $S^+|\alpha\rangle = 0$, $S^Z|\alpha\rangle = j|\alpha\rangle$. When $q^p = \pm 1$, two new features arise. First one can show that $S^{\pm}$ become nilpotent

$$(S^{\pm})^p = 0 \tag{1.15}$$

hence highest weight representations can only have spins $0 \leq j \leq \frac{p-1}{2}$. Second the Casimir values C_j become invariant under an affine Weyl Group

$$\begin{cases} j' & = j \bmod p \\ j' & = -1 - j \bmod p \end{cases} \tag{1.16}$$

C_j is thus not sufficient to label representations, and sets of states associated to different $j's$ representations for q generic can mix in the root of unity case when (1.16) holds. Accordingly $\mathcal{H} = \rho_{1/2}^{\otimes L}$ splits into[9,22]

- Type I representations which are indecomposable but not irreducible, are not highest weight and are obtained by gathering two $q-$generic $sl(2)$ representations with spins j, $j' = -1 - j \bmod p$.

- Type II representations which are still isomorphic to $sl(2)$ ones, with spins $0 \leq j < \frac{p-1}{2}$.

Representations with $j = \frac{p-1}{2}$ are also isomorphic to $sl(2)$ ones. Since their $q-$dimension $D_j = \sum q^{2S^z} = (2j+1)_q = 0$, they are nevertheless called type I by convention. Then, D_j changing of sign in the second of the transformations (1.16), type I representations are fully characterized by their $q-$dimension being zero. Type II representations have $D_j \neq 0$ and their highest weights obey

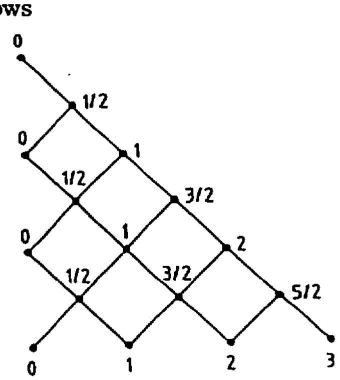
$$|\alpha_j\rangle \in \frac{\text{Ker } S^+}{\text{Im } (S^+)^{p-1}} \tag{1.17}$$

A metric can be introduced in $\mathcal{H}$ such that $(S^+)^+ = S^-$. Type I representations contains then null states. Type II representations are all unitary when $q = e^{i\pi/p}$ only, a result analogous to the unitary theorem in Virasoro representations[13].

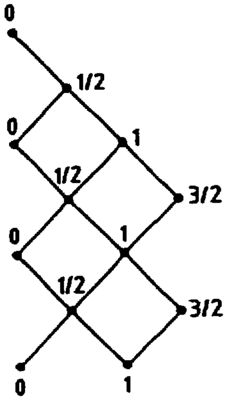
If $\Gamma_j^{(L)} = \begin{pmatrix} L \\ \frac{L}{2} - j \end{pmatrix} - \begin{pmatrix} L \\ \frac{L}{2} - j - 1 \end{pmatrix}$ is the number of spin j representations in the generic case, the number of type II ones for q a root of unity is given by

$$\Omega_j^{(L)} = \Gamma_j^{(L)} - \Gamma_{p-1-j}^{(L)} + \Gamma_{j+p}^{(L)} - \Gamma_{p-1-j+p}^{(L)} + \dots \tag{1.18}$$

It coincides with the number of paths on a truncated Bratteli diagram. The latter for q generic looks as follows



The vertical scale is the number L of spin $\frac{1}{2}$ tensorized representations. Numbers on the diagram are spins of representations outo which $\rho_{1/2}^{\otimes L}$ decomposes, while their multiplicity is the number of paths from j to the origin. For $q^p = \pm 1$ it turns out that the diagram truncated to exclude all $j \geq \frac{p-1}{2}$ still gives the type II representations onto which the space decomposes together with their multiplicity. If $q = e^{i\pi/4}$ for instance one has



Remarkably, the mapping $a = 2j + 1$ transforms allowed spins into heights on the A_{p-1} diagram, while $\Omega_j^{(L)}$ is also the number of states in the RSOS space with boundary conditions such that the height is 1 on the top row and $2j + 1$ on the bottom one[6b,8].

Having elucidated the representation theory for the symmetry algebra of our model we can consider the spectrum of (1.7). The set of eigenstates belonging to type II representations can be shown to have a natural basis built as follows[8]. We couple the spins in such away that for any n the first n spins of a state belong to some spin j_n representation (for q generic). Hence we get states $|j_1 j_2 ... j_L, m\rangle$ with $j_1 = \frac{1}{2}, -j_L \leq m \leq j_L$. To obtain the action of e matrices on these states we use the fact that e_i is identical to the projector P_0 on 0-spin representation when combining the i^{th} and $(i+1)^{th}$ spin. Knowledge of $6j$ coefficients[8,24]

$$\begin{pmatrix} j & \frac{1}{2} & j+\frac{1}{2} \\ \frac{1}{2} & j & 0 \end{pmatrix} = \left[\frac{(2j+2)_q}{(2)_q(2j+1)_q}\right]^{1/2}$$
$$\begin{pmatrix} j & \frac{1}{2} & j-\frac{1}{2} \\ \frac{1}{2} & j & 0 \end{pmatrix} = \left[\frac{(2j)_q}{(2)_q(2j+1)_q}\right]^{1/2} \tag{1.19}$$

gives

$$\langle \{j\}, m | e_i | \{j\}, m' \rangle = \delta_{mm'} \delta(j_1, j_1') ... \delta(j_{i-1}, j_{i-1}') \delta(j_{i-1}, j_{i+1}) \frac{[(2j_i+1)_q (2j_i'+1)_q]^{1/2}}{(2)_q (2j_{i-1}+1)_q} \delta(j_{i+1}, j_{i+1}') ... \delta(j_L, j_L') \tag{1.20}$$

When $q^p = \pm 1$, due to $(2j+1)_q = 0$ for $j = \frac{p-1}{2}$, the set of states with all $j's$ satisfying $0 \leq j < \frac{p-1}{2}$ is stable by $\epsilon' s$. Their number is the same than the number of eigenstates belonging to type II representations, and in fact the two corresponding subspaces are identical.

Hence the vertex space can be restricted to type II representations of $U_q sl(2)$ using the BRS characterization (1.17). Up to multiplicities inside representations, this subspace is isomorphic to the configuration space of the RSOS model, the change of basis being given by the above procedure. Inside this subspace, the action of vertex and RSOS transfer matrices are identical. For instance the partition function of the RSOS model with boundary conditions such that heights = 1 on top row and heights = $1+2j$ on bottom row is related to the partition function of the vertex model restricted to type II spin representations by

$$z^{\text{vertex}}_{\text{Type II},j} = (2j+1)_q z^{\text{RSOS}}_{1,1+2j} \tag{1.21}$$

Various other cases and generalizations to D, E models are described in[6b,8,25].

The vertex $\longrightarrow$ RSOS restriction procedure is very similar to the free field $\longrightarrow$ minimal model restriction as formulated in[16]. Though there is an essential difference since

$$[S^Z, S^+] + [S^Z, S^-] = 0$$

while[26]

$$[Q^3, Q^+] + [Q^3, Q^-] \neq 0 \tag{1.22}$$

Whether there is a quantum group structure hidden in Feigin Fuchs construction is an open question.

II. Use of $U_q sl(2)$ to obtain conformal weights

We discussed so far results for the finite lattice case. To complete the identification with conformal theories we need to know also the scaling behaviour of levels in the continuum limit[27]. This can be solved using Bethe ansatz calculations. One finds in particular, restricting to $p' = p-1$ for simplicity

$$C = 1 - \frac{6}{p(p-1)} \tag{2.1}$$

while the ground state of spin S^Z sector scales like[6b]

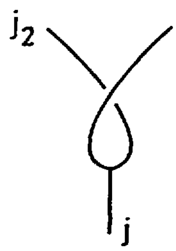
$$h_{1,1+2S^z} = \frac{\left(1 - 2(p-1)\left|S^Z\right|\right)^2 - 1}{4p(p-1)} \tag{2.2}$$

A result like (2.2) can be obtained without calculation using some quantum group arguments. First we notice that $H(q)$ is equivalent to $-H(-q^{-1})$ so two values $q = e^{i\pi/p}$ and $q = -e^{-i\pi/p} = e^{i\pi(p-1)/p}$ should in fact be associated to some A_{p-1}

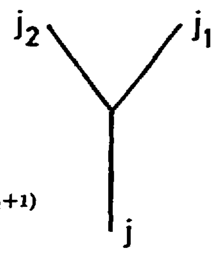
RSOS model. Then because of the left right dissymmetry in the coproduct (1.11) (non cocommutativity of U_q) it is not equivalent to combine $\rho_{j_1} \otimes \rho_{j_2}$ or $\rho_{j_2} \otimes \rho_{j_1}$. The two possible Clebsch Gordan are related by the identity[24]

$$\sum_{m_1' m_2' m_1 m_2} \left(\mathcal{R}^{j_1 j_2}\right)_{m_1' m_2'} \begin{pmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{pmatrix} = (-)^{j_1+j_2-j} q^{j(j+1)-j_1(j_1+1)-j_2(j_2+1)} \begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{pmatrix} \tag{2.3}$$

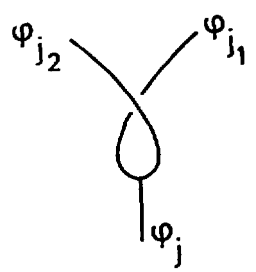
where $\mathcal{R}$ is the universal $\mathcal{R}$ matrix[13]. If we suppose that in the continuum limit is associated to ρ_j a field φ_j with dimension h_j, the analog of (2.3)



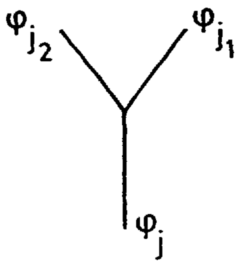
$$= (-)^{j_1+j_2-j} q^{j(j+1)-j_1(j_1+1)-j_2(j_2+1)} \tag{2.4}$$



is the relation



$$= (-)^{h_j - h_{j_1} - h_{j_2}} \tag{2.5}$$



due to

$$\varphi_{j_1}(z)\varphi_{j_2}(w) \sim (z-w)^{h_j - h_{j_1} - h_{j_2}} \varphi_j(z) \tag{2.6}$$

depending on the choice of q we find

$$h_j = \frac{j(j-p+1)}{p} \tag{2.7}$$

or

$$h_j = \frac{j[(p-1)j-1]}{p}$$

The first h_j is not relevant in our problem (it would correspond to hamiltonian $H = -\sum e_i$) while the second is indeed (2.2). Similar arguments were given in[34]. In this reference the $(-)^{j_1+j_2-j}$ was not taken into account to calculate $h's$, giving values of the WZW model.

The RSOS partition function in the isotropic ($u = \frac{\gamma}{2}$) case is

$$Z^{\text{RSOS}}_{1,1+2j} = \chi_{1,1+2j}(e^{i\pi T/L}) \tag{2.8}$$

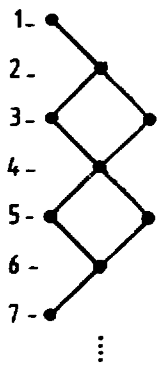
in the continuum limit. On the other hand, due tothe above vertex model analysis, the associated states form an irreducible representation of the $U_q sl(2)$ commutant, known as the Temperley lieb algebra[12]

$$\begin{cases} e_i^2 & = e_i \\ e_i e_{i\pm 1} e_i & = \dfrac{1}{(q+q^{-1})^2} e_i \end{cases} \tag{2.9}$$

Since the Virasoro algebra commutes with the screening operators, it is indeed natural that its lattice analog may appear in the commutant of $U_q sl(2)$.

III. Use of Temperley lieb algebra to build a lattice Virasoro algebra

Besides (2.8) the identification can be pushed further in the easy $C = \frac{1}{2}$ case[7]. There we have to deal with the A_3 Bratteli diagram, with e matrices (1.20) reading



$$\begin{aligned} (e_{2i})_{aa'} &= \delta\left(a_{2i-1}, a_{2i+1}\right) \\ (e_{2i+1})_{aa'} &= \frac{1}{2}\left[1 + \delta\left(a_{2i+1}, 1\right)\delta\left(a'_{2i+1}, 3\right)\right. \\ &\quad \left. + \delta\left(a_{2i+1}, 3\right)\delta\left(a'_{2i+1}, 1\right)\right] \end{aligned} \tag{3.1}$$

From corner transfer matrices analysis[6a 20], an express ... known

$$L_0 |a\rangle = \sum_{i\geq 1} \frac{i}{4} \left|a_{i+2} - a_i\right| |a\rangle \tag{3.2}$$

which corresponds to

$$L_0 = \sum_{i\geq 0} \left(i + \frac{1}{2}\right)(1 - e_{2i+2}) \tag{3.3}$$

The ground state is a path $(1, 2, 1, 2, ..., 1)$ (for L odd) with $h = 0$. It is known that $\text{Tr} x^{L_0}$ inside the set of paths with both extremities in $a = 1$ is $\chi_0(x)$. Similarly $\text{Tr} x^{L_0}$ for paths with $a_1 = 1$, $a_L = 3$ is $\chi_{1/2}(x)$ for $L \longrightarrow \infty$.

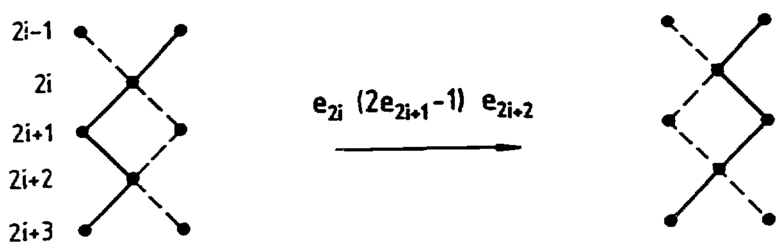
Using the relation

$$e_i + e_{i+1} - e_i e_{i+1} - e_{i+1} e_i - \frac{1}{2} = 0 \tag{3.4}$$

satisfied by (3.1) one can build all the over $L'_n s$. For instance

$$
\begin{aligned}
L_{-1} &= \sum_{i\geq 1} i\ e_{2i}(2e_{2i+1}-1)e_{2i+2} \\
L_1 &= \sum_{i\geq 1} i\ e_{2i+2}(2e_{2i+1}-1)e_{2i}
\end{aligned}
\tag{3.5}
$$

with the following action on paths



(3.6)

Similarly

$$
\begin{aligned}
L_{-2} &= \sum_{i\geq 1} -(i+\frac{1}{2})e_{2i}(2e_{2i+1}-1)(2e_{2i+2}-1)(2e_{2i+3}-1)e_{2i+4} \\
&+\frac{1}{2}(e_2-1)(2e_3-1)e_4 \\
L_2 &= \sum_{i\geq 1} -(i+\frac{1}{2})e_{2i+4}(2e_{2i+3}-1)(2e_{2i+2}-1)(2e_{2i+1}-1)e_{2i} \\
&-\frac{1}{2}e_2(2e_3-1)(1-e_4)
\end{aligned}
\tag{3.7}
$$

and acting on $\mid 0>$ we check graphically $L_2 L_{-2} \mid 0 >= \frac{1}{4} \mid 0>$, thus $C=\frac{1}{2}$.

IV. Toroïdal geometry - Modular invariance for finite systems

We can address the same question of restricting the vertex model to the RSOS model in toroïdal geometry[29]. In such a case, due to the left right dissymmetry of (1.8), the XXZ hamiltonian does not commute with $U_q sl(2)$. It is however possible to draw a commutative diagram. Introducing e_L which acts in the tensor product of the L^{th} and first spin $\frac{1}{2}$ representations as

$$
e_L = \frac{1}{q+q^{-1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\tag{4.1}
$$

We notice that e_L obeys also relations (2.9) with $e_{L+1} = e_1$. The associated hamiltonian is

$$
H(\varphi) = \sum_{i=1}^{L} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \frac{q+q^{-1}}{2} S_i^Z S_{i+1}^Z + \frac{e^{i\varphi}}{2} S_1^+ S_L^- + \mathrm{cc}
$$

with $S_{L+1} = S_1$. In the following we restrict to

$$e^{i\varphi} = q^{2k} \tag{4.2}$$

with k integer or half integer. in which case, denoting $H\left(e^{i\varphi} = q^{2k}\right) = H^k$ and the subspace of $\rho_{1/2}^{\otimes L}$ having $S^Z = k'$ by $\mathcal{H}_{k'}$, $k < k'$

$$\begin{array}{ccc} \mathcal{H}_k & \xrightarrow{S^{+(k'-k)}} & \mathcal{H}_{k'} \\ \Big\downarrow H^{k'} & & \Big\downarrow H^k \\ \mathcal{H}_k & \xrightarrow{S^{+(k'-k)}} & \mathcal{H}_{k'} \end{array} \tag{4.2}$$

is commutative. For q a root of unity - let us restrict to $q = e^{i\pi/p}$ here - we thus have

$$\begin{array}{ccccccc} \mathcal{H}_{k'-p} & \xrightarrow{(S^+)^{p-k'+k}} & \mathcal{H}_k & \xrightarrow{(S^+)^{k'-k}} & \mathcal{H}_{k'} & \xrightarrow{(S^+)^{p-k'+k}} & \mathcal{H}_{k+p} \\ \dots\Big\downarrow H^k & & \Big\downarrow H^{k'} & & \Big\downarrow H^k & & \Big\downarrow H^{k'}\dots \\ \mathcal{H}_{k'-p} & \xrightarrow{(S^+)^{p-k'+k}} & \mathcal{H}_k & \xrightarrow{(S^+)^{k'-k}} & \mathcal{H}_{k'} & \xrightarrow{(S^+)^{p-k'+k}} & \mathcal{H}_{k+p} \end{array} \tag{4.3}$$

which involves only two hamiltonians. For $1 \leq k + k' \leq p - 1$ the sequence is exact except for $\mathcal{H}_k$. Restricting there to Ker $(S^+)^{k'-k}$/Im $(S^+)^{p-k'+k}$ with $1 \leq k' - k \leq p - 1$ we obtain irreducible representations of the periodic Temperley lieb algebra (4.1)

$$\rho_{ab}, \quad a = k' + k, \quad b = k' - k, \quad 1 \leq a, b \leq p - 1 \tag{4.4}$$

involving in this toroïdal case two indices instead of one in the fixed boundary case.

ADE lattice models provide on the other hand representations[30] of TL algebra (4.1) by a formula analogous to (1.13)

$$_{acd}e_{abd} = \frac{1}{(2)_q} \delta_{ad} \frac{[v_b v_c]^{1/2}}{v_a} \tag{4.5}$$

where $abcd$ are points on a Dynkin diagram of incidence matrix C and

$$Cv = (q + q^{-1})v \tag{4.6}$$

In the unitary case in particular $q = e^{i\pi/H}$ where H is the Coxeter number.

It is shown in[9] that representations ρ^{ADE} of (4.1) furnished by (4.5) are reducible, and decompose as

$$\rho^{\mathrm{ADE}} = \sum_{a,b=1}^{p-1} \gamma_{ab}\rho_{ab} \tag{4.7}$$

where the $j's$ are the same integers than the ones of the corresponding modular invariant[31]

$$Z^{\mathrm{ADE}} = \sum_{a,b=1}^{p-1} \gamma_{ab} \left(\sum_{r=1}^{p-2} \chi_{ra}\bar{\chi}_{rb}\right) \tag{4.8}$$

Hence the coefficients in (4.8) have a natural interpretation as branching coefficients in Temperley lieb algebra representation theory.

Moreover one can show that in the isotropic case ($u = \frac{\gamma}{2}$ in 1.12) the restricted traces

$$z_{ab} = \mathrm{Tr}_{\rho_{ab}}(\mathcal{T})^{\mathrm{time}} \tag{4.9}$$

have the same modular properties than their continuum limit in (4.8)

$$Z_{ab} = \sum_{r=1}^{p-2} \chi_{ra}\bar{\chi}_{rb} \tag{4.10}$$

in the transformation $\tau \longrightarrow -1/\tau$. Hence the modular invariance program can as well be accomplished for finite lattice systems.

V. Other characters

We see from (1.17), (2.8) and (4.10) that the consideration of simple systems gives a lattice meaning to the right index of the Kac labels only or in other words to the "lower" $SU(2)$ in the coset construction[32] of theories (2.1)

$$\frac{SU(2)_{p-3} \otimes SU(2)_1}{SU(2)_{p-2}} \tag{5.1}$$

or to the "right" quantum group in braiding analysis[33].

A few steps towards identification of the missing degrees of freedom are accomplished in[6b]. For instance, turning to fixed boundary conditions as in part.1 it turns out that the RSOS partition function with heights $b, c = b \pm 1$ on the upper row and $a = 1 + 2j$ on the lower one is

$$Z^{\mathrm{RSOS}}_{b,c,1+2j} = \chi_{\inf(b,c),1+2j}(e^{i\pi T/L}) \tag{5.2}$$

In the vertex model, we have for instance associated to (5.2)

$$Z^{\mathrm{vertex}}_{\mathrm{Type\ II}.j',j} = (2j+1)_q \chi_{2j',1+2j} \tag{5.3}$$

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