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LINEARIZED FERMION - GRAVITON SYSTEM IN A (2+1)-DIMENSIONAL SPACE-TIME WITH CHERN-SIMONS TERM

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ABSTRACT

We study the fermion-graviton system at linearized level in a (2+1)-dimensional space-time with the gravitational Chern-Simons term. In this approximation it is shown that this system presents anomalous rotational properties and spin, in analogy with the gauge field-matter system.

I - INTRODUCTION

It has been recently analyzed by several authors the (2+1)-dimensional Abelian gauge theories with a topological (Chern-Simons) term as the action for this field. Because this special feature, the gauge field is totally specified by the matter field, and it is shown that the gauge field-matter system presents rotational anomalies[1], and explicit fractional statistics[2]. In this paper we would like to analyze these phenomena in the system composed by fermions coupled with gravitational field having the gravitational Chern-Simons (GCS) term as the action. We shall see that at linearized level, or weak field approximation, the gravitational fields $h_{\mu\nu}$ have no independent degrees of freedom, and the fermion-graviton system presents a rotational anomaly. The exotic statistic problem is also analyzed, but unfortunately no conclusion is presented.

This paper is organized as follows. In Section II we present our system and its linear approximation. It is shown that the field $h_{\mu\nu}$ can be expressed by the fermionic enery-momentum tensor $T_{\mu\nu}$. Because the GCS term alone is conformally invariant, it can only couple to matter with vanishing T^{μ}_{μ}, so we deal with massless fermions. In Section III we obtain the rotational anomalies by the explicit computation of the commutator between the angular momentum operator and the fermion field Ψ_a, and an anomalous spin for Ψ_a, that depend on arbitrary parameter, is also found. This result give us indication that we have a gravitational anyon. In Section IV we present our conclusion and some discussion about the consistency of our method of approximation, and the statistic problem for the fermion fields, when they are redefined by a specific "gauge" transformation, is also touched.

II - FERMION - GRAVITON SYSTEM

The massless fermion-graviton system is defined by the following Lagrangian:

$$L_{\Psi} = \frac{ie}{2} e^{\mu}_{a} \bar{\Psi} (\gamma^{a} \overrightarrow{D}_{\mu} - \overleftarrow{D}_{\mu} \gamma^{a}) \Psi, \tag{2.1a}$$

where the forward and backward covariant derivative are defined by

$$\vec{D}_\mu \Psi = (\partial_\mu + \omega_\mu)\Psi \; ; \; \bar{\Psi}\overleftarrow{D}_\mu = \bar{\Psi}(\overleftarrow{\partial}_\mu - \omega_\mu). \tag{2.1b}$$

From (2.1) one can derive the Dirac equation

$$ie^\mu_a \gamma^a D_\mu \Psi = 0, \tag{2.2a}$$

and the classical enery-momentum tensor

$$T_{\mu\nu} = \frac{i}{2}[\bar{\Psi}\gamma_{(\mu}\vec{D}_{\nu)}\Psi - \bar{\Psi}\overleftarrow{D}_{(\mu}\gamma_{\nu)}\Psi] - ig_{\mu\nu}L_F, \tag{2.2b}$$

In (2.2b) the notation $(\mu\nu)$ means $1/2(\mu\nu + \nu\mu)$, and $\gamma_\mu = e_{\mu a}\gamma^a$.

In the Dirac equation, γ^a, for $a = 0,1,2$, are given by

$$\gamma^a = (\sigma^3, i\sigma^1, i\sigma^2), \quad \gamma^a\gamma^b = \eta^{ab} - i\epsilon^{abc}\gamma_c,$$
$$\eta^{ab} = \mathrm{diag}(1, -1, -1), \tag{2.3}$$

with σ^i being the Pauli matrices, and Ψ a two-component spinor

The action for the matter field defined by

$$S_F = \int d^3x\, L_F, \tag{2.4}$$

depends, besides the fermionic field Ψ, on the dreibein that is related with the metric tensor $g_{\mu\nu}$ by the relation $g_{\mu\nu} = e^a_\mu e_{a\nu}$, and $e^{a\mu} = g^{\mu\nu}e^a_\nu$. $e = \det e_{\mu a} = \sqrt{g} = (\det g_{\mu\nu})^{1/2}$. The spin-connection ω_μ is given by $\omega_\mu = 1/8[\gamma^b, \gamma^c]e_{b\nu}\nabla_\mu e^\nu_c$.

The linear decomposition of the metric tensor $g_{\mu\nu}$, consists to separate into its asymptotic, Minkowski, part $\eta_{\mu\nu}$ plus a deviation $h_{\mu\nu}$; $g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu}$, where the parameter k^2 is defined, for our posterior convenience, as the Newton's constant. This decomposition in the $g_{\mu\nu}$, induces an expansion for the dreibein $e_{a\mu} = \eta_{a\mu} + \kappa h_{a\mu}$ (+ higher order correction). (The gravitational field $h_{\mu\nu}$ has, in this decomposition, dimension of $(\mathrm{mass})^{1/2}$.)

The approximation that we want to obtain to (2.1), consists to do an expansion in the parameter κ, and retain only terms linear in this parameter. As we shall see this approximation simplify enormously our calculation when we try to express the gravitational field $h_{\mu\nu}$ by the matter field. So, we have for L_Ψ and equations (2.2a,b), the following expressions:

$$L = \frac{ie}{2}[\bar{\Psi}\gamma^\mu\partial_\mu\Psi - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi] - \frac{i\kappa}{4}h^\mu_\nu[\bar{\Psi}\gamma^\nu(\partial_\mu\Psi) - (\partial_\mu\bar{\Psi})\gamma^\nu\Psi], \quad (2.5a)$$

where $e = 1 + \frac{1}{2}\kappa h$, $h = \mathrm{Tr}\, h_{\mu\nu}$,

$$i\not{\partial}\Psi - \frac{i}{4}\kappa h_{\mu\nu}[\gamma^\mu(\partial^\nu\Psi) + \gamma^\mu(\partial^\nu\Psi)] = 0, \quad (2.5b)$$

and

$$T_{\mu\nu} = \frac{i}{2}[\bar{\Psi}\gamma_{(\mu}\partial_{\nu)}\Psi - (\partial_{(\mu}\bar{\Psi})\gamma_{\nu)}\Psi] - \frac{i\eta_{\mu\nu}}{2}[\bar{\Psi}\gamma^\sigma(\partial_\sigma\Psi) - (\partial_\sigma\bar{\Psi})\gamma^\sigma\Psi]. \quad (2.5c)$$

One can see in (2.5) that, in the linear approximation we did not distinguish world and tangent space indices, and that the $T_{\mu\nu}$ is one order less, in the parameter κ, than L_Ψ as it should be by the definition $\delta S_\Psi = \frac{1}{2}\int d^3x\, \delta g^{\mu\nu}T_{\mu\nu}$. One can also see that $T_{\mu\nu}$ is conserved and traceless, until $O(\kappa)$, if the Dirac equation is obeyed, so $\partial^\mu T_{\mu\nu} = O(\kappa)$ and $T^\mu_\mu = O(\kappa)$. We shall use this approximation in our calculation, and we shall verify later that the error in our equation will be of order κ^2.

The GCS term that we shall use as the action for the gravitational field expressed by

$$S_{c-s} = -\frac{1}{4\mu\kappa^2} \epsilon^{\mu\nu\lambda} \int d^3x (R_{\mu\nu ab}\, \omega_\lambda^{\ ab} + \frac{2}{3} \omega_{\mu a}{}^{b} \omega_{\nu b}{}^{c} \omega_{\lambda c}{}^{a}), \tag{2.6a}$$

presents as basic dynamical variable the dreibein e^a_μ . The spin connection $\omega_\mu^{\ ab}$, and the curvature tensor $R_{\mu\nu ab}$, are expressed in terms of e^a_μ [3]. The dimensionless parameter $\mu\kappa^2$ in (2.6) will be assumed to be of order smaller than one, in order the perturbation theory makes sense in our analyze. The linearized GCS term, Eq. (2.6a), is given by

$$S_{c-s} = \frac{1}{4\mu} \epsilon_{\mu\alpha\gamma} \int d^3x\, h^{\mu\nu} \partial^\gamma (\partial_\nu \partial_\delta h^{\delta\alpha} - \Box h^\alpha_\nu). \tag{2.6b}$$

(This term could appear in our system induced by a graviton-fermion interaction after we have integreted out the fermionic degrees of freedom[4]; it is worth while to mention that this induced term comes from the second order correction of the graviton-fermion coupling given in (2.5a)).

The Cotton tensor $C_{\mu\nu}$, defined by the variation of S_{c-s} under the chaging in the metric tensor is identically traceless, symmetric and conserved. At linearized level we have

$$C_{\mu\nu} = -\frac{1}{\mu\kappa} \epsilon_{\mu\alpha\gamma} \partial^\gamma (\partial_\nu \partial_\delta h^{\delta\alpha} - \Box h^\alpha_\nu). \tag{2.7}$$

Considering now the total system defined by $S_I = S_f + S_{c-s}$, given in (2.5a) and (2.6b) we get by the variation of S_I under $\delta g^{\mu\nu}$

$$T_{\mu\nu} = \frac{1}{\mu\kappa} \epsilon_{(\mu\alpha\gamma} \partial^\gamma (\partial_{\nu)} \partial_\delta h^{\delta\alpha} - \Box h^\alpha_{\nu)}), \tag{2.8}$$

where $T_{\mu\nu}$ is given by (2.5c).

In order to obtain the physical content of this theory most clearly, it is convenient to study the components T_{00} and T_{0i}, using the following decomposition for the gravitational field $h_{\mu\nu}$.

$$h^{\mu\nu} = (h^{00} = N,\ h^{0i} = N^i,\ h^{ij}) , \qquad (2.9a)$$

where $i,j = 1,2$. Gauge fixing the potential $h_{\mu\nu}$, and considering only the transverse part of $h^{\mu i}$, we can choose[3],

$$N^i = -\epsilon^{ij}\partial_j W,\ \ h^{ij} = (\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2})\, V. \qquad (2.9b)$$

Using this gauge and (2.8) we can obtain only two independents differential equations

$$T_{00} = \frac{1}{\mu\kappa}\nabla^4 W , \qquad (2.10a)$$

$$T_{0i} = \frac{1}{\mu\kappa} G_i \nabla^2 \dot{W} + \frac{1}{2\mu\kappa}\epsilon_{ij}\partial_i(\nabla^2 N - \Box V) . \qquad (2.10b)$$

These equations above have in principal three variables (N,W,V). However the field equation $T_{\mu\nu} + C_{\mu\nu} = 0$, has just two degrees of freedom. In the purely topological theory, without S_r, W and $\lambda = \nabla^2 N - \Box V$ vanish, and V is undetermined, being a (Weyl) gauge variable[3]. Here, we certainly have W and λ different from zero, however, V is still undetermined, so, let us choose $V = 0$ and solve (2.10), obtaining for the (2+1)-dimensional space-time the metric tensor defined by

$$ds^2 = (1 + \kappa N(x))(dx^0)^2 + 2\kappa\epsilon_{ij}(\partial^i W(x))dx^0 dx^i - (dx^i)^2 , \qquad (2.11a)$$

with

$$N(x) = \mu\kappa \int d^2x' G(\vec{x}-\vec{x}') T_{00}(x'), \qquad (2.11b)$$

and

$$N(x) = 2\mu\kappa\epsilon^{ik} \int d^2x' G(\vec{x}-\vec{x}') (\partial'_{\kappa} T_{0i}(x')), \qquad (2.11c)$$

where $G(\vec{x})$ is defined by the equation

$$\nabla^4 G(\vec{x}) = \delta^2(\vec{x}), \qquad (2.12a)$$

which has solution

$$G(\vec{x}) = \frac{\vec{x}^2}{16\pi} \ln \vec{x}^2 + \frac{c}{8\pi} \vec{x}^2, \qquad (2.12b)$$

for any arbitrary constant "c". From (2.12b) we can see that $\nabla^2 G(\vec{x}) = D(\vec{x}) = (1/4\pi)\ln \vec{x}^2 + (1/2\pi)(1+c)$, that is well known solution of [illegible]).

In the weak field approximation it is assumed that for $r \to \infty$, the field $h_{\mu\nu} \to 0$, so, if we are inclined to make an integration by parts in (2.11c) we get

$$N(x) = 2\mu\kappa\epsilon^{ik} \partial_{\kappa} \int d^2x' G(\vec{x}-\vec{x}') T_{0i}(x'). \qquad (2.13)$$

Finally we would like to conclude this section noting that the spatial part of the metric tensor in (2.11a) is Minkowski, and because $h_{0i} \neq 0$, we have a rotation in the metric-space that comes from the parity odd GCS term.

III - ROTATIONAL ANOMALY

Let us now discuss about rotational features in our model. Under a rotation around the missing z-axis, the spinor field changes by

$$\delta\Psi_a = [J, \Psi_a]. \qquad (3.1)$$

The total angular momentum operator in a (2+1)-dimensional arbitrary metric space-time is

$$J = \epsilon^{ij} \int d^2x \, x_i T_{0j}(x) \sqrt{g}. \qquad (3.2)$$

The term $T_{0j}(x)\sqrt{g}$ is to be regarded in general as the spatial density of the momentum[5].

The commutator given in (3.1) can be obtained via the basic anti-commutator relation which the fermions field must obey.

$$\{\Psi_a(x), \Psi_b^{*}(y)\}\,\delta(x_0 - y_0) = \delta^3(x-y)\,\delta_{a,b}, \text{ for } a,b = 1,2.$$

In the linear approximation, we can write for our system

$$g^{1/2}(x) = 1 + \mu\kappa^2 \epsilon^{ik}\partial_k \int d^2x' G(\vec{x}' - \vec{x}) T_{0i}(x'), \qquad (3.3)$$

and see that the correction to $g^{1/2}$ is of order $\mu\kappa^2$, so, this result is in agreement with our initial assumption about our approximation.

Because the explicit dependence of the metric tensor on the fermion field, via T_{0i} in (3.3), the commutator $[J, \Psi_a]$ will present, besides the usual terms, an extra contribution coming from the commutator of Ψ_a with $g^{1/2}$. This extra contribution we name anomaly. Now let us write down the full result of $[J, \Psi_a]$.

$$[J, \Psi_a(x)] = \epsilon^{ij} \int d^2x' x'_i [T_{0j}(x'), \Psi_a(x)] g^{1/2}(x') +$$

$$\epsilon^{ij} \int d^2x' x'_i T_{0j}(x') [g^{1/2}(x'), \Psi_a(x)],$$

where

$$T_{0j} = \frac{i}{4}[\Psi^*(\partial_j\Psi) - (\partial_j\Psi^*)\Psi - \Psi^*\alpha_j\alpha^a(\partial_a\Psi) + (\partial_a\Psi^*)\alpha^a\alpha_j\Psi] + O(\kappa).$$

Now, all that we have to do is to calculate the commutator

$$[T_{0j}(x'), \Psi_a(x)] = -\frac{i}{4}\{\delta^2(\vec{x}-\vec{x}')(\partial_j\Psi_a(x)) - (\partial_j'\delta^2(\vec{x}-\vec{x}'))\Psi_a(x') -$$

$$\delta^2(\vec{x}-\vec{x}')(\partial_a\Psi_b(x))(\alpha_j\alpha^a)_{ab} + (\partial_a'\delta^2(\vec{x}-\vec{x}'))\Psi_b(x')(\alpha^a\alpha_j)_{ab}\}.$$

After some steps we get

$$[J, \Psi_a(x)] = -i\hat{x}\times(\hat{\nabla}\Psi_a(x))g^{1/2}(x) - \frac{1}{2}(\gamma_0\Psi(x))_a g^{1/2}(x) - \frac{i}{2}\Psi_a(x)\hat{x}(\hat{\nabla}g^{1/2}(x))$$

$$-\frac{1}{4}(\gamma_0\Psi(x))_a\hat{x}.(\hat{\nabla}g^{1/2}(x)) - i\mu\kappa^2\epsilon^{ij}\epsilon^{\ell k}\int d^2x' x_i' T_{0j}(x')(\partial_k' G(\vec{x}'-\vec{x}))(\partial_\ell\Psi_a(x))$$

$$+\frac{\mu\kappa^2}{16\pi}\epsilon^{ij}\int d^2x' x_i' T_{0j}(x')\ln|\vec{x}-\vec{x}'|^2(\gamma_0\Psi(x))_a +$$

$$\frac{\mu\kappa^2}{8\pi}(1+c)\,\epsilon^{ij}\int d^2x' x_i' T_{0j}(x')(\gamma_0\Psi(x))_a. \qquad (3.4)$$

At this point let us make some comments about the result above. First of all, besides the usual terms to $\delta\Psi_a(x)$ comming from the total angular moment of the field itself, there are contributions coming from angular momentum of the gravitation field; secondly, the last three terms come from the extra commutator between the matter field and the square root of the metric's determinant; finally we can observe that unless for a particular value for the parameter c, i.e, $c \neq -1$, there exist an extra (anomalous) spin for the spinor field

$$\frac{\mu\kappa^2}{8\pi}(1+c)J_0,$$

where $J_0 = \epsilon^{ij}\int d^2x\, x_i T_{0j}(x)$ is, up to the factor $g^{1/2}$, that is not relevant in our approximation to this anomalous spin, the angular momentum operator. Because there is no ultimate way to define a valur to "c" in (2.12b), this extra spin can assume any value.

IV - CONCLUSION AND DISCUSSION

We have studied the linearized fermion-graviton system in a three dimensional space-time with the gravitational Chern-Simons term as the action for the gravitational field $h_{\mu\nu}$. Our linear approximation consists in keeping terms until order κ in the expansion of the total action $S_T = S_f + S_{c-s}$, after we have assumed the weak field approximation: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. For S_{s-c} we have considered only quadratic term in the h-field and this contribution is of 0(1) in κ. If we had considered $0(\kappa)$ in S_{c-s}, as we did for S_f, we would have tri-linear contribution in the h-field; this κh^3-term would modify our Einstein equation $T_{\mu\nu} + C_{\mu\nu} = 0$, and in this case we may infer by inspection that two solutions for the h-field will appear, one of $0(\mu\kappa)$ and another $0(\kappa^{-1})$. Because we are in the weak field approximation, the fermionic action, given by $S_f = S_0 + \kappa S_1$, accepts only solutions of order $\mu\kappa$ in the perturbative regime, and for this solution we can write $S_f = S_0 + \mu\kappa^2 \hat{S}$, that is an acceptable expression for the fermionic action. The solution given in Sect. II for the h-field is in agreement with the consideration above.

As we have seen in Sect. II, only two independent degrees of freedom are left for the h-field; by (2.10a), h_{0i} is defined in a unique way, however by (2.10b), only the variable $\lambda = \nabla^2 N - \Box V$ is defined. Choosing the "gauge" $V = 0$, we get h_{00}, and h_{ij}, defined as well. In this gauge, $V = 0$, we have shown that the commutator $[J, \Psi_a]$ presents an anomalous contributions coming from the commutator $[h_{00}, \Psi_a]$, and an anomalous spin for the fermion appear. Because this extra spin can assume any value, and contributes additively to the "bare" spin, 1/2, of the fermionic field, there appear a particle with arbitrary spin that we identify as a gravitational anyon.

Another interesting point to study in this system is the exotic statistic problem. The total lagrangian of our system, $L_T = L_F + L_{c-s}$ with L_F and L_{c-s} given by (2.5a) and (2.2b) respectively, after some simplifications, reads

$$L_T = \frac{i}{2}[\bar{\Psi}\gamma^\mu(\partial_\mu\Psi) - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi] - \frac{i}{4}\kappa h^0_i[\bar{\Psi}\gamma^i(\partial_0\Psi) - (\partial_0\bar{\Psi})\gamma^i\Psi] -$$

$$\frac{i}{4}\kappa h^i_0[\bar{\Psi}\gamma^0(\partial_i\Psi) - (\partial_i\bar{\Psi})\gamma^0\gamma^0\Psi] + \text{surface terms.} \qquad (4.1)$$

As for a gauge theory (see Ref. (1)) where the matter field can be redefined by a specific gauge transformation in order the total lagrangian is described by free fields with explicit fractional statistic, we are tempted to try a similar procedure in (4.1). Under a space-time dependent Lorentz rotation on the tetrad, given by $e^\mu_a \to e'^\mu_a = \Lambda_a{}^b e^\mu_b$, the spinor field transforms as $\Psi \to \Psi' = L\Psi$, where L is the (space-time dependent) spinor representation of a tetrad rotation, and $L^{-1}\gamma^a L = \Lambda^a_b\gamma^b$ [6]. So, let us try a specific (infinitesimal) "gauge" rotation with $L = 1-(i/4)\kappa\epsilon_{ij}h^i_0\gamma^j$ in the spinor field Ψ, Eq. (4.1) reduces to

$$L = \frac{i}{2}[\bar{\Psi}'\gamma^{\mu}(\partial_{\mu}\Psi') - (\partial_{\mu}\bar{\Psi}')\gamma^{\mu}\Psi'] - \frac{i}{2}\kappa h_{i}^{0}[\bar{\Psi}'\gamma^{i}(\partial_{0}\Psi') - (\partial_{0}\bar{\Psi}')\gamma^{i}\Psi'] . \quad (4.2)$$

From (4.2) we can see that only h_0^i was eliminated from (4.1), by inspection one can see that is not possible to eliminate h_0^i and h_i^0 from (4.1) at the same time. Now, trying to analyze what (anti-)commutation relation the fermions field Ψ obey, we got to no conclusion about the evidence of exotic statistic in (4.2).

Another point that could be analyzed is including the Einstein action in our model. In this case some modification can, immediately, be observed: (i) the fermions may be massive now, and (ii) the variable V is now different from zero. After we have finished our calculation we received a paper by Deser[7], where a structureless massive particle couples with topologically massive gravity, there was also found the appearence of gravitational anyons.

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