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WESS–ZUMINO–WITTEN MODEL AS A THEORY OF FREE FIELDS III. THE CASE OF ARBITRARY SIMPLE GROUP

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WESS-ZUMINO-WITTEN MODEL AS A THEORY OF FREE FIELDS.
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Bosonization of Wess-Zumino-Witten model and free field representation of KAC-MOODY algebra on the lines of ref.[1] is worked out for any simple algebra of any complex level.

Fig. - 4, ref. - 10

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4. Bosonization in the case of arbitrary group

4.1. $s\ell(3)_k$

Expressions for currents in terms of free fields χ_α, w_α (labelled by three positive root vectors $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$ of $s\ell(3)$ algebra) and two-component scalar field $\vec{\phi}$, lying in the root plane, were presented in the last paper of ref. [2]. If roots are normalized by standard condition $(\vec{\alpha}_1,\vec{\alpha}_1) = (\vec{\alpha}_2,\vec{\alpha}_2) = (\vec{\alpha}_3,\vec{\alpha}_3) = 2$, then currents look like:

$$
\begin{aligned}
J_{\alpha_1} &= J_{12} = w_1 + \chi_3 w_2 \\
J_{\alpha_2} &= J_{13} = w_2 \\
J_{\alpha_3} &= J_{23} = w_3
\end{aligned}
\qquad (4.1.1)
$$

$$J_{-\alpha_1} = J_{21} = -\chi_1^2 w_1 + \chi_2 w_3 - (2-q^2)\partial\chi_1 + iq\chi_1\vec{\alpha}_1\partial\vec{\phi}$$

$$
\begin{aligned}
J_{-\alpha_2} = J_{31} = &-\chi_1\chi_2 w_1 - \chi_2^2 w_2 - \chi_2\chi_3 w_3 + \chi_1^2\chi_3 w_1 + \\
&+ (2-q^2)\chi_3\partial\chi_1 - (3-q^2)\partial\chi_2 - iq\chi_1\chi_3\vec{\alpha}_1\partial\vec{\phi} + iq\chi_2\vec{\alpha}_2\partial\vec{\phi}
\end{aligned}
$$

$$
\begin{aligned}
J_{-\alpha_3} = J_{32} = &\chi_1\chi_3 w_1 - \chi_2\chi_3 w_2 - \chi_3^2 w_3 - \chi_2 w_1 - \\
&- (3-q^2)\partial\chi_3 + iq\chi_3\vec{\alpha}_3\partial\vec{\phi}
\end{aligned}
$$

$$
\begin{aligned}
H_{\alpha_1} = H_{\mu_2} - H_{\mu_1} = J_{22} - J_{11} = &-2\chi_1 w_1 - \chi_2 w_2 + \chi_3 w_3 + \\
&+ iq\vec{\alpha}_1\partial\vec{\phi}
\end{aligned}
$$

$$H_{\alpha_2} = H_{\mu_3} - H_{\mu_1} = J_{33} - J_{11} = -\chi_1 w_1 - 2\chi_2 w_2 - \chi_3 w_3 + \\ + iq\,\vec{\alpha}_2 \partial\vec{\phi}$$

$$H_{\alpha_3} = H_{\mu_3} - H_{\mu_2} = J_{33} - J_{22} = \chi_1 w_1 - \chi_2 w_2 - 2\chi_3 w_3 + \\ + iq\,\vec{\alpha}_3 \partial\vec{\phi}$$

q is the parameter, related to the central charge:

$$k + C_V = k + 3 = q^2 \tag{4.1.2}$$

$\vec{\mu}_1$, $\vec{\mu}_2$, $\vec{\mu}_3$ are weight vectors of the fundamental representation (see fig.3a).

Operator expansions are:

$$W_\alpha(z)\chi_\beta(0) = \frac{\delta_{\alpha\beta}}{z} + \ldots \tag{4.1.3}$$

$$\vec{\alpha}\vec{\phi}(z)\,\vec{\beta}\vec{\phi}(0) = -(\vec{\alpha},\vec{\beta})\log z + \ldots$$

and

$$J_{ij}(z)J_{p\ell}(0) = k\,\frac{\delta_{i\ell}\delta_{jp} - \frac{1}{2}\delta_{ij}\delta_{p\ell}}{z^2} + \\ + \frac{\delta_{jp}J_{i\ell} - \delta_{i\ell}J_{pj}}{z} + \ldots \tag{4.1.4}$$

Energy-momentum tensor

$$T = \frac{1}{2(k+3)} :\sum_{\alpha\in\Delta_+}\left(J_\alpha J_{-\alpha} + J_{-\alpha}J_\alpha - \frac{1}{3}H_\alpha H_\alpha\right): \tag{4.1.5}$$

$$= W_1 \partial \chi_1 + W_2 \partial \chi_2 + W_3 \partial \chi_3 - \frac{1}{2}(\partial \phi_\perp)^2 -$$
$$- \frac{1}{2}(\partial \phi_{\parallel})^2 - i \frac{\sqrt{2}}{q} \partial^2 \phi_{\parallel}$$

Here $\phi_\parallel$ and $\phi_\perp$ stand for two components of ϕ, which are parallel and perpendicular to the vector $\vec{\rho} = \frac{1}{2}(\vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3) = \vec{\alpha}_2$

There are various versions of the representation (4.1.1): it may be replaced by any other one, obtained by a change of fields

$$\begin{aligned} W_1 &\to W_1 + S\chi_3 W_2 \\ W_3 &\to W_3 + S\chi_1 W_2 \\ \chi_2 &\to \chi_2 - S\chi_1\chi_3 \end{aligned} \qquad (4.1.6)$$

(with arbitrary S), which leaves T in (4.1.5) and operator expansions (4.1.4) invariant. (This is the origin of a and b parameters in the last paper of ref.[2]).

Our purpose now is to eliminate central charge dependence from (4.1.1), as it was done in sect.2.4 in the case of $s\ell(2)_k$. Now it is achieved by a more complicated change of variables:

$$\begin{aligned} W_1 &= k\widetilde{W}_1 \\ W_2 &= k\widetilde{W}_2 \\ W_3 &= k(\widetilde{W}_3 + \chi_1 \widetilde{W}_2) \end{aligned} \qquad (4.1.7)$$

$$\vec{\mu}_i \vec{\phi} = \frac{ik}{q} \vec{\mu}_i \vec{\widetilde{\phi}} = \frac{ik}{q} \widetilde{\phi}_i \qquad (4.1.8)$$

$$J_{\alpha_1} = k\widetilde{J}_{\alpha_1} \quad , \quad J_{\alpha_2} = k\widetilde{J}_{\alpha_2}$$

$$
\begin{aligned}
J_{\alpha_3} &= k\,\widetilde{J}_{\alpha_3} \\
J_{-\alpha_1} &= k\,\widetilde{J}_{-\alpha_1} + \partial\chi_1 \\
J_{-\alpha_2} &= k\,\widetilde{J}_{-\alpha_2} - \chi_3\,\partial\chi_1 \\
J_{-\alpha_3} &= k\,\widetilde{J}_{-\alpha_3} \\
H_{\vec{\mu}} &= -k\,\widetilde{H}_{\vec{\mu}}
\end{aligned}
\qquad (4.1.9)
$$

The sets of currents J or $\widetilde{J}$ may be represented in a matrix form:

$$
\widetilde{J} = \begin{pmatrix} \widetilde{H}_{\mu_1} & \widetilde{J}_{\alpha_1} & \widetilde{J}_{\alpha_2} \\ \widetilde{J}_{-\alpha_1} & \widetilde{H}_{\mu_2} & \widetilde{J}_{\alpha_3} \\ \widetilde{J}_{-\alpha_2} & \widetilde{J}_{-\alpha_3} & \widetilde{H}_{\mu_3} \end{pmatrix} = \qquad (4.1.10)
$$

$$
= \begin{bmatrix}
\chi_1\widetilde{W}_1 + \chi_2\widetilde{W}_2 + \partial\widetilde{\phi}_1 & \widetilde{W}_1 + \chi_3\widetilde{W}_2 & \widetilde{W}_2 \\
\begin{gathered} -\chi_1^2\widetilde{W}_1 + \chi_2\widetilde{W}_3 + \chi_1\chi_2\widetilde{W}_2 + \\ + \partial\chi_1 + \chi_1\partial(\widetilde{\phi}_2 - \widetilde{\phi}_1) \end{gathered} & \begin{gathered} -\chi_1\widetilde{W}_1 + \chi_3\widetilde{W}_3 + \\ + \chi_1\chi_3\widetilde{W}_2 + \partial\widetilde{\phi}_2 \end{gathered} & \begin{gathered} \widetilde{W}_3 - \\ -\chi_1\widetilde{W}_2 \end{gathered} \\
\begin{gathered} -\chi_1\chi_2\widetilde{W}_1 - \chi_2^2\widetilde{W}_2 - \chi_2\chi_3\widetilde{W}_3 - \\ -\chi_1\chi_2\chi_3\widetilde{W}_2 + \chi_1^2\chi_3\widetilde{W}_1 - \\ -\chi_3\partial\chi_1 + \partial\chi_2 + \chi_1\chi_3\partial(\widetilde{\phi}_1 - \widetilde{\phi}_2) + \\ + \chi_2\partial(\widetilde{\phi}_3 - \widetilde{\phi}_1) \end{gathered} & \begin{gathered} \chi_1\chi_3\widetilde{W}_1 - \chi_2\chi_3\widetilde{W}_2 - \\ -\chi_3^2\widetilde{W}_3 - \chi_1\chi_3^2\widetilde{W}_2 - \\ -\chi_2\widetilde{W}_1 + \partial\chi_3 + \\ + \chi_3\partial(\widetilde{\phi}_3 - \widetilde{\phi}_1) \end{gathered} & \begin{gathered} -\chi_2\widetilde{W}_2 - \\ -\chi_3\widetilde{W}_3 - \\ -\chi_1\chi_3\widetilde{W}_2 + \\ + \partial\widetilde{\phi}_3 \end{gathered}
\end{bmatrix} -
$$

This matrix $\tilde{J}(\chi, \tilde{w}, \tilde{\phi})$ may be rewritten as

$$\tilde{J} = g_L^{-1}(\chi)\, \tilde{J}_{(0)}(\tilde{w}, \tilde{\phi})\, g_L(\chi) + \\ + g_L^{-1}(\chi)\, \partial g_L(\chi) \tag{4.1.11}$$

with

$$g_L(\chi) = \begin{pmatrix} 1 & 0 & 0 \\ \chi_1 & 1 & 0 \\ \chi_2 & \chi_3 & 1 \end{pmatrix} \tag{4.1.12}$$

$$\tilde{J}_{(0)}(\tilde{w}, \tilde{\phi}) = \begin{pmatrix} \partial\tilde{\phi}_1 & \tilde{w}_1 & \tilde{w}_2 \\ 0 & \partial\tilde{\phi}_2 & \tilde{w}_3 \\ 0 & 0 & \partial\tilde{\phi}_3 \end{pmatrix} \tag{4.1.13}$$

Of course, if one recalls Gauss product from sect.3.1, it is clear, that

$$\tilde{J}_{(0)}(\tilde{w}, \varphi) = \left[g_U(\psi) g_D(\varphi)\right]^{-1} \partial \left[g_U(\psi) g_D(\varphi)\right] = \\ = g_D^{-1}(g_U^{-1}\partial g_U) g_D + g_D^{-1}\partial g_D \tag{4.1.14}$$

with

$$g_U(\psi) = \begin{pmatrix} 1 & \psi_1 & \psi_2 \\ 0 & 1 & \psi_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1.15}$$

$$g_D(\varphi) = \begin{pmatrix} e^{\varphi_1} & 0 & 0 \\ 0 & e^{\varphi_2} & 0 \\ 0 & 0 & e^{\varphi_3} \end{pmatrix} \tag{4.1.16}$$

$$\varphi_i = \vec{\mu}_i \vec{\varphi}\,, \qquad \varphi_1 + \varphi_2 + \varphi_3 = 0$$

$$\partial\psi_1 = e^{\varphi_1-\varphi_2}\widetilde{W}_1 = \frac{1}{k}e^{\varphi_1-\varphi_2}W_1$$
$$\partial\psi_2 - \psi_1\partial\psi_3 = e^{\varphi_1-\varphi_3}\widetilde{W}_2 = \frac{1}{k}e^{\varphi_1-\varphi_3}W_2 \qquad (4.1.17)$$
$$\partial\psi_3 = e^{\varphi_2-\varphi_3}\widetilde{W}_3 = \frac{1}{k}e^{\varphi_2-\varphi_3}(W_3 - \chi_1 W_2)$$

and (4.1.11) is nothing but a special choice of gauge for the current matrix $\widetilde{J} = g^{-1}\partial g,\ g = g_U(\psi)g_D(\widetilde{\phi})g_L(\chi)$.
However, this is a usefull gauge, since it provides representation of current algebra in terms of free fields. It is also easy to check up, that nontrivial change of variables (4.1.7), which was not required in the case of $s\ell(2)$ but is crucial for all other groups, may be encoded in the following rule:

$$k\,\mathrm{Tr}\, g_L^{-1}\bar\partial g_L(\chi)\widetilde{J} = k\,\mathrm{Tr}\,\bar\partial g_L g_L^{-1}(\chi)\widetilde{J}_{(0)} = \sum_{\alpha\in\Delta_+}\bar\partial\chi_\alpha W_\alpha \qquad (4.1.18)$$

On the left hand side a linear combination of monomials $k\, q_{\alpha\beta}(\chi)\bar\partial\chi_\alpha\widetilde{W}_\beta$ arises, and (4.1.18) simply represents (4.1.7) in the form of

$$W_\alpha = k\sum_{\beta\in\Delta_+} q_{\alpha\beta}(\chi)\widetilde{W}_\beta$$

Of course one may repeat all the reasoning of sect.2.4 concerning anomalies, which arise under the change of variables (4.1.17), and apply it to the case of $s\ell(3)$ (or any other group), see sect.4.3. below. Then one will realize that classical fields φ_i in (4.1.16) are related to quantum $\widetilde{\phi}_i$ and $\vec{\phi}$ in (4.1.1 - 4.1.13) as

$$\varphi_i = \frac{k}{k+3}\widetilde{\phi}_i = -\frac{i}{q}\vec{\mu}_i\vec{\phi} \qquad (4.1.19)$$

This means, that KM commutation relations (4.1.3) are valid provided

$$\vec{\alpha}\vec{\phi}(z)\,\vec{\beta}\vec{\phi}(0) = -(\vec{\alpha},\vec{\beta})\log z + \ldots \qquad (4.1.20)$$

while quantum WZW action implies, that

$$\varphi_i(z)\varphi_j(0) = \frac{(\vec{\mu}_i,\vec{\mu}_j)}{q^2}\log z + \ldots \qquad (4.1.21)$$

Eqs.(4.1.17) together with (4.1.19) determines the structure of screening operator insertions. These have the form of

$$\oint w_1 e^{\varphi_1-\varphi_2} = \oint w_1 e^{-i\vec{\alpha}_1\vec{\phi}/q} \quad (\text{for } \partial\psi_1) \qquad (4.1.22)$$

$$\oint\left[w_2 e^{\varphi_1-\varphi_3}(1-\chi_1\psi_1 e^{\varphi_2-\varphi_1}) + \psi_1 w_3 e^{\varphi_2-\varphi_3}\right] = \qquad (4.1.23)$$
$$= \oint\left[w_2 e^{-i\vec{\alpha}_2\vec{\phi}/q}(1-\chi_1\psi_1 e^{i\vec{\alpha}_1\vec{\phi}/q}) + \psi_1 w_3 e^{-i\vec{\alpha}_3\vec{\phi}/q}\right] (\text{for } \partial\psi_2)$$

$$\oint(w_3-\chi_1 w_2)e^{\varphi_2-\varphi_3} = \oint(w_3-\chi_1 w_2)e^{-i\vec{\alpha}_3\vec{\phi}/q} \quad (\text{for } \partial\psi_3) \qquad (4.1.24)$$

It seems that only (4.1.22) and (4.1.24) should be considered as independent screening operators, at least in some applications it is enough to use only these two kinds of insertions, leaving the complicated object (4.1.23) aside. (Note, that exponents $e^{-i\vec{\alpha}_1\vec{\phi}/q}$ and $e^{-i\vec{\alpha}_3\vec{\phi}/q}$ in (4.1.22) and (4.1.24) expressed through <u>simple</u> roots $\vec{\alpha}_1$ and $\vec{\alpha}_3$ have vanishing dimensions, e.g.

$$\Delta\left(e^{-i\vec{\alpha}_1\vec{\phi}/q}\right) = \frac{1}{2q^2}(-\vec{\alpha}_1)(2\vec{\rho}-\vec{\alpha}_1) = 0$$

(see (3.1.10)

The last thing we need to explain is the origin of transformation law (4.1.9) relating proper KM currents J with their classical analogues $\tilde{J}$ (another piece of this relation is eq.(4.1.19) just discussed). Additional terms $\partial\chi_1$ and $\chi_3\partial\chi_1$ in (4.1.9) arise in fact because of the transformation (4.1.7), which diagonalizez Lagrangian form (4.1.18). One should recall only that χ and W are quantum fields and one cannot change variables in such a simple way as if they were ordinary functions. In such changes one should carefully follow the normal ordering, which is implicit in all formulae including quantum fields. For example, if one substitutes $W_3 = k(\widetilde{W}_3 + \chi_1 \widetilde{W}_2)$ in $\chi_2 W_3$ in expression (4.1.1) for J_{21}, the answer is not simply $k\chi_2(\widetilde{W}_3 + \chi_1 \widetilde{W}_2)$ Instead

$$\chi_2 W_3 \to k \lim_{z \to 0} \left[\chi_2(0) (\widetilde{W}_3 + \chi_1 \widetilde{W}_2)(z) - \text{singularity} \right] =$$

$$= k\chi_2 \widetilde{W}_3 + k\chi_1\chi_2 \widetilde{W}_2 + \partial\chi_1 \qquad (4.1.25)$$

(since $k\chi_2(0)\widetilde{W}_2(z) = \chi_2(0) W_2(z) = \frac{1}{z} + \ldots$

In other words, $J_{21}(W) = k\widetilde{J}_{21}(\widetilde{W}) + \partial\chi_1$. Analogously

$$-k\chi_2\chi_3 W_3 \to -k \lim_{z \to 0} \left(\chi_2\chi_3(0) (\widetilde{W}_3 + \chi_1 \widetilde{W}_2)(z) - \right. \qquad (4.1.26)$$

$$\left. - \text{singularity} \right) = -k\chi_2\chi_3 \widetilde{W}_3 - k\chi_1\chi_2\chi_3 \widetilde{W}_2 - \chi_3 \partial\chi_1$$

and

$$J_{31}(W) = k\widetilde{J}_{31}(\widetilde{W}) - \chi_3\partial\chi_1$$

as stated in (4.1.9). Note, that the same kind of reasoning is required, when transformations (4.1.6) are performed.

4.2. General prescription

In fact the case of $sl(3)_k$ exhausts almost all possible problems, which can arise for arbitrary simple group. Let us formulate the procedure, required to find a representation of KM algebra G in terms of $\beta\gamma$ -systems χ_α, W_α labelled by all positive roots $\alpha \in \Delta_+$ and scalar fields $\vec{\phi}$, which take values in Cartan torus. In fact we are going to repeat the content of sect.4.1 in inverse sequence.

1) Fix a system of positive roots Δ_+ and introduce two fields χ_α and $\widetilde{W}_\alpha$ for each $\alpha \in \Delta_+$

2) Represent $g^{-1}\partial g$ in the form

$$g^{-1}\partial g = \widetilde{J} = g_L^{-1}(\chi)\widetilde{J}_{(0)}(\widetilde{W},\widetilde{\phi})g_L(\chi) + \qquad (4.2.1)$$

$+g_L^{-1}(\chi)\partial g_L(\chi)$, so that $\widetilde{J}_{(0)}(\widetilde{W},\widetilde{\phi})$ is linear matrix function of $\widetilde{W}$ and $\partial\widetilde{\phi}$. (This kind of representation is usually provided by Gauss product

$$g = g_U(\psi)\, g_D(\varphi)\, g_L(\chi) \qquad (4.2.2)$$

with an appropriate change of variables $\psi, \varphi \to \widetilde{W}, \widetilde{\phi}$).

3) Redefine $\widetilde{W} \to W(\widetilde{W},\chi)$ according to the rule (4.1.18)

$$k\,\mathrm{Tr}\, g_L^{-1}\bar{\partial} g_L \widetilde{J} = k\,\mathrm{Tr}\,\bar{\partial} g_L\, g_L^{-1}(\chi)\widetilde{J}_{(0)}(\widetilde{W},\widetilde{\phi}) = \qquad (4.2.3)$$
$$= \sum_{\alpha\in\Delta_+} W_\alpha \bar{\partial}\chi_\alpha$$

4) 1-form $d^{-1}\Omega = k\,\mathrm{Tr}\, g_L^{-1}\bar{\partial} g_L \widetilde{J}$ should be considered as integrated symplectic structure ($\Omega \sim dp\,dq$, $d^{-1}\Omega \sim p\,dq$) which dictates operator expansions for the fields $\widetilde{W}$, W, χ. In particular, (4.2.3) implies, that W and χ are Darbouxvariables[3], and

$$W_\alpha(z)\chi_\beta(0) = \delta_{\alpha\beta}/z + \ldots \qquad (4.2.4)$$

5) Express carefully $\widetilde{J}$ (which is originally defined in terms of $\widetilde{W}$) through W, taking into account the rules like (4.1.25), (4.1.26). Then

$$k\tilde{J}(\tilde{w},\chi,\tilde{\phi}) = J^{[k]}(w,\chi,\tilde{\phi}) + F(\chi) \tag{4.2.5}$$

$F(\chi)$ being some matrix function of χ. (We stressed the fact that $J^{[k]}$ depends on k explicitly, while $\tilde{J}$ and F contain no explicit k-dependence).

6) The fields $\tilde{\phi}_i$ are free and they are naturally labelled by basis vectors $\vec{e}_i$ in Cartan plane. If $\partial\tilde{\phi}_i$ appear in $\tilde{J}_{(0)}$ as diagonal elements with unit coefficients their operator expansion is postulated in the form of

$$\tilde{\phi}_i(z)\tilde{\phi}_j(0) = (\vec{e}_i,\vec{e}_j)\,\frac{q^2}{k^2}\log z + \ldots \tag{4.2.6}$$

with

$$q^2 = k + C_V \tag{4.2.7}$$

with this normalization of $\tilde{\phi}$ $\tilde{J}$ does not depend on k. Free scalar fields with natural operator expansion

$$\vec{\alpha}\vec{\phi}(z)\,\vec{\beta}\vec{\phi}(0) = -(\vec{\alpha},\vec{\beta})\log z + \ldots \tag{4.2.8}$$

are related to $\tilde{\phi}_i$ through (4.1.8),

$$\vec{e}_i\vec{\phi} = \frac{ik}{q}\tilde{\phi}_i \tag{4.2.9}$$

Then $J(w,\chi,\phi)$ form a level k KM algebra and Sugawara's energy-momentum tensor is quadratic in these fields:

$$T = \frac{1}{2(k+C_V)}:\mathrm{Tr}\,J^2: = \frac{1}{2q^2}\sum_{\alpha\in\Delta_+}:(J_\alpha J_{-\alpha} + J_{-\alpha}J_\alpha - \frac{1}{C_V}H_\alpha H_\alpha): = \sum_{\alpha\in\Delta_+} w_\alpha\partial\chi_\alpha - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{i}{q}\vec{\rho}\,\partial^2\vec{\phi} \tag{4.2.10}$$

with

$$\vec{\rho} = \frac{1}{2}\sum_{\alpha\in\Delta_+}\vec{\alpha} \tag{4.2.11}$$

This corresponds to the free field Lagrangian of the form (in conformal gauge for 2-dimensional metric)

$$4\pi L_q = -\sum_{\alpha\in\Delta_+} w_\alpha \bar\partial \chi_\alpha + \frac{1}{2}\partial\vec\phi\,\bar\partial\vec\phi + \frac{i}{q} R\vec\rho\cdot\vec\phi \qquad (4.2.12)$$

The only thing which remains is to find a representation, required in eq.(4.2.1) for all simple groups. As we already said, it is provided by Gauss product formulae, discussed in sect.3.2 . In sect.4.5 below we shall give a few more explicit examples. However, although the whole set of currents J can not be presented in a simple and general form, some ingredients look quite universal. For example Cartan current, labelled by Cartan vector $\vec\mu$, is

$$H_\mu = -\sum_{\alpha\in\Delta_+} (\vec\mu,\vec\alpha)\, w_\alpha \chi_\alpha + iq\,\vec\mu\,\partial\vec\phi \qquad (4.2.13)$$

One can easily check, that this formula is in accordance with the value $k = q^2 - C_V$ of central charge. Casimir eigenvalue may be defined as (3.2.11)

$$(\vec\mu,\vec\nu)\,C_V = \sum_{\alpha\in\Delta_+} (\vec\mu,\vec\alpha)(\vec\alpha,\vec\nu) \qquad (4.2.14)$$

for any $\vec\mu$ and $\vec\nu$.

Another universal formula is representation (4.2.10) for Sugawara energy-momentum tensor. From that formula one easily deduces that central charge of Virasoro algebra involved is

$$c_{WZW} = \sum_{\alpha\in\Delta_+} c_{w_\alpha\chi_\alpha} + \sum_{\vec\phi\perp\vec\rho} c_{\vec\phi} + c_{\parallel} =$$

$$= \frac{D-r}{2}\cdot 2 + (r-1)\cdot 1 + 1\cdot\left(1 - \frac{12\vec\rho^{\,2}}{k+C_V}\right) = \qquad (4.2.15)$$

$$= D - \frac{DC_V}{k+C_V} = \frac{Dk}{k+C_V}$$

D and r are dimension and rank of algebra G. The "strange formula" of Freudental (see[4 , eq.(12.1.8)] and eq.(6.2.31) below)

$$12\vec{\rho}^{\,2} = \mathcal{D} C_V \qquad (4.2.16)$$

is used.

4.3. Comments on Lagrangian approach in generic case

Let us remind the main group property of classical WZW Lagrangian

$$L_{cl}[g_1 g_2] = L_{cl}[g_1] + L_{cl}[g_2] + \frac{k}{4\pi}\mathrm{Tr}\,\bar{\partial} g_2 g_2^{-1} g_1^{-1} \partial g_1 \qquad (4.3.1)$$

(it follows directly from equations of motion expressed in the form of Lagrangian variation, $\delta L = \frac{k}{4\pi} g^{-1}\delta g\, \partial(g^{-1}\bar{\partial} g))$

The main feature of Gauss product $g = g_U(\psi) g_{\mathcal{D}}(\varphi) g_L(\chi)$ in triangular matrices, which is exploited in our construction, is

$$L_{cl}[g_U] = L_{cl}[g_L] = 0 \qquad (4.3.2)$$

Also

$$4\pi L_{cl}[g_{\mathcal{D}}] = -\frac{k}{2}\mathrm{Tr}\,|g_{\mathcal{D}}^{-1}\partial g_{\mathcal{D}}|^2 \qquad (4.3.3)$$

$$\mathrm{Tr}\,\partial_\mu g_L g_L^{-1} g_{\mathcal{D}}^{-1} \partial_\nu g_{\mathcal{D}} = \mathrm{Tr}\,\partial_\mu g_{\mathcal{D}} g_{\mathcal{D}}^{-1} g_U^{-1} \partial_\nu g_U = 0 \qquad (4.3.4)$$

Therefore

$$\begin{aligned} 4\pi L_{cl}[g] &= 4\pi L_{cl}[g_U g_{\mathcal{D}} g_L] = \\ &= -\frac{k}{2}\mathrm{Tr}\,|g_{\mathcal{D}}^{-1}\partial g_{\mathcal{D}}|^2 + k\,\mathrm{Tr}\,\bar{\partial} g_L g_L^{-1} \tilde{J} \end{aligned} \qquad (4.3.5)$$

with

$$\tilde{J}(\psi,\varphi) \equiv (g_U g_{\mathcal{D}})^{-1}\partial(g_U g_{\mathcal{D}}) \qquad (4.3.6)$$

For appropriate choice of parametrization $g_{\mathcal{D}}(\varphi)$ the first term in (4.3.5) becomes

$$-\frac{k}{2}\operatorname{Tr}|g_{\infty}^{-1}\partial g_{\infty}|^2 = -\frac{k}{2}\partial\vec{\varphi}\,\bar{\partial}\vec{\varphi} \qquad (4.3.7)$$

and the main prescription is to use the fields W instead of ψ, such that (4.3.5) becomes diagonal and quadratic:

$$k\operatorname{Tr}\bar{\partial}g_L g_L^{-1}(\chi)\tilde{J}(\psi,\varphi) = \sum_{\alpha\in\Delta_+} W_\alpha \bar{\partial}\chi_\alpha \qquad (4.3.8)$$

As explained in sect.2.4 one should take into account Jacobian of the change of variables $\psi_\alpha \to W_\alpha$, including anomaly contribution, which changes the action $L_{cl} \to L_{q}$.

Generically this change of variables is defined by a matrix product

$$W_\alpha = k\sum_{\beta\in\Delta_+}\left[X_L(\chi)C(\varphi)Y_U(\psi)\right]_{\alpha\beta}\partial\psi_\beta \qquad (4.3.9)$$

($X(\chi)$ specifies the relation between W and $k\tilde{W}$, see sect.4.2 above). Eq.(4.3.9) is again a sort of Gauss product: X_L and Y_U are a lower and upper triangle matrices respectively with units at diagonal.

Matrix $C(\varphi)$ is diagonal and has $e^{-\vec{\alpha}\vec{\varphi}}$ as an entry, corresponding to the root α. The same matrices X, C, Y defines the classical part of original measure of integration over the fields ψ, according to

$$\|\delta g\|^2 = \frac{1}{2}\int G\operatorname{Tr}(g^{-1}\delta g)^2 = \int G\Big[\frac{1}{2}\operatorname{Tr}(g_\infty^{-1}\delta g_\infty)^2 + \qquad (4.3.10)$$

$$+\operatorname{Tr}\delta g_L g_L^{-1} g_\infty^{-1} g_U^{-1}\delta g_U g_\infty\Big] =$$

$$= \int G\Big[\frac{1}{2}(\delta\vec{\varphi})^2 + \sum_{\alpha,\beta}\delta\chi_\alpha\left(X_L(\chi)C(\varphi)Y_U(\psi)\right)_{\alpha\beta}\delta\psi_\beta\Big]$$

In the case of $s\ell(2)$ matrices X and Y are unit. However, for other groups they are non-trivial and thus classical measure is highly non-linear. Therefore the proper quantum measure in functional integral is subtle in this case. Naively (i.e. provided that the measure coincides with the classical one) the relevant anomaly is

$$\log \mathrm{Det}_{anom}\left[G^{-1}(XCY)^{-1}\bar\partial(XCY)\partial\right] \tag{4.3.11}$$

However, we prefer to omit matrices X, Y from this expression, attributing their contribution to quantum measure. Then, according to (2.4.11)

$$4\pi \log \mathrm{Det}_{anom}\left[G^{-1}C(\varphi)^{-1}\bar\partial C(\varphi)\partial\right] \sim \tag{4.3.12}$$

$$\sim \frac{1}{12}\int \mathrm{Tr}\left[\,|\partial \log C|^2 + |\partial \log CG|^2 + 4\partial\log CG \times \bar\partial \log C\right] = \frac{1}{12}S_{Liouv}(G) + \frac{1}{2}\int \mathrm{Tr}\left[|\partial \log C|^2 - \log C \times \partial\bar\partial \log G\right] = \frac{1}{12}S_{Liouv}(G) + \frac{1}{2}\int\Big[\sum_{\alpha,\beta\in\Delta_+}(\vec\alpha\partial\vec\varphi)(\vec\beta\bar\partial\vec\varphi) + \sum_{\alpha\in\Delta_+}(\vec\alpha\vec\varphi)\mathcal{R}\Big] =$$

$$= \frac{1}{12}S_{Liouv}(G) + \int\left[\frac{c_v}{2}|\partial\vec\varphi|^2 + (\vec\rho\vec\varphi)\mathcal{R}\right] \tag{4.3.13}$$

$S_{Liouv}(G)$ is the Liouville action, c_v and $\vec\rho$ are defined by (4.2.14) and (4.2.11) respectively. Starting from (4.3.11) instead of (4.3.12) we would obtain extra contributions (like $|\partial \log(1-\chi_1\psi_1 e^{-\vec\alpha_1\vec\varphi})|^2$ in the case of $s\ell(3)$) to the quantum action which should be cancelled by quantum measure in order to make Lagrangian theory (2.4.1) consistant with WZW conformal model defined by KM algebra and Sugawara stress tensor.

Combining (4.3.5), (4.3.8) and (4.3.13), one obtains the final quantum WZW action

$$4\pi L_q = 4\pi L_{cl} - \frac{1}{2} c_v |\partial\vec{\varphi}|^2 - \vec{\rho}\vec{\varphi}\mathcal{R} =$$

$$= -\Big[\sum_{\alpha\in\Delta_+} w_\alpha \bar{\partial}\chi_\alpha + \frac{k+c_v}{2}\partial\vec{\varphi}\bar{\partial}\vec{\varphi} + \vec{\rho}\vec{\varphi}\mathcal{R}\Big] = \tag{4.3.14}$$

$$= -\sum_{\alpha\in\Delta_+} w_\alpha \bar{\partial}\chi_\alpha + \frac{1}{2}\partial\vec{\phi}\bar{\partial}\vec{\phi} + \frac{i\vec{\rho}\vec{\phi}}{q}\mathcal{R}$$

with

$$\vec{\varphi} = -\frac{i}{q}\vec{\phi}$$
$$q^2 = k + c_v \tag{4.3.15}$$

in accordance with (4.2.12).

The transformation rule (4.3.9) implies the form of screening operator insertions. In fact for simple roots α_s $\Upsilon(\psi)_{\alpha_s\alpha} = 0$ for all $\alpha \neq \alpha_s$ (as it happened with α_1 and α_3 in the case of $s\ell(3)$ in sect.4.1). In this case the insertions have particularly simple form. For $d\psi_{\alpha_s}$:

$$Q_{\alpha_s} = \oint e^{\vec{\alpha}_s\vec{\varphi}}\Big(\sum_{\beta\in\Delta_+} \mathcal{X}(\chi)_{\alpha_s\beta} w_\beta\Big) = \tag{4.3.16}$$

$$= \oint e^{-i\vec{\alpha}_s\vec{\phi}/q}\Big(\sum_{\beta\in\Delta_+} \mathcal{X}(\chi)_{\alpha_s\beta} w_\beta\Big)$$

Since

$$\langle\vec{\alpha}_s,\vec{\rho}\rangle \equiv \frac{2(\vec{\alpha}_s,\vec{\rho})}{(\vec{\alpha}_s,\vec{\alpha}_s)} = 1 \tag{4.3.17}$$

for any simple root, dimension of exponent is vanishing, $\Delta(e^{-i\vec{\alpha}_s\vec{\phi}/q}) = \frac{1}{2q^2}(-\vec{\alpha}_s)(-\vec{\alpha}_s + 2\vec{\rho}) = 0$. We shall demonstrate in the next section, that insertions of Q_{α_s} (for simple roots α_s only) appear sufficient to reproduce all known correlators for $s\ell(N)_k$ - algebras.

4.4. Correlation functions in $sl(N)$ WZW theories

Now let us turn to the computation of the correlators in general case of the $sl(N)$ WZW theory. The formulae for Cartan's currents of the algebra and the stress-tensor are coincide with (4.2.10) and (4.2.13):

$$\vec{H} = i\sqrt{C_v+k}\,\partial\vec{\phi} - \sum_{\alpha\in\Delta_+} \vec{\alpha}\, w_\alpha \chi_\alpha \tag{4.4.1}$$

$$T = \sum_{\alpha\in\Delta_+} w_\alpha \chi_\alpha - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{i}{\sqrt{C_v+k}}\vec{\rho}\,\partial^2\vec{\phi}$$

where:

$$C_v = N \quad , \quad \vec{\alpha}^2 = 2 \quad , \quad \phi_i(z)\phi_j(0) = -\delta_{ij}\log z + \ldots \tag{4.4.2}$$

(All useful information about the root system of the $sl(N)$ and other simple algebras can be found in ref.[1]). The formulae for the other currents are more complicated.

In this section we restrict ourselves by considering the correlation functions of the vertex operators of the fundamental representations of $sl(N)$ which have the dimensions equal to N. The highest weight vectors of these representations are:

$$\hat{V} = \exp\left(i\frac{\vec{\lambda}}{q}\vec{\phi}\right), \quad \hat{V}^* = \exp\left(i\frac{\vec{\lambda}^*}{q}\vec{\phi}\right), \quad q^2 = C_v + k \tag{4.4.3}$$

where $\vec{\lambda}$ and $\vec{\lambda}^*$ satisfy:

$$\begin{aligned} \vec{\alpha}_1\vec{\lambda} &= 1 \;, & \vec{\alpha}_i\vec{\lambda} &= 0 & i &= 2,\ldots,\ell \\ \vec{\alpha}_\ell\vec{\lambda}^* &= 1 \;, & \vec{\alpha}_k\vec{\lambda}^* &= 0 & k &= 1,\ldots,\ell-1 \end{aligned} \tag{4.4.4}$$

Vectors $\vec{\alpha}_i$ $(i=1,\ldots,\ell)$ are simple roots of the $sl(N)$, $\ell = N-1 = \mathrm{Rank}\; sl(N)$. The other vectors of the representations $N(\bar{N})$ can be obtained by taking the products of highest weight vectors $\hat{V} \in N$, $\hat{V}^{*} \in \overline{N}$ with some monomials $\chi_{\alpha_1} \dots \chi_{\alpha_k}$: $\vec{\alpha}_1 + \vec{\alpha}_2 + \dots + \vec{\alpha}_k = \vec{\alpha} \in \Delta_+$. For the $s\ell(3)$ algebra the fundamental representations 3 and $\overline{3}$ have the following form (see fig 3a, $\vec{\lambda} = \vec{\lambda}_1$, $\vec{\lambda}^{*} = -\vec{\mu}_3$):

$$3: \quad \hat{V}, \ \hat{V}\chi_1, \ \hat{V}(\chi_2 + a\chi_1\chi_3) \tag{4.4.5}$$

$$\overline{3}: \quad \hat{V}^{*}, \ \hat{V}^{*}\chi_3, \ \hat{V}^{*}(\chi_2 - b\chi_1\chi_3)$$

where $a + b = 1$. One can find these parameters in the general form of the $s\ell(3)$ currents in (the last paper of) ref.[9].) Here we choose $a = 0$, $b = 1$ and consider the two-point correlation function, following the way proposed in sect.2.

The vacuum charge $V_s(R)$ in general case will be of the form:

$$V_s(R) = e^{\frac{i}{q} 2\vec{\rho}\vec{\phi}(R)} \prod_{\alpha \in \Delta_+} \chi_\alpha(R) \tag{4.4.6}$$

Then, the two-point correlator equals:

$$\left\langle V_A(z)\, \widetilde{V}_B^{*}(0)\, V_s(R) \right\rangle_0 = \frac{1}{z^{2\Delta}}\, \delta_{AB} \tag{4.4.7}$$

where the operator V belongs to the representation N and $\widetilde{V}^{*}$ is waved operator of the representation $\overline{N}$, which, in fact, is defined by the equality (4.4.7). The dimension Δ :

$$\Delta = \frac{1}{2q^2}\vec{\lambda}(\vec{\lambda} + 2\vec{\rho}) = \frac{1}{2q^2}\vec{\lambda}^{*}(\vec{\lambda}^{*} + 2\vec{\rho}) = \frac{C_g}{2q^2} \tag{4.4.8}$$

is the dimension of the operators (4.4.5). C_g is the value of the second Casimir in the fundamental representation, which for the $s\ell(N)$ will be:

$$C_g = \frac{N^2 - 1}{N} \tag{4.4.9}$$

The indices A and B in (4.4.7) coincide when the sum of weights or Cartan's eigenvalue vectors of the V_A and V_B^* is zero, i.e. they have opposite directions. For example, the operators $V_1 = \hat{V}$ and $V_1^* = \hat{V}^*(\chi_2 - \chi_1\chi_3)$ have opposite weight vectors, thus

$$\tilde{V}_1^* = \exp\left[-\frac{i}{q}(\vec{\lambda}+2\vec{\rho})\vec{\phi}\right]\prod_{\alpha\in\Delta_+} e^{-u_\alpha + i v_\alpha} . \qquad (4.4.10)$$

Then the equality (4.4.7) is satisfied by inserting $V_1(z)$, $\tilde{V}_1^*(0)$ and $V_S(R)$.

Let us remind that the "vacuum" insertion (4.4.6) is the consequence of the fact that metric on sphere has a singular point R, i.e. it can be written in the form $ds^2 = |\omega(z)|^2$, where meromorphic differential $\omega(z) = (z-R)^{-2}dz$ has pole of order 2. It is necessary to point out that all above formulae for the vertex operators deserve some explanation, concerning the normal ordering implied. The naive normal ordering implies that the dimension of the vertex operator $V_\mu = :\exp(i\vec{\mu}\vec{\phi}):$ is simply $\frac{1}{2}\vec{\mu}^2$. But due to the presence of the term $\mathcal{R}\phi$ in the Lagrangian our ϕ is not quite ordinary free scalar field. This leads, in particular, to the fact that the correct normal ordering prescription differs from the naive normal ordering by the following way

$$V_\mu = :\exp(i\vec{\mu}\vec{\phi}):\left(\omega(z)\right)^{\frac{1}{q}\vec{\mu}\vec{\rho}} \qquad (4.4.11)$$

Later on we shall continue to write the expressions for the vertex operators in a slightly vulgar manner, omitting $\omega(z)$, but one should remember that it is the second factor in (4.4.11) which leads to the absence of the dependence on point R in the correlators. The vulgar form of the vertex operators gives correct results provided the point of the singularity R is taken in infinity.

For the four-point function this is not the whole story. As it has been explained above, one should make some insertions with "screening" charges which are of the form of a one-dimensional operator integrated over a noncontractable contour. Let us consider the correlator

$$\langle V_1(0)V_1^*(x)V_1(1)\tilde{V}_1^*(\infty)V_s(R)Q\ldots Q\rangle \qquad (4.4.12)$$

We choose (in the $s\ell(3)$ case):

$$V_1(z)=\hat{V}(z) \qquad (4.4.13)$$

$$V_1^*(z)=\hat{V}^*(z)\chi_2(z) \qquad (4.4.14)$$

and $\tilde{V}_1^*$ was defined in (4.4.10). It is necessary to insert ℓ = Rank $s\ell(N)$ = N-1 contour operators (ℓ =2 for $s\ell(3)$) which have the form:

$$Q_1=\oint w_1 e^{-\frac{i}{q}\vec{\alpha}_1\vec{\phi}}, \quad Q_2=\oint \chi_1 w_2 e^{-\frac{i}{q}\vec{\alpha}_3\vec{\phi}} \qquad (4.4.15)$$

in the $s\ell(3)$ case, where $\vec{\alpha}_1$ is a simple root (one can change $\vec{\alpha}_1$ for $\vec{\alpha}_3$ which is the other simple root in the $s\ell(3)$ case). Then $\vec{\alpha}_2=\vec{\alpha}_1+\vec{\alpha}_3=\vec{\theta}$ is the highest root of the $s\ell(3)$ algebra. The charge neutrality is satisfied because of the following equality:

$$\vec{\theta}=\vec{\lambda}+\vec{\lambda}^* \qquad (4.4.16)$$

which takes place for the general $s\ell(N)$ case. It should be stressed that the $\vec{\theta}$, appearing from the charge balance (4.4.16) of (4.4.12) should be expanded into the sum of the <u>simple</u> roots

$$\vec{\theta}=\vec{\alpha}_1+\ldots+\vec{\alpha}_\ell \qquad (4.4.17)$$

because (we need) only the operator of the form: $W\chi\ldots\chi\exp\left(-\frac{i}{q}\vec{\alpha}_s\vec{\phi}\right)$ where $\vec{\alpha}_s$ is a simple root, which has unit dimension since

$$\Delta\Big(\underset{s\ell(3)}{\exp}\left(-\frac{i}{q}\vec{\alpha}_s\vec{\phi}\right)\Big) = -\frac{1}{2q^2}\vec{\alpha}_s\left(-\vec{\alpha}_s+2\vec{\rho}\right)=0 \qquad (4.4.18)$$

Thus, the $s\ell(3)$ correlator (4.4.12), (4.4.13), (4.4.14) is proportional to:

$$\oint dt_1 dt_2\, t_1^{-\frac{1}{q^2}}(t_1-1)^{-\frac{1}{q^2}}(t_2-x)^{-\frac{1}{q^2}-1}(t_1-t_2)^{-\frac{1}{q^2}-1} \qquad (4.4.19)$$

(see fig.4).

Integration over t_2 leads to the result

$$\oint dt_1\, t_1^{-\frac{1}{q^2}}(t_1-1)^{-\frac{1}{q^2}}(t_1-x)^{-\frac{2}{q^2}-1}\sim \qquad (4.4.20)$$

$$\sim F\left(\frac{1}{q^2},1-\frac{1}{q^2},1-\frac{3}{q^2},x\right)$$

which is a linear combination of the Knizhnik-Zamolodchikov equation solutions, given in ref.[5].

For the general $s\ell(N)$ case the prescription suggested above, gives for the 4-point correlator:

$$\oint dt_1\ldots dt_\ell\, t_1^{-\frac{1}{q^2}}(t_1-1)^{-\frac{1}{q^2}}(t_\ell-x)^{-\frac{1}{q^2}}\cdot$$

$$\cdot\left[(t_1-t_2)(t_2-t_3)\ldots(t_{\ell-1}-t_\ell)\right]^{-\frac{1}{q^2}-1}\frac{1}{t_\ell-x}\sim$$

$$\sim\oint dt_1\, t_1^{-\frac{1}{q^2}}(t_1-1)^{-\frac{1}{q^2}}(t_1-x)^{-\frac{\ell}{q^2}-1}\sim$$

$$\sim F\left(\frac{1}{q^2},1-\frac{1}{q^2},1-\frac{N}{q^2},x\right)$$

(4.4.21)

since $N = \ell + 1$. The result (4.4.21) is also a linear combination of the Knizhnik-Zamolodchikov equation solutions [5]. To prove this fact one should use the following relation for hypergeometric functions:

$$(\gamma-\beta)F(\alpha,\beta-1,\gamma,z)+(2\beta-\gamma-\beta z+\alpha z)F(\alpha,\beta,\gamma,z)+ \\ +\beta(z-1)F(\alpha,\beta+1,\gamma,z)=0 \tag{4.4.22}$$

The other correlators in the $s\ell(N)$ WZW theory may be computed in a similar manner.

4.5. TWO MORE COMPLICATED EXAMPLES: FREE FIELDS FOR WZWM WITH GROUPS SL(4) AND Sp(2)

Here we elaborate explicitely the free field representation for the cases of the group $Sp(2)\simeq so(5)$ of rank 2 and the group $sl(4)$ of rank 3.

4.5.1. $Sp(2)_k$: We can use the parametrization, introduced in ss.3.3:

$$\tilde{J}=\left(\begin{array}{cc|cc} d\varphi_1 & \tilde{w}_1 & \tilde{w}_2 & \tilde{w}_3 \\ 0 & d\varphi_2 & \tilde{w}_4 & \tilde{w}_2 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -d\varphi_2 & -\tilde{w}_1 \\ & & 0 & -d\varphi_1 \end{array}\right), \quad g_L=\left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \chi_1 & 1 & & \\ \hline \chi_2 & \chi_4 & 1 & 0 \\ \chi_3 & \chi_2-\chi_1\chi_4 & -\chi_1 & 1 \end{array}\right) \tag{4.5.1}$$

where the relation between fields $\widehat{w}_j$ and root subspaces $\mathcal{G}_{\alpha_j}$ is illustrated in Fig. 3b. At the same time $\tilde{J}=(g_u g_D)^{-1}\partial(g_u g_D)$ where $g_D=diag(e^{\varphi_1}, e^{\varphi_2}, e^{-\varphi_2}, e^{-\varphi_1})$

$$g_u(\psi)=\left(\begin{array}{cc|cc} 1 & \psi_1 & \psi_2 & \psi_3 \\ 0 & 1 & \psi_4 & \psi_2-\psi_1\psi_4 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & -\psi_1 \\ & & 0 & 1 \end{array}\right) \tag{4.5.2}$$

and

$$\begin{aligned} \tilde{w}_1 &= e^{\varphi_2-\varphi_1} d\psi_1 \\ \tilde{w}_2 &= e^{-\varphi_2-\varphi_1}(d\psi_2-\psi_1 d\psi_4) \\ \tilde{w}_3 &= e^{-2\varphi_1}(d\psi_3+\psi_2 d\psi_1-\psi_1 d\psi_2+\psi_1^2 d\psi_4) \\ \tilde{w}_4 &= e^{-2\varphi_2} d\psi_4 \end{aligned} \tag{4.5.3}$$

These fields are related to free fields χ_j, w_k as follows:

$$\kappa \widetilde{w}_1 = \frac{1}{2}(w_1 - \chi_2 w_3), \quad \kappa \widetilde{w}_2 = \frac{1}{2}(w_2 + \chi_1 w_3)$$
$$\kappa \widetilde{w}_3 = w_3, \quad \kappa \widetilde{w}_4 = w_4 + \chi_1 w_2, \quad q^2 = 2\alpha^2, \phi_i = q\varphi_i \qquad (4.5.4)$$

with central charge $k = -3 + q^2$. The currents, expressed in terms of free fields look like: $\mathcal{J}_{jk} = (k\, g^{-1} dg)_{jk}$

$$\mathcal{J}_{11} = \frac{1}{2}\chi_1 w_1 + \frac{1}{2}\chi_2 w_2 + \chi_3 w_3 + q\partial\varphi_1, \quad \mathcal{J}_{12} = \frac{1}{2} w_1 + \frac{1}{2}\chi_4 w_2 +$$
$$+ \frac{1}{2}\chi_2 w_3 - \frac{1}{2}\chi_1\chi_4 w_3, \quad \mathcal{J}_{13} = \frac{1}{2} w_2 - \frac{1}{2}\chi_1 w_3, \quad \mathcal{J}_{14} = w_3$$
$$\mathcal{J}_{21} = \frac{1}{2}\chi_3 w_2 + \chi_2 w_4 - \frac{1}{2}\chi_1^2 w_1 + \frac{1}{2}\chi_1\chi_2 w_2 - \frac{1}{2}\chi_1\chi_3 w_3$$
$$+ q\chi_1\partial(\varphi_2 - \varphi_1) + (k + \frac{3}{2})\partial\chi_1, \quad \mathcal{J}_{22} = -\frac{1}{2}\chi_1 w_1 + \frac{1}{2}\chi_2 w_2 +$$
$$+ \chi_4 w_4 + q\partial\varphi_2, \quad \mathcal{J}_{23} = w_4, \quad \mathcal{J}_{24} = \frac{1}{2} w_2 - \frac{1}{2}\chi_1 w_3 \qquad (4.5.5)$$
$$\mathcal{J}_{31} = -\frac{1}{2}\chi_3 w_1 - \frac{1}{2}\chi_1\chi_2 w_1 + \frac{1}{2}\chi_1^2\chi_4 w_1 - \frac{1}{2}\chi_2^2 w_2 - \frac{1}{2}\chi_3\chi_4 w_2 -$$
$$- \frac{1}{2}\chi_1\chi_2\chi_4 w_2 - \frac{1}{2}\chi_2\chi_3 w_3 + \frac{1}{2}\chi_1\chi_3\chi_4 w_3 - \chi_2\chi_4 w_4 - q\chi_2\partial(\varphi_1 + \varphi_2)$$
$$+ q\chi_1\chi_4\partial(\varphi_1 - \varphi_2) + (k + \frac{1}{2})\partial\chi_2 - (k + \frac{3}{2})\chi_4\partial\chi_1; \quad \mathcal{J}_{32} = -\chi_2 w_1 +$$
$$+ \chi_1\chi_4 w_1 - \chi_2\chi_4 w_2 - \chi_4^2 w_4 - 2q\chi_4\partial\varphi_2 + k\partial\chi_4, \quad \mathcal{J}_{41} = -\chi_1\chi_3 w_1$$
$$- \chi_2\chi_3 w_2 - \chi_1\chi_2^2 w_2 - \chi_3^2 w_3 - \chi_2^2 w_4 - 2q\chi_1\chi_2\partial\varphi_2 - 2q\chi_3\partial\varphi_1$$
$$+ k\partial\chi_3 + (k+1)\chi_1\partial\chi_2 - (k+3)\chi_2\partial\chi_1.$$
$$\mathcal{J}_{24} = \mathcal{J}_{13}, \quad \mathcal{J}_{33} = -\mathcal{J}_{22}, \quad \mathcal{J}_{34} = -\mathcal{J}_{12}, \quad \mathcal{J}_{42} = \mathcal{J}_{31}, \quad \mathcal{J}_{43} = -\mathcal{J}_{21},$$
$$\mathcal{J}_{44} = -\mathcal{J}_{11},$$

The energy-momentum tensor is

$$T = \frac{1}{2(k+3)}\, \mathrm{tr} :J^2: = - w_1\partial\chi_1 - w_2\partial\chi_2 - w_3\partial\chi_3 - \qquad (4.5.6)$$
$$- w_4\partial\chi_4 - \frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}(\partial\varphi_2)^2 - \frac{\sqrt{2}}{2\sqrt{k+3}}\partial^2(2\varphi_1 + \varphi_2).$$

4.5.2. <u>$Sl(4)_k$</u>: This case is direct generalization of that of $sl(3)_k$ (see ss.4.1). In terms of original fields $\widetilde{W}_\alpha$, $\widetilde{\Phi}_i$ the current matrix $J = \kappa g^{-1} dg$ looks like:

$$
\begin{aligned}
\kappa^{-1} J_{11} &= d\widetilde{\Phi}_1 + \chi_1 \widetilde{W}_1 + \chi_2 \widetilde{W}_2 + \chi_4 \widetilde{W}_4 \\
\kappa^{-1} J_{12} &= \widetilde{W}_1 + \chi_3 \widetilde{W}_3 + \chi_5 \widetilde{W}_4 \\
\kappa^{-1} J_{13} &= \widetilde{W}_2 + \chi_6 \widetilde{W}_4 \\
\kappa^{-1} J_{14} &= \widetilde{W}_4 \\
\kappa^{-1} J_{21} &= \chi_2 \widetilde{W}_3 + \chi_4 \widetilde{W}_5 - \chi_1^2 \widetilde{W}_1 - \chi_1 \chi_2 \widetilde{W}_2 - \chi_1 \chi_4 \widetilde{W}_4 + \chi_1 d(\widetilde{\Phi}_2 - \widetilde{\Phi}_1) + d\chi_1 \\
\kappa^{-1} J_{22} &= d\widetilde{\Phi}_2 + \chi_3 \widetilde{W}_3 + \chi_5 \widetilde{W}_5 - \chi_1 \widetilde{W}_1 - \chi_1 \chi_3 \widetilde{W}_2 - \chi_1 \chi_5 \widetilde{W}_4 \\
\kappa^{-1} J_{23} &= \widetilde{W}_3 + \chi_6 \widetilde{W}_5 - \chi_1 \widetilde{W}_2 - \chi_1 \chi_6 \widetilde{W}_4 \qquad\qquad (4.5.7) \\
\kappa^{-1} J_{24} &= \widetilde{W}_5 - \chi_1 \widetilde{W}_4 \\
\kappa^{-1} J_{31} &= \chi_4 \widetilde{W}_6 - \chi \chi_3 \widetilde{W}_3 - \chi_3 \chi_4 \widetilde{W}_5 - \chi_1 \chi_2 \widetilde{W}_1 - \chi_2^2 \widetilde{W}_2 - \chi_2 \chi_4 \widetilde{W}_4 + \chi_1^2 \chi_3 \widetilde{W}_1 + \\
&\quad + \chi_1 \chi_2 \chi_3 \widetilde{W}_2 + \chi_1 \chi_3 \chi_4 \widetilde{W}_4 + \chi_2 d(\widetilde{\Phi}_3 - \widetilde{\Phi}_1) + \chi_1 \chi_3 d(\widetilde{\Phi}_1 - \widetilde{\Phi}_2) + d\chi_2 - \chi_3 d\chi_1 \\
\kappa^{-1} J_{32} &= \chi_5 \widetilde{W}_6 - \chi_3^2 \widetilde{W}_3 - \chi_3 \chi_5 \widetilde{W}_5 - \chi_2 \widetilde{W}_1 - \chi_2 \chi_3 \widetilde{W}_2 - \chi_2 \chi_5 \widetilde{W}_4 + \chi_1 \chi_3 \widetilde{W}_1 + \chi_1 \chi_3^2 \widetilde{W}_2 + \\
&\quad + \chi_1 \chi_3 \chi_5 \widetilde{W}_4 + \chi_3 d(\widetilde{\Phi}_3 - \widetilde{\Phi}_2) + d\chi_3 \\
\kappa^{-1} J_{33} &= d\widetilde{\Phi}_3 + \chi_6 \widetilde{W}_6 - \chi_3 \widetilde{W}_3 - \chi_3 \chi_6 \widetilde{W}_5 - \chi_2 \widetilde{W}_2 - \chi_1 \chi_6 \widetilde{W}_4 + \chi_1 \chi_3 \widetilde{W}_2 + \chi_1 \chi_3 \chi_6 \widetilde{W}_4 \\
\kappa^{-1} J_{34} &= \widetilde{W}_6 - \chi_3 \widetilde{W}_5 - \chi_2 \widetilde{W}_4 + \chi_1 \chi_3 \widetilde{W}_4 \\
\kappa^{-1} J_{41} &= -\chi_4 \chi_6 \widetilde{W}_6 - \chi_2 \chi_5 \widetilde{W}_3 - \chi_4 \chi_5 \widetilde{W}_5 + \chi_2 \chi_3 \chi_6 \widetilde{W}_3 + \chi_3 \chi_4 \chi_6 \widetilde{W}_5 - \chi_1 \chi_4 \widetilde{W}_1 - \chi_2 \chi_4 \widetilde{W}_2 - \\
&\quad - \chi_4^2 \widetilde{W}_4 + \chi_1 \chi_2 \chi_6 \widetilde{W}_1 + \chi_2^2 \chi_6 \widetilde{W}_2 + \chi_2 \chi_4 \chi_6 \widetilde{W}_4 + \chi_1^2 \chi_5 \widetilde{W}_1 + \chi_1 \chi_2 \chi_5 \widetilde{W}_2 + \chi_1 \chi_4 \chi_5 \widetilde{W}_4 - \\
&\quad - \chi_1^2 \chi_3 \chi_6 \widetilde{W}_1 - \chi_1 \chi_2 \chi_3 \chi_6 \widetilde{W}_2 - \chi_1 \chi_3 \chi_4 \chi_6 \widetilde{W}_4 + \chi_4 d(\widetilde{\Phi}_4 - \widetilde{\Phi}_1) + \chi_2 \chi_6 d(\widetilde{\Phi}_1 - \widetilde{\Phi}_3) + \\
&\quad + \chi_1 \chi_5 d(\widetilde{\Phi}_1 - \widetilde{\Phi}_2) + \chi_1 \chi_3 \chi_6 d(\widetilde{\Phi}_2 - \widetilde{\Phi}_1) + d\chi_4 - \chi_6 d\chi_2 - \chi_5 d\chi_1 + \chi_3 \chi_6 d\chi_1 \\
\kappa^{-1} J_{42} &= -\chi_5 \chi_6 \widetilde{W}_6 - \chi_3 \chi_5 \widetilde{W}_3 - \chi_5^2 \widetilde{W}_5 + \chi_3^2 \chi_6 \widetilde{W}_3 + \chi_3 \chi_5 \chi_6 \widetilde{W}_5 - \chi_3 \chi_4 \widetilde{W}_2 - \chi_4 \chi_5 \widetilde{W}_4 + \chi_2 \chi_3 \chi_6 \widetilde{W}_2 + \\
&\quad + \chi_2 \chi_5 \chi_6 \widetilde{W}_4 + \chi_1 \chi_3 \chi_5 \widetilde{W}_2 + \chi_1 \chi_5^2 \widetilde{W}_4 - \chi_1 \chi_3^2 \chi_6 \widetilde{W}_2 - \chi_1 \chi_3 \chi_5 \chi_6 \widetilde{W}_4 + \chi_5 d(\widetilde{\Phi}_4 - \widetilde{\Phi}_2) + \\
&\quad + \chi_3 \chi_6 d(\widetilde{\Phi}_2 - \widetilde{\Phi}_3) + d\chi_5 - \chi_6 d\chi_3 - \chi_4 \widetilde{W}_1 + \chi_2 \chi_6 \widetilde{W}_1 + \chi_1 \chi_5 \widetilde{W}_1 - \chi_1 \chi_3 \chi_6 \widetilde{W}_1 \\
\kappa^{-1} J_{43} &= -\chi_6^2 \widetilde{W}_6 - \chi_5 \widetilde{W}_3 - \chi_5 \chi_6 \widetilde{W}_5 + \chi_3 \chi_6 \widetilde{W}_3 + \chi_3 \chi_6^2 \widetilde{W}_5 - \chi_4 \widetilde{W}_2 - \chi_4 \chi_6 \widetilde{W}_4 + \chi_2 \chi_6 \widetilde{W}_2 + \chi_2 \chi_6^2 \widetilde{W}_4 + \\
&\quad + \chi_1 \chi_5 \widetilde{W}_2 + \chi_1 \chi_5 \chi_6 \widetilde{W}_4 - \chi_1 \chi_3 \chi_6 \widetilde{W}_2 - \chi_1 \chi_3 \chi_6^2 \widetilde{W}_4 + \chi_6 d(\widetilde{\Phi}_4 - \widetilde{\Phi}_3) + d\chi_6 \\
\kappa^{-1} J_{44} &= d\widetilde{\Phi}_4 - \chi_6 \widetilde{W}_6 - \chi_5 \widetilde{W}_5 + \chi_3 \chi_6 \widetilde{W}_5 - \chi_4 \widetilde{W}_4 + \chi_2 \chi_6 \widetilde{W}_4 + \chi_1 \chi_5 \widetilde{W}_4 - \chi_1 \chi_3 \chi_6 \widetilde{W}_4 .
\end{aligned}
$$

Relation between two independent free fields is:

$$k\widetilde{W}_1 = W_1\ ;\ k\widetilde{W}_2 = W_2\ ;\ k\widetilde{W}_3 = W_3+\chi_1 W_2\ ;\ k\widetilde{W}_4 = W_4\ ;\ k\widetilde{W}_5 = W_5+\chi_1 W_4$$

$$k\widetilde{W}_6 = W_6+\chi_3 W_5+\chi_2 W_4\ ;\ k\widetilde{\phi}_i = q\phi_i\ ;\ k=-4+q^2\ ;\ q^2 = 2\lambda^2,\ \phi_i = \vec{\mu}_i\vec{\phi} \qquad (4.5.8)$$

Substituting these expressions into eq.(4.5.7) after appropriate normal ordering we obtain:

$$J_{11} = \chi_1 W_1+\chi_2 W_2+\chi_4 W_4+q\partial\phi_1$$

$$J_{12} = W_1+\chi_3 W_2+\chi_5 W_4$$

$$J_{13} = W_2+\chi_6 W_4$$

$$J_{14} = W_4$$

$$J_{21} = \chi_2 W_3+\chi_4 W_5-\chi_1^2 W_1+q\chi_1\partial(\phi_2-\phi_1)+(k+2)\partial\chi_1$$

$$J_{22} = -\chi_1 W_1+\chi_3 W_3+\chi_5 W_5+q\partial\phi_2$$

$$J_{23} = W_3+\chi_6 W_5$$

$$J_{24} = W_5$$

$$J_{31} = \chi_4 W_6-\chi_1\chi_2 W_1-\chi_2\chi_3 W_3-\chi_2^2 W_2+\chi_1^2\chi_3 W_1+q\chi_2\partial(\phi_3-\phi_1)+q\chi_1\chi_3\partial(\phi_1-\phi_2)+(k+1)\partial\chi_2-(k+2)\chi_3\partial\chi_1$$

$$J_{32} = -\chi_2 W_1+\chi_5 W_6+\chi_1\chi_3 W_1-\chi_2\chi_3 W_2-\chi_3^2 W_3+q\chi_3\partial(\phi_3-\phi_2)+(k+1)\partial\chi_3$$

$$J_{33} = -\chi_2 W_2-\chi_3 W_3+\chi_6 W_6+q\partial\phi_3$$

$$J_{34} = W_6$$

$$\begin{aligned} J_{41} = &-\chi_1\chi_4 W_1-\chi_2\chi_4 W_2-\chi_2\chi_5 W_3-\chi_4^2 W_4-\chi_4\chi_5 W_5-\chi_4\chi_6 W_6+\chi_1\chi_4\chi_3 W_1+\chi_1^2\chi_5 W_1- \\ &-\chi_1^2\chi_3\chi_6 W_1+\chi_2^2\chi_6 W_2+\chi_2\chi_3\chi_6 W_3+q\chi_4\partial(\phi_4-\phi_1)+q\chi_2\chi_6\partial(\phi_1-\phi_3)+q\chi_1\chi_5\partial(\phi_1-\phi_2)+ \\ &+q\chi_1\chi_3\chi_6\partial(\phi_2-\phi_1)+k\partial\chi_4-(k+1)\chi_6\partial\chi_2-(k+2)\chi_5\partial\chi_1+(k+2)\chi_3\chi_6\partial\chi_1 \end{aligned} \qquad (4.5.9)$$

$$\begin{aligned} J_{42} = &-\chi_4 W_1+\chi_2\chi_6 W_1+\chi_1\chi_5 W_1-\chi_3\chi_4 W_2-\chi_3\chi_5 W_3-\chi_4\chi_5 W_4-\chi_5^2 W_5-\chi_5\chi_6 W_6- \\ &-\chi_1\chi_3\chi_6 W_1+\chi_2\chi_3\chi_6 W_2+\chi_3^2\chi_6 W_3+q\chi_5\partial(\phi_4-\phi_2)+q\chi_3\chi_6\partial(\phi_2-\phi_3)+ \\ &+k\partial\chi_5-(k+1)\chi_6\partial\chi_3 \end{aligned}$$

$$\begin{aligned} J_{43} = &-\chi_4 W_2-\chi_5 W_3+\chi_2\chi_6 W_2+\chi_3\chi_6 W_3-\chi_4\chi_6 W_4-\chi_5\chi_6 W_5-\chi_6^2 W_6+ \\ &+q\chi_6\partial(\phi_4-\phi_3)+k\partial\chi_6 \end{aligned}$$

$$J_{44} = -\chi_4 W_4-\chi_5 W_5-\chi_6 W_6+q\partial\phi_4$$

$$\sum_{j=1}^{4}\phi_j = 0$$

4.6. HAMILTONIAN APPROACH, OR THE WZW LAGRANGIAN AS d^{-1} OF KIRILLOV-KOSTANT FORM ON COADJOINT ORBIT OF KM ALGEBRA.

As we have already mentioned in ss.3.6, there is a remarkable relation between the structure of Lie algebra $\mathfrak{g}$ and certain mechanical systems, interpreted as the motion of point-like objects or strings on homogeneous spaces M=G/H, which are orbits of coadjoint representation of G. This relation is described by Kirillov-Kostant construction. In the case of particles this movement is described by natural Lagrangian, and in the case of strings the action of WZWM (and not of ordinary non-chiral sigma-model) arises. In other words,Lagrangian of WZWM may be naturally considered as d^{-1} of Kirillov-Kostant form on the orbit of KM coadjoint representation. Moreover,the free field representation of WZWM naturally arises for appropriate choice of coordinates on the orbit,dictated by Gauss decomposition. (Analogous construction in the case if Virasoro algebra leads to a free-field representation of Liouville theory,see the second paper of ref.[3])

Generalization of the orbit approach for finite dimensional algebras,presented in ss.3.5. to infinite-dimensional case is straightforward. Consider a Kac-Moody algebra $\tilde{\mathfrak{g}}$ elements of which are triples $u(z) \oplus \mathbb{C}c \oplus \mathbb{C}d$ where $u(z) \in L\mathfrak{g}$ - the loop space of the Lie algebra $\mathfrak{g}$, c is the central element, $d = z\, d/dz$ (see also section 6). The dual space is $\tilde{\mathfrak{g}}^* = \{v(z) \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0\}$ with invariant pairing

$$<u,v> = tr\frac{1}{2\pi}\int dz\, u(z)\, v(z)$$

$$<c,\Lambda_0> = \Lambda_0(c) = 1 \qquad (4.6.1)$$

$$<d,\delta> = \delta(d) = 1$$

From commutation rules

$$[u_1 \oplus \lambda_1 c \oplus \mu_1 d,\ u_2 \oplus \lambda_2 c \oplus \mu_2 d] =$$

$$= [u_1,u_2] + \mu_1 z\partial_z u_2 - \mu_2 z\partial_z u_1 \oplus tr\frac{1}{2\pi}\int dz\, u_2 \partial_z u_1 \oplus 0d \qquad (4.6.2)$$

we may derive the formulae for adjoint and coadjoint action of the loop group LG :

$$Ad_g\{u\oplus\lambda c \oplus \mu d\} = \{gug^{-1} - \mu g^{-1}\partial_z g \oplus (\lambda +$$

$$+ tr\frac{1}{2\pi}\int dz\, u(z) g^{-1}(z)\partial_z g)c, \mu d\} \qquad (4.6.3)$$

$$Ad^*_g\{v \oplus \mu'\delta \oplus \lambda'\Lambda_0\} = \{g^{-1}vg + \frac{\lambda'}{2\pi} g^{-1}\partial_z g \oplus \qquad (4.6.4)$$

$$\oplus (\mu' + \frac{1}{2\pi}\int dz\, vg^{-1}\partial_z v) \oplus \lambda'\Lambda_0\}$$

Let us consider the orbit $\mathcal{O}_{X_0}$, which contains a vector X_0 of the form

$$X_0 = (0,0,k) \qquad (4.6.5)$$

General element of this orbit X looks like

$$X = (kg^{-1}\partial g, 0, k) \qquad (4.6.6)$$

The stationary subgroup of the element X_0 is the finite dimensional group G, and

$$\mathcal{O}_{X_0} = LG/G \qquad (4.6.7)$$

Let us calculate now Kirillov-Kostant form (3.6.8). First of all,

$$Y = g^{-1}dg \qquad (4.6.8)$$

and

$$[Y,Y] = [g^{-1}dg, g^{-1}dg] \oplus (tr\frac{1}{2\pi}\int dz\, g^{-1}dg\, \partial_z g^{-1}dg)c =$$

$$= [g^{-1}dg, g^{-1}dg] \oplus \left(\mathrm{tr} \frac{1}{2\pi} \int dz \left(g^{-1}dg\, g^{-1}\partial_z dg - \right.\right. \tag{4.6.9}$$
$$\left.\left. - g^{-1}dg\, g^{-1}\partial_z g\, g^{-1}dg \right)\right) c$$

Then

$$\Omega = \frac{K}{2\pi} \mathrm{tr} \int dz \left(g^{-1}dg\, g^{-1}\partial_z dg - g^{-1}dg\, g^{-1}\partial_z g\, g^{-1}dg \right) \tag{4.6.10}$$

Thus the canonical action $\mathcal{A}$ (3.6.) is the integral of the form α ,which naturally has a form of two-fold integral:

$$\mathcal{A} = \frac{K}{2\pi} \mathrm{tr} \iint d^2x \left(-g^{-1}dg\, g^{-1}dg + d^{-1}(g^{-1}dg\, g^{-1}dg\, g^{-1}dg \right) \tag{4.6.11}$$

As it has been explained above, Gauss decomposition leads to diagonalization of $\mathcal{A}$ and thus of the Kirillov-Kostant form on the whole orbit $\mathcal{O}_{x_0}$,except for a set of measure zero, where this decomposition becomes unvalid.

4.7 TOWARDS BOSONIZATION OF COSET MODELS

Formal application of bosonization scheme [2] to coset models $M=G/H$ is straightforward. However, generically it does not seem to lead to quadratic stress tensors and thus is not absolutely satisfactory. Surprisingly enough somewhat more sophisticated embeddings of H into G may exist, which give rise to a slightly different coset construction, leading to quadratic stress tensors. Let us discuss several simple examples.

We begin from standard coset models. The simplest possible example is $M = G/H = U(1)_{k_1} \times U(1)_{k_2} / U(1)_{k_1+k_2}$. KM currents are expressed in terms of two scalar fields ϕ_1, ϕ_2, taking values in circles of radia $1/\sqrt{k_1}$ and $1/\sqrt{k_2}$ respectively,

$$J_1 = i\sqrt{k_1}\,\partial\phi_1 \quad ; \quad J_2 = i\sqrt{k_2}\,\partial\phi_2 \tag{4.7.1}$$

The H-subalgebra is generated by

$$J_H = i\sqrt{k_1}\,\phi_1 + i\sqrt{k_2}\,\phi_2 = i\sqrt{k_1+k_2}\,\phi_3 \tag{4.7.2}$$

Primary vertex operators of original WZWM G are

$$V_{n_1,n_2} = \exp\left(in_1\sqrt{k_1}\,\phi_1 + in_2\sqrt{k_2}\,\phi_2\right) \tag{4.7.3}$$

with arbitrary integer n_1, n_2. Vertex operators of coset model M are those commuting with J_H, thus they obey the constraint

$$n_1 k_1 + n_2 k_2 = 0 \tag{4.7.4}$$

This constraint may be resolved only for rational k_1 and k_2 (thus original WZWM should be rational), and

$$n_1 = n k_2 / \{k_1, k_2\} \qquad n_2 = -n k_1 / \{k_1, k_2\} \tag{4.7.5}$$

$\{k_1,k_2\}$ standing for an obviously defined analogue of the largest common divisor of rational k_1 and k_2. Thus primary fields of coset model M are associated with vertices

$$V_n = \exp\left(i\frac{\sqrt{k_1k_2}}{\{k_1,k_2\}}\, n\left(\sqrt{k_2}\phi_1-\sqrt{k_1}\phi_2\right)\right) \qquad (4.7.6)$$

i.e. with those of another WZWM $U(1)_l$, $l = \frac{k_1k_2(k_1+k_2)}{\{k_1,k_2\}^2}$ quite as it should be [6]. One may obtain the same result, considering a current, orthogonal to J_H in (4.7.2),

$$J_M = i\gamma\left(\sqrt{k_2}\partial\phi_1-\sqrt{k_1}\partial\phi_2\right) = i\gamma\sqrt{k_1+k_2}\,\partial\phi_M \qquad (4.7.7)$$

with γ defined from the integral valuedness condition for all contour integrals $\frac{1}{2\pi i}\oint J_M$ along non-contractable cycles on Riemann surfaces ($\frac{\sqrt{k_1}}{2\pi}\oint\partial\varphi_1$ and $\frac{\sqrt{k_2}}{2\pi}\oint\partial\varphi_2$ are arbitrary integer winding numbers, thus $\gamma\sqrt{\frac{k_1}{k_2}}$ and $\gamma\sqrt{\frac{k_2}{k_1}}$ should be integer, and

$$\gamma = \frac{\sqrt{k_1k_2}}{\{k_1,k_2\}} \qquad (4.7.8)$$

for rational k_1,k_2). In the case of this abelian model Sugawara's stress tensor of WZWM g naturally splits into two orthogonal quadratic parts, associated with $U(1)_{k_1+k_2}$ and $M=U(1)_l$:

$$\begin{aligned} T &= \frac{1}{2k_1}:J_1^2: + \frac{1}{2k_2}:J_2^2: = -\tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 = \\ &= -\tfrac{1}{2}(\partial\phi_H)^2 - \tfrac{1}{2}(\partial\phi_M)^2 = \frac{1}{2(k_1+k_2)}:J_H^2: + \frac{1}{2\gamma^2(k_1+k_2)}:J_M^2: \end{aligned} \qquad (4.7.9)$$

Another example is $M = G/H = sl(2)_{k_1}\times sl(2)_{k_2}/\,sl(2)_{k_1+k_2}$.

Six independent bosonized currents are:

$$\begin{array}{ll} J_1^+ = w_1\cdot\frac{i}{\sqrt{2}} & J_2^+ = w_2\cdot\frac{i}{\sqrt{2}} \\ H_1^0 = \left[-\chi_1w_1+\frac{i}{\sqrt{2}}q_1\partial\phi_1\right]\frac{i}{\sqrt{2}} & H_2^0 = \left[-\chi_2w_2+\frac{i}{\sqrt{2}}q_2\partial\phi_2\right]\frac{i}{\sqrt{2}} \\ J_1^- = \left[\chi_1^2w_1-i\sqrt{2}q_1\chi_1\partial\phi_1-k_1\partial\chi_1\right]\frac{i}{\sqrt{2}} & J_2^- = \left[\chi_2^2w_2-i\sqrt{2}q_2\chi_2\partial\phi_2-k_2\partial\chi_2\right]\frac{i}{\sqrt{2}} \\ q_1^2 = k_1+2 & q_2^2 = k_2+2 \end{array} \qquad (4.7.10)$$

and subalgebra H is generated by $J_H = J_1 + J_2$:

$$
\begin{aligned}
J_H^+ &= (W_1 + W_2)\frac{i}{\sqrt{2}} \\
H_H &= \left(-\chi_1 W_1 - \chi_2 W_2 + \frac{i}{\sqrt{2}} q_1 \partial\phi_1 + \frac{i}{\sqrt{2}} q_2 \partial\phi_2\right)\frac{i}{\sqrt{2}} \\
J_H^- &= \left(\chi_1^2 W_1 + \chi_2^2 W_2 - i q_1 \sqrt{2}\chi_1 \partial\phi_1 - i q_2 \sqrt{2}\chi_2 \partial\phi_2 - k_1 \partial\phi_1 - k_2 \partial\phi_2\right)\frac{i}{\sqrt{2}}
\end{aligned}
\tag{4.7.11}
$$

The central charge of this subalgebra is $k_H = k_1 + k_2$. One easily verifies, that (W-independent) vertex operators of coset model M, commuting with all J_H, have the form of

$$
V = (\chi_1 - \chi_2)^u \exp - \frac{iu}{\sqrt{2}}\left(\frac{\phi_1}{q_1} + \frac{\phi_2}{q_2}\right) \tag{4.7.12}
$$

Thus coset model is easily bosonized. Unpleasent thing, however, is that stress tensor of H is no longer quadratic, even at classical level $J_H^+ J_H^- + J_H^- J_H^+ - 2H_H^2$ contains a term like

$$
(\chi_1 - \chi_2)^2 W_1 W_2 \tag{4.7.13}
$$

Therefore the stress tensor of $M = G/H$ also contains higher powers of fields. This is the reason, why we do not find this construction for coset models quite satisfactory (though it obviously provides a very simple bosonization of <u>any</u> coset model). We shall try to demonstrate a possible outcome with the help of one more example, $M = G/H = sl(2)/U(1)$.

Generators of KM algebra G are now:

$$
\begin{aligned}
J^+ &= W\frac{i}{\sqrt{2}} \\
H &= \left[-\chi W + \frac{iq}{\sqrt{2}}\partial\phi\right]\frac{i}{\sqrt{2}} \\
J^- &= \left[\chi^2 W - iq\sqrt{2}\chi\partial\phi - k\partial\chi\right]\frac{i}{\sqrt{2}} \qquad q^2 = k+2
\end{aligned}
\tag{4.7.14}
$$

According to the general scheme above, subalgebra H is generated by $H = \left[-\chi W + \frac{iq}{\sqrt{2}}\partial\phi\right]\frac{i}{\sqrt{2}}$ and has the same central charge $k = -2 + q^2$. Vertex operators of coset model M, commuting with the current H, have the form of

$$V_n = \chi^n \exp - \frac{in\sqrt{2}}{q}\phi \tag{4.7.15}$$

(again, for the sake of brevity, we present only W-independent vertices). The stress tensor $T_H = \frac{1}{(k+2)} :H^2:$ is no longer quadratic (in variance with $T_G = \frac{1}{(k+2)} :J^+J^- + J^-J^+ - 2H^2:$), it contains a classical term $\chi^2 W^2$, analogous to (4.7.13). Thus T_M also is non-quadratic.

However, one may embed subalgebra H = U(1) into G = = sl(2) in a quite different way. Let generator of H be

$$J_H = \frac{iq}{\sqrt{2}} \partial\phi \tag{4.7.16}$$

Then central charge

$$k_H = q^2 \neq k_G = -2 + q^2, \tag{4.7.17}$$

but instead a real separation of variables takes place: H is described entirely in terms of the field ϕ , while $M = G/H$ - in terms of χ and W . KM algebra (4.7.14) acquires a form of

$$\begin{aligned} J^+ &= I^+ = W\tfrac{i}{\sqrt{2}} \\ J^0 &= I^0 + j^0 = \chi W \tfrac{i}{\sqrt{2}} + j^0 \\ J^- &= I^- - 2\chi j^0 = \tfrac{i}{\sqrt{2}}[\chi^2 W - k\partial\chi] - 2\chi j^0 \end{aligned} \tag{4.7.18}$$

Vertex operators of this $M = sl(2)_k/U(1)_{k+2}$ are made from χ and W only, e.g. $V_n = \chi^n$ and both stress tensors T_H and T_M are quadratic:

$$\begin{aligned} T_G &= \frac{1}{2(k+2)} :J^+J^- + J^-J^+ - 2H^2: = -\left[\tfrac{1}{2}(\partial\phi)^2 + \frac{i}{q\sqrt{2}}\partial^2\phi\right] + W\partial\chi \\ &= \frac{1}{2q} :J_H^2: + T_M \end{aligned} \tag{4.7.19}$$

One may also note, that currents I in (4.7.18) form a closed KM algebra $sl(2)_{-2}$ themselves, if $k = -2$ $(k+2 = q^2 = 0)$.

We consider this kind of construction a somewhat more beatiful generalisation of abelian theory (4.7.1)-(4.7.9).

It may be a bit surprising, but this non-standard construction can be applied to other non-trivial manifolds $M = G/H$. Before we give a one more example of how this works, let us comment briefly on intrinsic meaning of this fact.

The classical part of bosonization [2] of KM algebra G in fact comes from the action of G on homogeneous spaces G/H as an algebra of vector fields. Then χ_α are related to (complex) coordinates on G/H, and $W_\alpha \sim \partial/\partial\chi_\alpha$. The fields ϕ are related to coordinates on H. "Classical" Killing vectors $J(W,\chi)$ do not depend on ϕ, but such decoupling no longer takes place, when W, χ, ϕ are considered z-dependent, and (central extended) KM algebra G arises instead of classical finite-dimensional G. Thus far we considered only flag manifolds with H being a product of $U(1)$ factors. ($SU(n)/U(1)^{n-1}$) in the case of $sl(n)$) and this provided us with bosonization of WZWM. Inclusion of non-abelian subgroups H provides a natural approach to arbitrary coset models. An important ingredient is splitting of KM algebra and, what is even more important, the splitting of Sugawara's stress tensor into mutually commuting quadratic pieces. Above we considered two examples of this kind: $U(1)_{k_1} \times U(1)_{k_2}/U(1)_{k_1+k_2}$ and $sl(2)_k/U(1)_{k+2}$. Let us present a really non-abelian example of $G/H = SU(3)/SU(2)\times U(1)$. We reserve the notation ϕ for scalar field, associated with coordinate on $U(1)$. Of course, in non-abelian situation not all coordinates on H are associated with scalar fields: some β,γ-pairs arise.

In the case of SU(2) which is now under consideration, one β,γ -pair of fields with spin 1 and one scalar field $\phi_{\parallel}$ arise. If we start from the $sl(3)_k$ algebra (4.1.1), this β,γ -pair may be identified with χ_1, W_1, and

$$\vec{\phi} = \frac{1}{\sqrt{(\alpha_1,\alpha_1)}}\vec{\alpha}_1\phi_{\parallel} - \frac{1}{\sqrt{(\lambda_3,\lambda_3)}}\vec{\mu}_3\phi_{\perp} = \frac{1}{\sqrt{2}}\vec{\alpha}_1\phi_{\parallel} - \sqrt{\frac{3}{2}}\vec{\mu}_3\phi_{\perp} \qquad (4.7.20)$$

($\vec{\mu}_3$ is a weight, orthogonal to $\vec{\alpha}_1$, see Fig. 3a). Generators of $sl(2)$ algebra, embedded into $sl(3)$, look like:

$$\begin{aligned} j^+ &= W_1 \\ j^0 &= -\chi_1 W_1 + \frac{i}{\sqrt{2}} q\,\partial\phi_{\parallel} \qquad (4.7.21) \\ j^- &= -[\chi_1^2 W_1 - i\sqrt{2}\, q\chi_1\partial\phi_{\parallel} + (2-q^2)\partial\chi_1] \end{aligned}$$

Original algebra (4.1.1) may be rewritten in terms of $j^{\pm,0}$ instead of $W_1, \chi_1, \phi_{\parallel}$:

$$\begin{aligned} J_{12} &= I_{12} + j^+ = \chi_3 W_2 + j^+ \\ J_{13} &= I_{13} = W_2 \\ J_{23} &= I_{23} = W_3 \\ J_{21} &= I_{21} + j^- = \chi_2 W_3 + j^- \qquad (4.7.22) \\ J_{31} &= I_{31} + \chi_2 j^0 - \chi_3 j^- = [-\chi_2^2 W_2 - \chi_2\chi_3 W_3 - (3-q^2)\partial\chi_2 - \\ &\qquad + i\sqrt{\tfrac{3}{2}}\, q\chi_2\partial\phi_{\perp}] + \chi_2 j^0 - \chi_3 j^- \\ J_{32} &= I_{32} - \chi_2 j^+ - \chi_3 j^0 = [-\chi_2\chi_3 W_2 - \chi_3^2 W_3 - (3-q^2)\partial\chi_3 + \\ &\qquad + i\sqrt{\tfrac{3}{2}}\, q\chi_3\partial\phi_{\perp}] - \chi_2 j^+ - \chi_3 j^0 \\ J_{11} &= I_{11} + j^0 = [\chi_2 W_2 + \frac{i}{\sqrt{6}} q\,\partial\phi_{\perp}] + j^0 \\ J_{22} &= I_{22} - j^0 = [\chi_3 W_3 + \frac{i}{\sqrt{6}} q\,\partial\phi_{\perp}] - j^0 \\ J_{33} &= I_{33} = -\chi_2 W_2 - \chi_3 W_3 - i\sqrt{\tfrac{2}{3}}\, q\,\partial\phi_{\perp} \end{aligned}$$

If central charge of $sl(3)$ algebra is $k = -3 + q^2$, then that of $sl(2)$-subalgebra (4.7.21) is $-2 + q^2 = k+1$. Sugawara's stress tensor

$$T[sl(3)_k] = W_1\partial\chi_1 + W_2\partial\chi_2 + W_3\partial\chi_3 - \tfrac{1}{2}(\partial\vec{\phi})^2 - \frac{i}{q}\vec{\rho}\,\partial^2\vec{\phi} =$$

$$= \left[W_1\partial\chi_1 - \frac{1}{2}(\partial\phi_\parallel)^2 - \frac{i\sqrt{2}}{2q}\partial^2\phi_\parallel\right] + \left[W_2\partial\chi_2 + W_3\partial\chi_3 - \tfrac{1}{2}(\partial\phi_\perp)^2 + \frac{i}{q}\sqrt{\tfrac{3}{2}}\,\partial^2\phi_\perp\right] = \tag{4.7.23}$$

$$= T[sl(2)_{k+1}] + T_{coset}$$

naturally splits into two quadratic pieces, depending on fields in H and G/H respectively. Currents I are natural objects in the $sl(3)_k/sl(2)_{k+1}$ coset model with the stress tensor T_{coset}. Note, that if $k+1 = 0$ (i.e. $q^2=2$) the central charge of $sl(2)_{k+1}$ vanishes, and the algebra of currents I closes by itself - the $sl(2)$-currents decouple - and I form an $sl(3)_{-1}$ KM-algebra, realized in terms of only 5 free fields. This suggestion may be easily verified by explicit calculation of O.P.E. of I's for $q^2=2$.

Generalization of this example is rather straightforward. Let us stress once more, that this coset is somewhat unfamiliar, since in the case of $G/\bigotimes_i H_i$ with simple group G all subgroups possess the same parameter q, thus the corresponding central charges are non-equal and mutually related through

$$K[H_i] + C_V[H_i] = q^2 = K[G] + C_V[G] \tag{4.7.24}$$

and the central charges of the Virasoro algebra for this coset model is

$$C_{G/H}=\frac{k[G]\mathcal{D}[G]}{k[G]+C_v[G]}-\sum_i\frac{k[H_i]\mathcal{D}[H_i]}{k[H_i]+C_v[H_i]}=$$
$$=\frac{k_G\mathcal{D}_G-\sum k_i\mathcal{D}_i}{q^2}=(\mathcal{D}_G-\sum\mathcal{D}_i)-\frac{\mathcal{D}_G C_{v,G}-\sum_i\mathcal{D}_i C_{v,i}}{q^2} \qquad (4.7.25)$$

Of course these approaches to cosets are not the only ones, suggested by bosonization scheme [2]. There is also a close relation with models, possessing higher spin symmetries (W-algebras) [7-9]. In particular, the stress tensor in the construction of ref.[8] for $sl(n)_k \times sl(n)_1/sl(n)_{k+1}$ -model is a fragment of bosonized stress tensor (4.2.10). The crucial restriction in ref.[8] is that one of Kac-Moody algebras in the direct product is of level 1. This is the reason, why bosonization in terms of scalars only appears possible. Making use of the full stress tensor (4.2.10), one should obtain an analogous construction for other G/H, however, β,γ -systems arise in generic situation.

Application of bosonization construction to quantum KM algebras in the spirit of ref.[10] (where only the case of k=1 was discussed) also seems straightforward and deserves investigation.

We shall return to bosonization of coset models and to related questions in another publication.

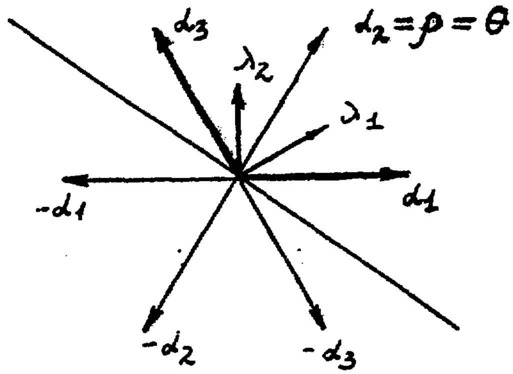
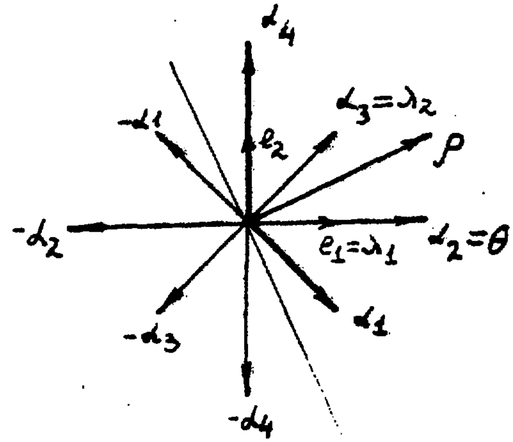


Fig.3:

a)The roots of the algebra sl(3)$\simeq A_2$

$\alpha_1=e_1-e_2$, $\alpha_2=e_1-e_3$, $\alpha_3=e_2-e_3$; $\alpha_1,\alpha_3\in\Pi$; $\rho=\alpha_2=e_1-e_3$;
$\lambda_1=e_1$, $\lambda_2=-e_3$; $ht_{\alpha_1}=ht_{\alpha_3}=1$, $ht_{\alpha_3}=2$; $\mu_j=e_j$.

Correspondence between the fields $\widetilde{w}_j$ and the positive root subspaces: $\widetilde{w}_j\rightarrow \mathfrak{g}_{\alpha_j}$



b) The roots of the algebra sp(2)$\simeq C_2$

$\alpha_1=e_1-e_2$, $\alpha_2=2e_1$, $\alpha_3=e_1+e_2$, $\alpha_4=2e_2$; $\alpha_1,\alpha_4\in\Pi$;
$\rho=2e_1+e_2$; $\lambda_1=e_1$, $\lambda_2=e_1+e_2$; $ht_{\alpha_1}=ht_{\alpha_4}=1$, $ht_{\alpha_3}=2$, $ht_{\alpha_2}=3$
$\widetilde{w}_j\rightarrow \mathfrak{g}_{\alpha_j}$

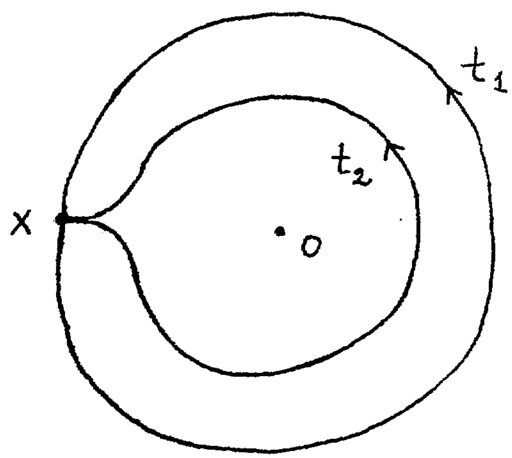


Fig.4: Contours of integration C_z in Felder's construction in the case sl(3).

REFERENCES

1. Gerasimov A. et al. Preprint ITEP 89-64.

2. Wakimoto M. //Comm.Math.Phys. 1986,104,605.
   Feigin B.,Frenkel E.//Usp.Mat.Nauk 1988,43,No.5,227.
   Dotsenko V.,Fateev V.,Zamolodchikov A. Preprint ITP 17-88.
   Morozov A. Preprint ITEP 89-43.

3. Alekseev A.,Faddeev L.,Shatashvili S.//Journ.Diff.Geom. 1989,111.
   Alekseev A.,Shatashvili S. Preprint LOMI E-16-88.

4. Kac V. "Infinite Dimensional Lie Algebras" Camb.Univ.Press London/NY, 1985.

5. Knizhnik V.,Zamolodchikov A.//Nucl.Phys. 1984,B247, 83.

6. Moore G.,Seiberg N. Preprint IASSNS-HEP-89/6.

7. Zamolodchikov A.//Theor.Math.Phys. 1985,65,347.

8. Fateev V.,Lukyanov S.//Int.J.Mod.Phys. 1988,/3A, 507.

9. Bais F.,Bouwknegt P.,Surrige M.,Shoutens K.//Nucl.Phys. 1988,B304,348,371.

10. Frenkel I.,Jing N. Preprint Yale Univ., 1988.

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