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WESS-ZUMINO-WITTEN MODEL AS A THEORY OF FREE FIELD.
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The free field representation of Wess-Zumino-Witten model /1,2/ is generalized to the case of arbitrary Riemann surface. The multiloop calculations for free fields on Riemann surfaces are discussed. The special attention is attracted to the bosonic $\beta\gamma$-system, which appears in the "bosonization" scheme for the Kac-Moody current algebras. We consider the general properties of the multiloop blocks of the WZWM and in particular we explain, how the one-loop characters are reproduced by our methods.

Fig. - 2, ref. - 21

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5. MULTILOOP CALCULATIONS FOR FREE FIELDS ON RIEMANN SURFACES [3,4,5]

In previous sections we discussed the free field representation of WZWM and represented it in terms of several scalar fields, which take values in a circle, and of several $\beta\gamma$-systems of bosonic fields w_α, χ_α with spin $j=1$. We tried to demonstrate, that this kind of representation simplifies considerably calculation of tree (genus 0) correlators in WZWM, just as it happens in analogous situation with minimal models [6]. However, the main advantage of free field representation is that it naturally gives rise to multiloop conforimal blocks (modulo a special projection, see p.c) in the Introduction). Before a brief and preliminary discussion of this subject in Section 6 below, let us remind the main information concerning multiloop calculations for free fields.

5.1 DIFFERENTIAL GEOMETRY OF RIEMANN SURFACES [7]

Here we collect some facts from the theory of Riemann surfaces, which appear usefull in multiloop calculations.

<u>Jacobian map</u>,

$$\xi \to \vec{\xi} = \int_{\xi_0}^{\xi} \vec{\omega} \quad ; \quad \vec{\xi} = \{\xi_1 \ldots \xi_p\} \qquad (5.1.1)$$

may be considered as a map of genus p Riemann surface S_p into p-dimensional torus (Jacobian), which is a factor of C^p over a group of translations $\xi_i \to \xi_i + \delta_{ij}; \xi_i \to \xi_i + T_{ij}$. The concrete choice of point ξ_0 in (5.1.1) is usually unessential.

The image of Riemann surface under the map (5.1.1) is described by Riemann's vanishing theorem in terms of theta-functions. On S_p there are p-1 points $R_1^*,\ldots,R_{p-1}^*$, such, that for arbitrary p-1 points on S_p

$$\Theta_*(\vec{\xi}_1+\ldots+\vec{\xi}_{p-1}-\vec{R}_1^*-\ldots-\vec{R}_{p-1}^*)=0 \qquad (5.1.2)$$

(parameter * is arbitrary non-singular half-integer characteristic).

From this theorem it is easy to derive, that holomorphic 1-differential

$$\mathcal{V}_*^2(\xi)=\sum_{i=1}^{p}\Theta_*(\vec{0})_{,i}\,\omega^i(\xi) \qquad (5.1.3)$$

has double zeroes at points $R_1^*,\ldots,R_{p-1}^*$ and is in fact a square of holomorphic ½-differential $\mathcal{V}_*(\xi)$. Another corollary is that Prime bidifferential,

$$E(\xi,\xi')=\frac{\Theta_*(\vec{\xi}-\vec{\xi}')}{\mathcal{V}_*(\vec{\xi})\mathcal{V}_*(\vec{\xi}')} \qquad (5.1.4)$$

possesses a simple zero when $\xi=\xi'$ and has no poles at all. $E(\xi,\xi')$ is invariant under the shift of ξ along any A-period, and changes under the shift of ξ along B_j-period as:

$$E(\xi+B_j,\xi')=E(\xi,\xi')\exp(2\pi i(\vec{\xi}-\vec{\xi}')_j+\pi i T_{jj}). \qquad (5.1.5)$$

There is another usefull object: a holomorphic p/2-differential without poles and zeroes,

$$\sigma_*(\xi)=\mathcal{V}_*(\xi)\Big/\prod_{\alpha=1}^{p-1}E(\xi,R_\alpha^*). \qquad (5.1.6)$$

For even non-singular theta-characteristic e Szego kernel is defined as

$$G_e^{(1/2)}(\xi,\xi')=\frac{\Theta_e(\vec{\xi}-\vec{\xi}')}{\Theta_e(\vec{0})E(\xi,\xi')}. \qquad (5.1.7)$$

It may be interpreted as Green function of ½-differentials (spinors) on Riemann surface with appropriate boundary conditions:

$$\frac{\langle\tilde{\psi}(\xi)\psi(\xi')\rangle_e}{\det\bar{\partial}_{1/2}} = \langle\langle\tilde{\psi}(\xi)\psi(\xi')\rangle\rangle_e = G_e^{(1/2)}(\xi,\xi'). \qquad (5.1.8)$$

For these Green functions the following analogue of Wick's theorem holds:

$$\langle\langle\tilde{\psi}(\xi_1)\ldots\tilde{\psi}(\xi_n)\psi(\xi_1')\ldots\psi(\xi_n')\rangle\rangle_e = \frac{\Theta_e(\vec{\xi}_1+\ldots+\vec{\xi}_n-\vec{\xi}_1'\ldots-\vec{\xi}_n')\prod_{i<j}E(\xi_i,\xi_j)\prod_{i<j}E(\xi_i',\xi_j')}{\Theta_e(\vec{0})\prod_{i,j}E(\xi_i,\xi_j')} = \qquad (5.1.9)$$

$$= \det_{(i,j)}\frac{\Theta_e(\vec{\xi}_i-\vec{\xi}_j')}{\Theta_e(\vec{0})E(\xi_i,\xi_j')} = \det_{(i,j)}\langle\langle\tilde{\psi}(\xi_i)\psi(\xi_j')\rangle\rangle_e .$$

We shall need also Green functions of Laplace operator Δ_0. Usually Green function $\langle\xi|\frac{1}{\Delta_0}|\xi'\rangle$ on a surface with metric $\gamma(\xi)$ is defined as a solution of the following equation:

$$\Delta_0^{\{\gamma\}}(\xi)\log G^{\{\gamma\}}(\xi,\xi') = 2\pi i\left(\delta^{\{\gamma\}}(\xi,\xi') - \frac{1}{\int\sqrt{\gamma(\xi)}d^2\xi}\right) \qquad (5.1.10)$$

$\delta^{\{\gamma\}}$-function here is normalized as follows:

$$\int\delta^{\{\gamma\}}(\xi,\xi')\sqrt{\gamma(\xi)}d^2\xi = 1 \qquad \text{i.e.} \quad \delta^{\{\gamma\}}(\xi,\xi') = \frac{\delta(\xi,\xi')}{\sqrt{\gamma(\xi)}} . \qquad (5.1.11)$$

The second term on the r.h.s. of (5.1.10) is due to zero modes: Green function $\log G(\xi,\xi') = \sum_{\lambda_n\neq 0}\phi_n(\xi)\overline{\phi_n(\xi')}/\lambda_n$ with normalised eigenfunctions $\phi_n(\xi)$, $\Delta_0\phi_n = \lambda_n\phi_n$ satisfies the equation

$$\Delta_0\log G = \sum_{\lambda_n\neq 0}\frac{\Delta\phi_n(\xi)}{\lambda_n}\overline{\phi_n(\xi')} = \sum_{\lambda_n\neq 0}\phi_n(\xi)\overline{\phi_n(\xi')} = \sum\phi_n(\xi)\overline{\phi_n(\xi')} -$$
$$- \sum_{\lambda_n=0}\phi_n(\xi)\overline{\phi_n(\xi')} = \left(2\pi i\delta(\xi,\xi') - \sum_{\lambda_n=0}\phi_n(\xi)\overline{\phi_n(\xi')}\right). \qquad (5.1.12)$$

It is hard to write down explicit formula for $G^{\{\gamma\}}$ for arbitrary metric γ. In string theory, however, we need slightly different Green functions, wich are solutions of two other equations: (5.1.13) and (5.1.18) below:

$$\Delta_0^{\{\gamma\}}\log G^{(0)}(\xi|\{A_I,\xi_I\}) = \sum_{I=1}^{N}A_I\delta^{\{\gamma\}}(\xi,\xi_I)2\pi i \qquad (5.1.13)$$

or, in conformal gauge,

$$\partial\bar\partial \log G^{(0)}(\xi|\{A_I,\xi_I\}) = \sum_{I=1}^{N} A_I \delta(\xi,\xi') 2\pi i \qquad (5.1.14)$$

with additional constraint

$$\sum_{I=1}^{N} A_I = 0 . \qquad (5.1.15)$$

Explicit solution of eq.(5.1.14) is:

$$G^{(0)}(\xi|\{A_I,\xi_I\}) = f\{A_I,\xi_I\}\prod_{I=1}^{N}|E(\xi,\xi_I)|^{2A_I}\exp\{\pi A_I \operatorname{Im}(\vec\xi-\vec\xi_I)_i \frac{1}{(\operatorname{Im}\Omega)_{ij}} \operatorname{Im}(\vec\xi-\vec\xi_I)_j\} . \qquad (5.1.16)$$

Eq.(5.1.16) defines single-valued function on S_p. The multiplyer $f\{A_I,\xi_I\}$ is equal to:

$$f\{A_I,\xi_I\} = \prod_{I=1}^{N}|\hat\sigma_*(\xi_I)|^{2A_I/p}\exp\{\frac{\pi A_I}{p(p-1)}\operatorname{Im}(\vec{\tilde\xi}_I)_i \frac{1}{(\operatorname{Im}\Omega)_{ij}}\operatorname{Im}(\vec{\tilde\xi}_I)_j\} = \prod_{I=1}^{N}\sqrt{g_{A_I}(\xi_I)}^{\,A_I/2} \qquad (5.1.17)$$

it accounts for the proper dependence of $G(\xi|\{A_I,\xi_I\})$ on ξ_I, in applications it is unessential. (Vector $\vec{\tilde\xi}_I$ and metric $g_{A_I}(\xi)$ entering eq.(5.1.17) are defined below, in eqs. (5.1.28) and (5.1.29)).

The second type of relevant Green functions is defined by the equation

$$\Delta_0^{\{\gamma\}} \log G^{\{R\}}(\xi,\xi') = 2\pi i\,(\delta^{\{\gamma\}}(\xi,\xi') + \alpha R\sqrt{\gamma}(\xi)) \qquad (5.1.18)$$

with $\alpha\int R\sqrt{\gamma} = -1$ i.e.

$$\alpha = -\frac{1}{4\pi i(p-1)} . \qquad (5.1.19)$$

Solution of (5.1.18) looks like

$$G^{\{R\}}(\xi,\xi') = F\{R^*\}\,|E(\xi,\xi')\sigma_*^{\frac{1}{p-1}}(\xi)\sigma_*^{\frac{1}{p-1}}(\xi')|^2 \exp\left(\frac{\pi}{(p-1)^2}\operatorname{Im}\vec{\tilde\xi}_i \frac{1}{(\operatorname{Im}\Omega)_{ij}}\operatorname{Im}\vec{\tilde\xi}'_j\right)\times \qquad (5.1.20)$$
$$\times\left(\sqrt{\gamma(\xi)\gamma(\xi')}\right)^{\frac{1}{2(p-1)}}$$

with

$$\vec{\tilde\xi} = (p-1)\vec\xi - \sum_{a=1}^{p-1}\vec R_a^* \qquad \vec{\tilde\xi}' = (p-1)\vec\xi' - \sum_{a=1}^{p-1}\vec R_a^* \qquad (5.1.21)$$

and is single-valued function of ξ and ξ'. The factor

$$F\{R_a^*\} = \prod_{a=1}^{p-1}\left(\sqrt{g_{A_I}(R_a^*)}\right)^{\frac{1}{p-1}}$$

turns $G^{\{R\}}$ into 0-differential in all R_a^*'s. Let us note, that eq.(5.1.20) provides a principal way to find out a Green function of the type (5.1.11). One should find a new metric $\tilde{\gamma}$ connected with original γ through

$$R_{\tilde{\gamma}} = \partial\bar{\partial}\log\tilde{\gamma} = (p-1)\gamma/\int\sqrt{\gamma}\,d^2\xi . \tag{5.1.22}$$

Then substituting this $\tilde{\gamma}$ into eq.(5.1.20) one gets $G^{\{\gamma\}}$.

Let us define Green function at coincident points as

$$\log G^{\{\gamma\}}(\xi,\xi') = \lim_{\epsilon\to 0}\left[\int_{\epsilon}^{+\infty} dt\langle\xi|e^{-t\Delta_0}|\xi\rangle - \log\frac{|\xi-\xi'|^2\sqrt{\gamma(\xi)}}{\epsilon}\right]. \tag{5.1.23}$$

Counterterm is chosen to maintain two-dimensional covariance. In fact we have

$$G^{\{\gamma\}}(\xi,\xi) = \frac{1}{\sqrt{\gamma(\xi)}}\lim_{\xi'\to\xi}\frac{G^{\{\gamma\}}(\xi,\xi')}{|\xi-\xi'|^2} . \tag{5.1.24}$$

The following metrics on Riemann surface are of special interest:

Bergmann metric:

$$g_{Berg}(\xi) = \frac{1}{2ip}\sum_{k,\ell}^{p}\omega_k(\xi)\frac{1}{(\operatorname{Im}T)_{k\ell}}\overline{\omega_\ell(\xi)} . \tag{5.1.25}$$

It is normalized so,that

$$\int\sqrt{g_{Berg}(\xi)}\,d^2\xi = 1 . \tag{5.1.26}$$

Arakelov metric,related to the Bergmann one,according to (5.1.22),

$$\tilde{g}_{Berg}(\xi) = g_{Ar}(\xi) \quad \text{i.e.} \quad R_{Ar}(\xi) = g_{Berg}(\xi) \tag{5.1.27}$$

$$g_{Ar}(\xi) = |\sigma_*(\xi)|^{4/p}\exp\left(\frac{\pi}{p(p-1)}\operatorname{Im}\vec{\tilde{\xi}}_i\frac{1}{(\operatorname{Im}T)_{ij}}\operatorname{Im}\vec{\tilde{\xi}}_j\right) \tag{5.1.28}$$

$$\vec{\tilde{\xi}} = (p-1)\vec{\xi} - \vec{\Delta}_* \quad ; \quad \vec{\Delta}_* = \sum_a \vec{R}_a \tag{5.1.29}$$

<u>Singular metrics:</u>

$$g_W(\xi) = |W(\xi)|^2 \qquad (5.1.30)$$

which are squares of moduli of holomorphic or meromorphic 1-differentials $W(\xi)$. These metrics have zeroes and poles at some points Q_a, P_a respectively. Curvature is concentrated in these points,

$$R_W = \partial\bar\partial \log g_W(\xi) = 2\pi i \left(\sum_a^{n_Q} \delta(\xi, Q_a) - \sum_b^{n_P} \delta(\xi, P_b)\right) \qquad (5.1.31)$$

There are constraints on Q_a and P_a:

$$n_Q - n_P = 2(p-1) \quad , \sum_a^{n_Q} \vec{Q}_a - \sum_b^{n_P} \vec{P}_b = 2\Delta_* \quad \forall * \qquad (5.1.32)$$

As a consequense we have:

$$\int R_W = 4\pi i (p-1) \qquad (5.1.33)$$

5.2. SCALAR FIELD ON RIEMANN SURFACE

5.2.1. Let us consider the functional integral

$$A\{k_I\} = \int \mathcal{D}\phi \exp\left[\frac{1}{4\pi i}\int \sqrt{g}\, g^{ab}\partial_a\phi\partial_b\phi + \sum_{I=1}^{N} k_I \phi(\xi_I)\right] \qquad (5.2.1)$$

where $\phi(\xi)$ is a scalar field on the surface S_p. Integration over zero mode ϕ =const gives rise to condition $\sum_{I=1}^{N} k_I = 0$. Thus, we may use Green functions (5.1.13). One equaly verifies, that this Gauss functional integral is equal to

$$A\{k_I\} = \left(\frac{\det N_c}{\det' \Delta_c}\right)^{1/2} \prod_{I<J}^{N} G(\xi_I,\xi_J)^{k_I k_J} \prod_{I=1}^{N}\left(\sqrt{g(\xi_I)}\right)^{-k_I^2/2} \delta\left(\sum_{I=1}^{N} k_I\right) \qquad (5.2.2)$$

Here

$$G(\xi,\xi') = |E(\xi,\xi')|^2 \exp\left[\pi\, \mathrm{Im}(\vec{\xi}-\vec{\xi}')_i \frac{1}{(\mathrm{Im}\,T)_{ij}} \mathrm{Im}(\vec{\xi}-\vec{\xi}')_j\right] \qquad (5.2.3)$$

5.2.2. Consider now a slightly more complicated functional integral,

$$A_\lambda\{k_I\} = \int \mathcal{D}\phi \exp\left[\frac{1}{4\pi i}\int \sqrt{g}\left(g^{ab}\partial_a\phi\partial_b\phi + 2\lambda R\phi\right) + \sum_{I=1}^{N} k_I\phi(\xi_I)\right]. \qquad (5.2.4)$$

Integration over zero mode leads to the following condition:

$$\sum_{I=1}^{N} k_I + \frac{\lambda}{2\pi i}\int\sqrt{g}R = \sum_{I=1}^{N} k_I + 2\lambda(p-1) = 0. \qquad (5.2.5)$$

It is usefull to shift variable $\phi \to \phi + \phi_0$, with ϕ_0 being solution of the equation

$$\partial\bar{\partial}\phi_0 = \lambda\partial\bar{\partial}\ln g(\xi,\bar{\xi}) + \pi i\sum_{I=1}^{N} k_I\,\delta(\xi,\xi_I). \qquad (5.2.6)$$

Let us introduce auxiliary singular metric $g_*(\xi,\bar{\xi}) = |\nu_*(\xi)|^4$ with double zeroes at points R_a^* . We may rewrite (5.2.6) in the following way:

$$\partial\bar{\partial}\phi_0(\xi,\bar{\xi}) = \lambda\partial\bar{\partial}\log\left[g/g_*\right] + \pi i\left(\sum_{I=1}^{N} k_I\,\delta(\xi,\xi_I) + 2\lambda\sum_{a}^{p-1}\delta(\xi,R_a^*)\right) \qquad (5.2.7)$$

Solution of this equation looks like

$$\phi_0(\xi,\bar{\xi}) = \lambda\ln\left[g/g_*\right] + \sum_{I=1}^{N} k_I \ln\left\{\frac{G(\xi,\xi_I)}{\prod_a G(\xi,R_a^*)^{\frac{1}{p-1}}}\cdot F(R^*,\xi_I)\right\} \qquad (5.2.8)$$

Insertion of $F(R_a, \xi_I)$ makes the whole expression scalar at points R_a, ξ_I and is unessential in what follows.

Now it is easy to calculate functional integral (5.2.4):

$$A_\lambda\{k_I\} = \left(\frac{\det N_0}{\det' \Delta_0}\right)^{1/2} \exp\left\{\frac{1}{4\pi i}\,\phi_0 \Delta_0 \phi_0\right\} \delta\left(\sum_{I=1}^{N} k_I + 2\lambda(p-1)\right). \quad (5.2.9)$$

Let us begin with evaluation of $(\phi_0 \frac{1}{\Delta_0} \phi_0)$:

$$\frac{1}{4\pi i}(\phi_0 \Delta_0 \phi_0) = 1/2\Big(\lambda \ln \frac{g}{g_*} + \sum_{I=1}^{N} k_I \ln\Big\{\frac{G(\xi,\xi_I)}{\prod_a G(\xi,R_a)^{\frac{1}{p-1}}}\cdot F(R_a^*,\xi_I)\Big\}\Big)\Big(\frac{\lambda}{2\pi i}\partial\bar\partial \ln g/g_* +$$

$$+ \sum_{a=1}^{p-1} 2\lambda\,\delta(\xi, R_a^*) + \sum_{I=1}^{N} k_I\,\delta(\xi,\xi_I)\Big) = -\frac{\lambda^2}{4\pi i} S_L[g/g_*] + \sum_a 2\lambda^2 \ln \frac{g}{g_*}(R_a^*) +$$

$$+ \sum_{I=1}^{N} \lambda k_I \ln g/g_*(\xi_I) + \sum_{I,J} \frac{k_I k_J}{2} \ln\Big\{\frac{G(\xi_I,\xi_J)}{\prod_a G(\xi_I,R_a^*)^{\frac{1}{p-1}}}\cdot F\Big\} + \sum_{I,a} \lambda k_I \ln\Big\{\frac{G(\xi_I,R_a^*)F}{\prod_b G(R_a^*,R_b^*)^{\frac{1}{p-1}}}\Big\} \quad (5.2.10)$$

Exponentiating this expression, one gets:

$$\exp\left\{\frac{1}{4\pi i}\phi_0\Delta_0\phi_0\right\} = \exp\left\{-\frac{\lambda^2}{4\pi i} S_L[g/g_*]\right\}\cdot \prod_a \left(\frac{g(R_a^*)}{g_*(R_a^*)}\right)^{2\lambda^2} \cdot \prod_I \left(\frac{g(\xi_I)}{g_*(\xi_I)}\right)^{\lambda k_I} \times$$

$$\times \prod_{I<J} G(\xi_I,\xi_J)^{k_I k_J} \cdot \prod_I G(\xi_I,\xi_I)^{k_I^2/2} \cdot \prod_{a,I} G(\xi_I,R_a^*)^{2k_I\lambda} \cdot \prod_{a<b} G(R_a^*,R_b^*)^{4\lambda^2} \prod_a G(R_a^*,R_a^*)^{2\lambda^2}. \quad (5.2.11)$$

Where $S_L[g/g_*] = \int |\partial \ln[g/g_*]|^2 d^2\xi$ Liouville action.

Taking into account regularization rule (5.I.24), we obtain the final answer:

$$A_\lambda\{k_I\} = \exp\left\{-\frac{\lambda^2}{4\pi i} S_L'[g/g_*]\right\}\left(\frac{\det N_0}{\det' \Delta_0}\right)^{1/2} \cdot \prod_{I<J} G(\xi_I,\xi_J)^{k_I k_J} \times$$

$$\times \prod_I \left(\frac{g_*(\xi_I)}{\prod_a G(\xi_I,R_a^*)^2}\right)^{-\lambda k_I} \cdot \prod_I g(\xi_I)^{-(k_I^2 - 2\lambda k_I)/2} \left[\frac{\prod_{a<b} G(R_a^*,R_b^*)}{|\nu_*'(R_a^*)|^2}\right]^{4\lambda^2}. \quad (5.2.12)$$

5.2.3. Consider now the scalar field ϕ which takes values in a circle of radius r: $\phi \sim \phi + 2\pi r$ On a non-simply-connected surface this field is not necessarily single-valued. Indeed, we have

$$\phi(\xi,\bar\xi) = \phi_S(\xi,\bar\xi) + i\pi r\left[(\vec m + \vec n \bar T)^T \frac{1}{(\operatorname{Im} T)}\vec\xi - (\vec m + \vec n T)\frac{1}{(\operatorname{Im} T)}\vec{\bar\xi}\right] \quad (5.2.13)$$

where $\phi_s(\xi,\bar\xi)$ is single-valued on S_p. The values of k_I are no longer arbitrary, instead

$$K_I 2\pi\tau = 2\pi i k_I \qquad k_I \in \mathbb{Z}. \tag{5.2.14}$$

Functional integral now is an infinite sum, with each item related to a definite homotopic class of mapping of S_p into a circle, Mapping classes are labelled by two p-vectors m_i, n_i, and

$$A(\tau) = \int \mathcal{D}\phi \exp\left[\frac{1}{4\pi i}\int\sqrt{g}\, g^{ab}\partial_a\phi\partial_b\phi + \frac{i}{\tau}\sum k_I\phi(\xi_I)\right] =$$

$$= \int\mathcal{D}\phi_s \exp\left[\frac{1}{4\pi i}\int\sqrt{g}\, g^{ab}\partial_a\phi_s\partial_b\phi_s + \frac{i}{\tau}\sum k_I\phi_s(\xi_I)\right] \tag{5.2.15}$$

$$\times\left[\sum_{\{m_i,n_i\}\in\mathbb{Z}^{2p}} \exp\left\{-\frac{\pi}{2}\tau^2(\vec m+\vec n T)^T\frac{1}{(\operatorname{Im}T)}(\vec m+\vec n T) - \pi\left[(\vec m+\vec n T)^T\frac{1}{\operatorname{Im}T}\vec Z - (\vec m+\vec n T)^T\frac{1}{\operatorname{Im}T}\vec{\bar Z}\right]\right\}\right]$$

where

$$\vec Z = \sum_{I=1}^{N} k_I\vec\xi_I. \tag{5.2.16}$$

We obtain the result of integration over ϕ_s ,making use of (5.2.2) and the condition $\sum k_I = 0$:

$$\left(\frac{\det N_0}{\det'\Delta_0}\right)^{1/2}\prod_{I<J}|E(\xi_I,\xi_J)|^{-k_Ik_J/\tau^2}\exp\left\{\frac{2\pi}{\tau^2}(\operatorname{Im}\vec Z)^T\frac{1}{\operatorname{Im}T}(\operatorname{Im}\vec Z)\right\}\delta\left(\sum k_I\right)\prod_I\left(\sqrt{g}(\xi_I)\right)^{k_I^2/2\tau^2}. \tag{5.2.17}$$

The sum in eq.(5.2.15) is usually reffered to as instantonic contribution [4] , because non-trivial solutions of equations of motion $\partial\bar\partial\phi = 0$ are known as instantons. Instanton contribution $I[z,\bar z]$ is calculated in Appendix to this subsection. According to eq.(A.3) (A.11) from this Appendix,we have:

$$I[z,\bar z] = \tau^{-3p/2}(\det\operatorname{Im}T)^{1/2}\exp\left(-\frac{2\pi}{\tau^2}(\operatorname{Im}\vec Z)^T\frac{1}{\operatorname{Im}T}(\operatorname{Im}\vec Z)\right)\tilde I(z,\bar z). \tag{5.2.18}$$

Taking into account,that det Im T = det $N_1^{(can)}$, we obtain:

$$A[\tau] = \left[\frac{\det'\Delta_0}{\det N_0\det N_1^{(can)}}\right]^{-1/2}\prod_{I<J}|E(\xi_I,\xi_J)|^{-k_Ik_J/\tau^2}\tilde I[z,\bar z]\prod_{I=1}^{N}\left(\sqrt{g(\xi_I)}\right)^{k_I^2/2\tau^2}\delta\left(\sum k_I\right) \tag{5.2.19}$$

In Appendix it is demonstrated, that whenever $\beta^2 = \gamma^2/2$ is rational number, the sum $\tilde{I}[z,\bar{z}]$ is finite bilinear combination of theta-functions.

The most important result of consideration of circle-valued scalars instead of ordinary scalar fields is the absence of non-holomorphic contributions like $\exp(\mathrm{Im}\,z)^T \frac{1}{\mathrm{Im}\,\tau}(\mathrm{Im}\,z)$ in final answers.

All this consideration is straightforwadly generalized to the case of a multiplet of scalar fields, taking values in a torus (see, for example, [8]). The main new thing is the occurrence of lattice theta-function, associated with the torus C^n/Γ (Γ being a translation group),

$$\Theta_\Gamma(\vec{z}|\tau) = \sum_{\vec{\Lambda}_i \in \Gamma} \exp\left[i\pi(\vec{\Lambda}_i^T \tau_{ij} \vec{\Lambda}_j) + 2\pi i(\vec{\Lambda}_i^T \vec{z}_i)\right] \tag{5.2.20}$$

APPENDIX

Let us consider the instantonic sum, depending on two real p-vectors μ_i, ν_i, one complex p-vector z_i and two parameters β and γ:

$$I_{\mu,\nu}[z,\bar{z}] = \sum_{\substack{m_i \in \mathbb{Z}+\mu_i \\ n_i \in \mathbb{Z}+\nu_i}} \exp\left\{-\pi\beta^2(\vec{m}+\vec{n}\bar{\tau})^T\frac{1}{\mathrm{Im}\,\tau}(\vec{m}+\vec{n}\tau) - 2\pi\gamma\left[(\vec{m}+\vec{n}\bar{\tau})\frac{1}{\mathrm{Im}\,\tau}\vec{z} - (\vec{m}+\vec{n}\tau)\frac{1}{\mathrm{Im}\,\tau}\vec{\bar{z}}\right]\right. \tag{A.1}$$

and express it in terms of theta-functions when β^2 is rational. Let as apply Poisson transformation w.r. to m_i,

$$\sum_{m_i \in \mathbb{Z}+\mu_i} f\{m_i\} = \sum_{\{m_i\} \in \mathbb{Z}^p} f\{m_i+\mu_i\} = \sum_{\{M_i\} \in} e^{2\pi i M_j \mu_j} \int d^p f\, e^{-2\pi i M_j f_j} f\{f_j\} \tag{A.2}$$

in order to obtain:

$$I_{\mu,\nu}[z,\bar{z}] = \beta^{-p}(\det \mathrm{Im}\,\tau)^{-1/2} \exp\left[-\frac{4\pi\gamma^2}{\beta^2}(\mathrm{Im}\,z)^T\frac{1}{\mathrm{Im}\,\tau}(\mathrm{Im}\,z)\right]\tilde{I}_{\mu,\nu}[z,\bar{z}]$$

$$\tilde{I}_{\mu,\nu}[z,\bar{z}] = \sum_{\substack{M_i \in \mathbb{Z} \\ n_i \in \mathbb{Z}+\mu_i}} \exp(2\pi i \vec{M}^T\vec{\mu}) \cdot \exp\frac{i\pi}{2}\left\{\left(\frac{\vec{M}}{\beta}+\beta\vec{n}\right)^T\tau\left(\frac{\vec{M}}{\beta}+\beta\vec{n}\right) - \right. \tag{A.3}$$

$$\left. - \left(\frac{\vec{M}}{\beta}-\beta\vec{n}\right)^T\bar{\tau}\left(\frac{\vec{M}}{\beta}-\beta\vec{n}\right)\right\} \cdot \exp\frac{2\pi i\gamma}{\beta}\left[\left(\frac{\vec{M}}{\beta}+\vec{n}_\beta\right)\vec{z} - \left(\frac{\vec{M}}{\beta}-\vec{n}_\beta\right)\vec{\bar{z}}\right]$$

If β^2 is rational,

$$\beta^2 = P/Q \tag{A.4}$$

further simplifications arise:

$$\tilde{I}_{\mu,\nu} = \sum_{\substack{M_i \in \mathbb{Z} \\ N_i \in \mathbb{Z}+\nu_i}} \exp(2\pi i \vec{M}^T \vec{\mu}) \exp\left[\frac{i\pi PQ}{2}\left(\frac{\vec{M}}{P}+\frac{\vec{N}}{Q}\right)^T T \left(\frac{\vec{M}}{P}+\frac{\vec{N}}{Q}\right) + \right.$$
$$\left. + 2\pi i \gamma Q\left(\frac{\vec{M}}{P}+\frac{\vec{N}}{Q}\right)^T \vec{z}\right] \exp\left[-\frac{i\pi PQ}{2}\left(\frac{\vec{M}}{P}-\frac{\vec{N}}{Q}\right)^T T\left(\frac{\vec{M}}{P}-\frac{\vec{N}}{Q}\right) - 2\pi i \gamma Q\left(\frac{\vec{M}}{P}-\frac{\vec{N}}{Q}\right)^T \bar{\vec{z}}\right]. \tag{A.5}$$

Let us use the following substitution:

$$\frac{\vec{M}}{P} = \frac{\vec{a}-\vec{b}}{2} + \vec{\epsilon}_P \qquad \frac{\vec{N}}{Q} = \frac{\vec{a}+\vec{b}}{2} + \vec{\epsilon}_Q + \frac{\vec{\nu}}{Q} \tag{A.6}$$

where a_i and b_i are simultaneously even or odd, and components of p-vectors ϵ_P and ϵ_Q take values $0, 1/P, \ldots, (P-1)/P$ and $0, 1/Q, \ldots, (Q-1)/Q$ respectively:

$$\epsilon_P \in \mathbb{Z}_P^p$$
$$\epsilon_Q \in \mathbb{Z}_Q^p \qquad \left(\mathbb{Z}_n = \frac{1}{n}\mathbb{Z} (\mathrm{mod}\, n)\right). \tag{A.7}$$

Restrictions on a_i and b_i may be encoded by δ-function:

$$\sum_{\text{even } m_i} \delta(a - l - m) = \frac{1}{2^p} \sum_{\zeta \in \mathbb{Z}_2^p} \exp\left(2\pi i (\vec{a}-\vec{b})^T \cdot \vec{\zeta}\right). \tag{A.8}$$

The sum (A.5) now turns into

$$\tilde{I}_{\mu,\nu} = 2^{-p} \sum_{\substack{\epsilon_P \in \mathbb{Z}_P^p \\ \epsilon_Q \in \mathbb{Z}_Q^p \\ \zeta \in \mathbb{Z}_2^p}} \sum_{\substack{a \in \mathbb{Z}^p \\ b \in \mathbb{Z}^p}} \exp\left(i\pi \frac{PQ}{2} \hat{a} T \hat{a} + 2\pi i \gamma Q \hat{a} z\right) \times \tag{A.9}$$
$$\times \exp\left(-\frac{i\pi PQ}{2} \hat{b} T \hat{b} - 2\pi i \gamma Q \hat{b} \bar{z}\right) \exp 2\pi i (\hat{a}-\hat{b})(\zeta + \frac{\mu P}{2}) \exp(-4\pi i \zeta \epsilon_P)$$

where $\quad \hat{a} \equiv a + \epsilon_P + \epsilon_Q + \nu/Q$

$\qquad\quad \hat{b} \equiv b - \epsilon_P + \epsilon_Q + \nu/Q.$

Making use of the definition of theta-function,

$$\Theta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right](z|T) = \sum_{a \in \mathbb{Z}^p} \exp\left(i\pi(a+\alpha)^T T (a+\alpha) + 2\pi i (a+\alpha)^T (z+\beta)\right) \tag{A.10}$$

we get the final expression:

$$\tilde{I}_{\mu,\nu} = \sum_{\epsilon_P, \epsilon_Q, \zeta} e^{-4\pi i \epsilon_P \zeta}\, \Theta\left[\begin{matrix}\epsilon_P + \epsilon_Q + \nu/Q \\ \zeta + \mu P/2\end{matrix}\right]\left(\gamma Q z \middle| \frac{PQ}{2} T\right) \Theta\left[\begin{matrix}\epsilon_Q - \epsilon_P + \nu/Q \\ \zeta + \mu P/2\end{matrix}\right]\left(-\gamma Q \bar{z} \middle| \frac{PQ}{2} \bar{T}\right) \tag{A.11}$$

5.3. CIRCLE-VALUED SCALAR FIELD WITH MODIFIED LAGRANGIAN

Let us consider now the functional integral

$$A_\lambda\{K_I\} = \int \mathcal{D}\phi \exp\left[\frac{1}{4\pi i}\int\sqrt{g}\,g^{ab}(\partial_a\phi)(\partial_b\phi) + \sum K_I\phi(\xi_I) + \frac{\lambda}{2\pi i}\int\sqrt{g}\,R\phi(\xi)\right] \qquad (5.3.1)$$

where scalar field ϕ takes values in a circle. With given value of coefficient λ in (5,3.1) possible values of radius of the circle and momenta k_I are restricted by the single-valuedness condition for $\exp S(\phi)$:

$$2\lambda r = in \quad ; \quad K_I = \frac{ik_I}{r} \qquad n, k_I \in \mathbb{Z}. \qquad (5.3.2)$$

Using these restrictions, we may write:

$$A_\lambda\{K_I\} = \int \mathcal{D}\phi \exp\left[\frac{1}{4\pi i}\int\sqrt{g}\,g^{ab}(\partial_a\phi)(\partial_b\phi) + \frac{i}{r}\left\{\sum_{I=1}^{N} k_I\phi(\xi_I) + \frac{n}{4\pi i}\int\sqrt{g}\,R\phi\right\}\right]. \qquad (5.3.3)$$

Divide the field ϕ into homotopically trivial ϕ_S and non-trivial $\phi_{m,n}$ parts, as we did in ss.5.2.3. Then:

$$A_\lambda\{K_I\} = A_\lambda^q\{K_I\} \cdot A_\lambda^{inst}\{K_I\} \qquad (5.3.4)$$

$$A_\lambda^q\{K_I\} = \int \mathcal{D}\phi_s \exp\left[\frac{1}{4\pi i}\int\sqrt{g}\,g^{ab}(\partial_a\phi_s)(\partial_b\phi_s) + \frac{i}{r}\left\{\sum_{I=1}^{N} k_I\phi_s(\xi_I) + \frac{n}{4\pi i}\int\sqrt{g}\,R\phi_s\right\}\right] \qquad (5.3.5)$$

$$A_\lambda^{inst}\{K_I\} = \sum_{k_i,m_i}\exp\left[\frac{1}{4\pi i}\int\sqrt{g}\,g^{ab}(\partial_a\phi_{k,m})(\partial_b\phi_{k,m}) + \frac{i}{r}\left\{\sum k_I\phi_{k,m}(\xi_I) + \frac{n}{4\pi i}\int\sqrt{g}\,R\phi_{k,m}\right\}\right] \qquad (5.3.6)$$

A_λ^q has been already calculated in (5.2.12). Now we shall discuss the instantonic contribution. To begin with let us note, that (5.3.1) is not generically a proper formula. The field ϕ itself and not only its derivative enters (5.3.1). However, the field ϕ is not single-valued and does not take definite value at any given point. To make the field ϕ single-valued we cut the surface S_p (Fig. 1) and define single-valued ϕ on this simply-connected surface S_p^c. Now A^{inst} is well defined, but it depends on the cuts. For example, in the case of p=2 (Fig. 2) small deformation of the cut change instantonic contribution as follows:

$$\delta S[\phi_{\kappa,m}] = \frac{k}{4\pi^2}\int_{V_2}\sqrt{g}R\phi_{\kappa,m} - \frac{k}{4\pi^2}\int_{V_1}\sqrt{g}R\phi_{\kappa,m} = \frac{k}{4\pi^2}\int_{V_1}\sqrt{g}R\delta\phi_{\kappa,m}\,. \qquad (5.3.7)$$

It deserves noting, that there is no difficulties of this kind with terms $\frac{i}{2}\sum k_I\phi(\xi_I)$ because of conditions (5.3.2). To make (5.3.6) correct we should add some boundary term. It is easy to verify, that proper expression is:

$$\tilde{S}[\phi_{\kappa,m}] = \frac{1}{4\pi i}\int_{S_p^c}(\partial\phi_{\kappa,m})^2 + \frac{i}{2}\Big[\sum k_I\phi_{\kappa,m}(\xi_I) + \frac{k}{4\pi i}\int_{S_p^c}(\partial\bar\partial\log(g))\phi_{\kappa,m} - \oint_{\partial S_p^c}\Omega\phi_{\kappa,m}\Big] \qquad (5.3.8)$$

where Ω is defined from

$$d\Omega = \frac{k}{4\pi i}\partial\bar\partial\log(g) - \sum_a l_a\,\delta(\xi - P_a) \qquad (5.3.9)$$

and $D = \sum l_a(P_a)$ is any divisor of appropriate degree. It is easy to show, that (5.3.8) does not depend on divisor D and metric g. Given a section $\omega(z)$ of linear bundle, associated with D, we may write down explicit expression for Ω :

$$\Omega = \frac{1}{2\pi i} * d\log\frac{g(z)^{m/2}}{|\omega(z)|^2}\,. \qquad (5.3.10)$$

If we change D for another divisor D' and section $\omega(z)$ for $\omega'(z)$, the difference is

$$\delta\tilde{S} = \frac{1}{2\pi^2}\oint_{\partial S_p^c} * d\log\left|\frac{\omega(z)}{\omega'(z)}\right|^2\phi(\xi) = \frac{1}{2\pi^2}\Big(\sum_i\oint_{a_i} * d\log\left|\frac{\omega(z)}{\omega'(z)}\right|^2\Delta_{b_i}\phi\ -$$
$$-\sum_i\oint_{b_i} * d\log\left|\frac{\omega(z)}{\omega'(z)}\right|^2\Delta_{a_i}\phi\Big) = 0\ (\mathrm{mod}\ 2\pi i) \qquad (5.3.11)$$

where $\Delta_{a_i}\phi$ and $\Delta_{b_i}\phi$ are jumps of the field $\phi_{\kappa,m}$ on the cuts. Let us choose one special divisor $K^{m/2}$ with section $(\nu_4(\xi))^m$. Let us show, that (5.3.6) really does not depend on the choice of metric g. Changing the metric $g \to g'$, we have:

$$\tilde{S}_{g'}-\tilde{S}_g=\frac{m}{4\pi\tau}\Big(\int_{S_p^c}(\partial\bar\partial\log[\tfrac{g}{g'}]\cdot\phi)\Big)-\frac{m}{4\pi\tau}\oint_{\partial S_p^c}*d\log[\tfrac{g}{g'}]=$$

$$=\frac{m}{4\pi\tau}\int_{\partial S_p^c}*d\log[\tfrac{g}{g'}]-\frac{m}{4\pi\tau}\int_{\partial S_p^c}*d\log[\tfrac{g}{g'}]=0\,. \qquad (5.3.12)$$

Let us choose metric to be $g_*=|\nu_*(\xi)|^4$ for the sake of convenience. Then we obtain:

$$A^{inst}(k_I)=\sum_{\kappa,m}\exp\Big[\frac{1}{4\pi i}\int(\partial\phi_{\kappa,m})^2+\frac{i}{\tau}\Big(\sum_{I=1}^{N}k_I\phi_{\kappa,m}(\xi_I)+h\sum_{a}^{p-1}\phi_{\kappa,m}(P_a^*)\Big)\Big] \qquad (5.3.13)$$

This sum has been already calculated in (5.2.)

The answer is bilinear combination of theta-functions (if $\tau^2/2 = P/Q$, $P,Q\in Z$). In order to obtain a single theta-function with given characteristic $\left[\frac{\vec\delta}{\vec\varepsilon}\right]$, one should consider a more general boundary condition of the type

$$\phi(\xi+A_\kappa)=\phi(\xi)+2\pi\tau\Big(m_\kappa+\frac{\alpha_\kappa}{PQ}\Big)$$
$$\phi(\xi+B_\kappa)=\phi(\xi)+2\pi\tau\Big(n_\kappa+\frac{\beta_\kappa}{PQ}\Big). \qquad (5.3\ 14)$$

Some linear combinations of $A(\alpha_\kappa,\beta_\kappa)$, which arise in this case instead of (5.3.13), with different $(\alpha_\kappa,\beta_\kappa)$ are equal to a square of module of a single theta-function. However, in this case $\exp\frac{ik_I}{\tau}\phi(\xi_I)$ is not well defined. Also our discussion above,concerning the term $\int\sqrt{g}R\phi$ appear incorrect. The proper prescription for (5.3.8) in this situation is:

$$\tilde{S}=\frac{1}{4\pi i}\int(\partial\phi)^2+\frac{i}{\tau}\int\Omega\wedge d\phi \qquad (5.3.15)$$

where $d\Omega=\sum_{I=1}^{N}k_I\delta(z-z_I)+\frac{1}{2\pi i}\frac{h}{2}\partial\bar\partial\log(g)$.

This expression is obviously invariant with respect to the shifts (5.3.14) and does not depend on any cut.

5.4. b-c-SYSTEMS WITH ARBITRARY HALF-INTEGER SPINS :

In this section we reproduce the formulae for conformal blocks of Grassmanian b-c - systems with spins $(j,1-j)$ for arbitrary $j \in \frac{1}{2}\mathbb{Z}$. We shall use the following strategy. First, we obtain correlation functions in the simplest case of $j=\frac{1}{2}$, using local bosonization. Then, by a change of variables in functional integral we shall treat the case of arbitrary j.

5.4.1. Let us consider a special case of b-c-system: fermions $\tilde{\psi}(\xi), \psi(\xi)$ with spins ½ and the following O.P,E.:

$$\tilde{\psi}(\xi)\psi(\xi') = \frac{1}{\xi-\xi'} + \text{r.t.} \qquad (5.4.1)$$

Stress tensor has the form of

$$T_\psi = \frac{1}{2}\left[\tilde{\psi}(\xi)\partial\psi(\xi) - \partial\tilde{\psi}(\xi)\psi(\xi)\right]. \qquad (5.4.2)$$

On the sphere this theory may be easily bosonized in terms of one scalar field, which takes values in a circle of unit radius:

$$\begin{aligned} &\psi = \exp(-i\phi) \\ &\tilde{\psi} = \exp(+i\phi) \qquad T_\psi = T_\phi = -\tfrac{1}{2}(\partial\phi)^2. \end{aligned} \qquad (5.4.3)$$

Indeed, let us compare correlation functions in the theory of fermionic spinors and in its bosonized version:

$$\left\langle \prod_{i=1}^{n}\psi(\xi_i)\prod_{j=1}^{n}\tilde{\psi}(\xi'_j)\right\rangle = \det_{(i,j)}\left(\frac{1}{\xi_i-\xi'_j}\right) \qquad (5.4.4)$$

$$\left\langle \prod_{i=1}^{n} e^{-i\phi(\xi_i)} \prod_{j=1}^{n} e^{+i\phi(\xi'_j)}\right\rangle = \frac{\prod_{i<j}(\xi_i-\xi_j)\prod_{i<j}(\xi'_i-\xi'_j)}{\prod(\xi_i-\xi'_j)}. \qquad (5.4.5)$$

It is easy to realize, that (5.4.4) and (5.4.5) possess the same zeroes and poles and thus coincide. Note also, that central charges of these fermionic and bosonic theories are the same:

$$2(6j^2-6j+1)_{j=\frac{1}{2}} = (-\tfrac{1}{2})\,2(6j^2-6j+1)_{j=0} \qquad (5.4.6)$$

($-\frac{1}{2}$ is due to the fact, that ϕ is real boson).

Let us consider now a couple of fermions on arbitrary Riemann surface. To define the theory on arbitrary surface, we have to choose phases, which fermions acquire when they move along non-contractable cycles. This freedom is fixed by the choice of some "characteristic", i.e. of two p-vectors $\vec{\epsilon}, \vec{\delta}$. When fermion is shifted along A_k (B_k) cycle, it becomes multiplied by $\exp i\pi(\epsilon_k+1)$; $\exp i\pi(\delta_k+1)$.

Before we discuss, how to bosonize fermionic correlators one comment is in order. When functional integral in bosonic theory is calculated, one should integrate over momenta p of intermediate states $F(\partial x)e^{ipx}$. However, from (5.4.3) we see, that only integer momenta are allowed, if one wants to make correspondence to fermionic theory. This is exactly the reason, why we should consider ϕ as a field, which takes values in a circle of unit radius, $\phi \sim \phi + 2\pi$.

We have already discussed in ss. 5.2, 5.3 how the correlators of circle-valued scalar fields are calculated. Thus we have:

$$A = \left\langle \prod_{i=1}^{n} e^{i\phi(\xi_i)} \prod_{j=1}^{n} e^{-i\phi(\xi'_j)} \right\rangle = \left| \frac{\prod_{i<j} E(\xi_i,\xi_j) \prod_{i<j} E(\xi'_i,\xi'_j)}{\prod_{i,j} E(\xi_i,\xi'_j)} \right|^2 \frac{\sum\limits_{\substack{\delta,\epsilon \\ (\delta,\epsilon)=\text{even}}} \left|\Theta\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right]\left(\sum \vec{\xi}_i - \sum \vec{\xi}'_j\right)\right|^2}{|\det \bar\partial_0|} \qquad (5.4.7)$$

In terms of fermions this formula may be interpreted as follows:

$$A = \sum_{\substack{\delta,\epsilon \\ (\delta,\epsilon)=\text{even}}} \left| \left\langle \prod_{i=1}^{n} \tilde\psi(\xi_i) \prod_{j=1}^{n} \psi(\xi'_j) \right\rangle_e \right|^2 \qquad (5.4.8)$$

Thus correlators in fermionic theory are:

$$\left\langle \prod_{i=1}^{n} \tilde\psi(\xi_i) \prod_{j=1}^{n} \psi(\xi'_j) \right\rangle_e = \frac{\prod_{i<j} E(\xi_i,\xi_j) \prod_{i<j} E(\xi'_i,\xi'_j)}{\prod_{i,j} E(\xi_i,\xi'_j)} \, \frac{\Theta_e\left(\sum \vec{\xi}_i - \sum \vec{\xi}'_j\right)}{(\det \bar\partial_0)^{1/2}} \qquad (5.4.9)$$

It is easy to verify, that (5.4.9) has proper transformation properties under the shifts of z_i or y_j along A or B-cycles.

Fay's identity,

$$\frac{\prod_{i<j} E(z_i,z_j)\prod_{i<j} E(z_i',z_j')}{\prod_{i,j} E(z_i,z_j')}\frac{\Theta_e(\sum\vec{z}_i-\sum\vec{z}_j')}{\Theta_e(\vec{0})}=\det_{(i,j)}\frac{\Theta_e(\vec{z}_i-\vec{z}_j')}{\Theta_e(\vec{0})E(z_i,z_j')} \quad (5.4.10)$$

acquires a natural interpretation as Wick's theorem:

$$\left\langle\prod_{i=1}^{n}\tilde{\psi}(\xi_i)\prod_{j=1}^{n}\psi(\xi_j')\right\rangle=\det_{(i,j)}\frac{\Theta_e(\vec{\xi}_i-\vec{\xi}_j')}{\Theta_e(\vec{0})E(\xi_i,\xi_j')}\cdot\frac{\Theta_e(\vec{0})}{(\det\bar{\partial}_e)^{1/2}} \quad (5.4.11)$$

where $$G_e^{(1/2)}(\xi_i,\xi_j')=\frac{\Theta_e(\vec{\xi}_i-\vec{\xi}_j')}{\Theta_e(\vec{0})E(\xi_i,\xi_j')} \quad (5.4.12)$$

and $$\det\bar{\partial}_{1/2}=\frac{\Theta_e(\vec{0})}{(\det\bar{\partial}_e)^{1/2}} \quad (5.4.13)$$

are fermionic propagator and determinant.

5.4.2. Let us discuss now the case of arbitrary $j\in\frac{1}{2}Z$. The simplest way to work out the answer makes use of the change of variables [9]:

$$b(\xi)=\Omega_{j-1/2}(\xi)\tilde{\psi}(\xi) \qquad c(\xi)=\Omega^{-1}_{j-1/2}(\xi)\psi(\xi) \quad (5.4.14)$$

where $\Omega_{j-1/2}$ is holomorphic $(j-\frac{1}{2})$-differential with zeroes, located at points $Q_1,\dots,Q_{n_j}$, $n_j=(2j-1)(p-1)$. It is obvious that O.P.E. for b and c has correct form:

$$b(\xi)c(\xi')=\frac{1}{\xi-\xi'}+\text{r.t.} \quad (5.4.15)$$

Ordinary norms of b and c correspond to the following norms for $\tilde{\psi}$ and ψ:

$$\|\delta b\|^2=\|\delta\tilde{\psi}\|^2|\Omega_{j-1/2}|^2 \; ; \quad \|\delta c\|^2=\|\delta\psi\|^2|\Omega_{j-1/2}|^{-2}. \quad (5.4.16)$$

Thus integration over regular b and c fields is equivalent to integration over $\tilde{\psi}$ possessing poles at $Q_1, \ldots, Q_{n_j}$ and ψ possessing zeroes at the same points. Therefore we have the following relation between measures:

$$\mathcal{D}b\,\mathcal{D}c = \mathcal{D}\tilde{\psi}\,\mathcal{D}\psi \prod_{i=1}^{n_j} \frac{\psi(Q_i)}{(\Omega'_{j-1/2}(Q_i))^{\frac{1}{2j+1}}} \tag{5.4.17}$$

where $\Omega_{j-1/2}(\xi) = \Omega'_{j-1/2}(Q_i)(\xi - Q_i) + O((\xi - Q_i)^2)$.

The action is:

$$S = \int d^2\xi\, (\tilde{\psi}\Omega_{j-1/2})\bar{\partial}(\psi\Omega^{-1}_{j-1/2}) = \int d^2\xi\, (\tilde{\psi}\bar{\partial}\psi) \tag{5.4.18}$$

and we obtain the following equality:

$$\left\langle \prod_{i=1}^{m} b(x_i) \prod_{\mu=1}^{n} c(y_\mu) \right\rangle = \left\langle \prod_{a=1}^{n_j} \frac{\psi(Q_a)}{\Omega'_{j-1/2}(Q_a)^{\frac{1}{2j+1}}} \prod_{i=1}^{m} \Omega_{j-1/2}(x_i)\tilde{\psi}(x_i) \prod_{\mu=1}^{n} \Omega^{-1}_{j-1/2}(y_\mu)\psi(y_\mu) \right\rangle. \tag{5.4.19}$$

The charge conservation in fermionic theory leads to the following restriction: $m = n + n_j$, or

$$m - n = (2j-1)(p-1). \tag{5.4.20}$$

The norms (5.4.16) are not exactly standard norms on the bundles of j and $1-j$ differentials, which have the form of

$$\int \|b\|^2 d^2z = \int |\underbrace{b_{z\ldots z}}_{j}|^2 (g^{z\bar{z}})^j \sqrt{g}\, d^2z \;;\; \int \|c\|^2 d^2z = \int |c^{\overbrace{z\ldots z}^{j-1}}|^2 (g_{z\bar{z}})^{j-1} \sqrt{g}\, d^2z \tag{5.4.21}$$

To make (5.4.16) and (5.4.21) the same, let us choose the metric g and $(j-½)$-differential $\Omega_{j-½}$ as follows:

$$g\,d^2z = |\nu_*(z)|^4 \;;\; \Omega_{j-1/2}(z) = \nu_*^{2j-1}(z). \tag{5.4.22}$$

Thus we obtain the following answer for correlators of b,c-systems in metric $g=|\nu_*(z)|^4$

$$\left\langle \prod_{a=1}^{n+n_j} b(z_a) \prod_{b=1}^{n} c(y_b) \right\rangle_e = \frac{\prod_{a<a'}^{n+n_j} E(z_a,z_{a'}) \prod_{b<b'}^{n} E(y_b,y_{b'})}{\prod_{a,b} E(z_a,y_b)} \left(\frac{\prod_a \sigma_*(z_a)}{\prod_b \sigma_*(y_b)} \right)^{(2j-1)} \times \tag{5.4.23}$$

$$\times \left[\frac{\prod_{\alpha<\beta} E(R_\alpha^*,R_\beta^*)}{\prod_\alpha \nu_*'(R_\alpha^*)} \right]^{(2j-1)^2} \Theta_e\left(\sum_{a=1}^{n+n_j} \vec{z}_a - \sum_{b=1}^{n} \vec{y}_b - (2j-1)\Delta_*\right) / (\det \bar\partial_c)^{1/2} .$$

Now we shall discuss, how a transformation from one characteristic to another may be performed (to * as a special case). We use the same trick - a change of variables:

$$\tilde{b}(z) = b(z)\, f_{e,e'}(z) \qquad \tilde{c}(z) = c(z)\, f_{e,e'}^{-1}(z) \tag{5.4.24}$$

where $f_{e,e'}$ is defined by the condition, that $\tilde{b}(z)$ has characteristic $e'(e)$. Explicit formula is:

$$f_{e,e'}(z) = \frac{\prod_{i=1}^{N} E(z,Q_i)}{\prod E(z,P_j)} , \quad \sum \vec{P}_j - \sum \vec{Q}_i = \frac{(\vec{\varepsilon}' - \vec{\varepsilon})}{2} + \frac{(\vec{\delta}' - \vec{\delta})^T}{2} T . \tag{5.4.25}$$

Changing variables in accordance with (5.4.24), we obtain the following relation:

$$\left\langle \prod_{a=1}^{n+n_j} b(z_a) \prod_{b=1}^{n} c(y_b) \prod_{a=1}^{n+n_j} f(z_a) \prod_{b} (f(y_b))^{-1} \cdot \prod_{i=1}^{N} \frac{b(P_i)\nu_*(P_i)^{1-2j}}{\left[[f^{-1}(P_i)]' \right]^{1/2}} \cdot \prod_{k=1}^{N} \frac{c(Q_k)\nu_*(Q_k)^{2j-1}}{\left[f(Q_k)' \right]^{1/2}} \right\rangle_e = \tag{5.4.26}$$

$$= \frac{\prod_{a<a'} E(z_a,z_{a'}) \prod_{b<b'} E(y_b,y_{b'})}{\prod_{a,b} E(z_a,y_b)} \cdot \left(\frac{\sigma_*(z_a)}{\sigma_*(y_b)} \right)^{2j-1} \frac{\Theta_{e'}\left(\sum \vec{z}_a - \sum \vec{y}_b - (2j-1)\Delta_*\right)}{(\det \bar\partial_c)^{1/2}} \prod_{k=1}^{p} [f(R_k)]^{-1} \exp\left(-i\pi \frac{(\delta'-\delta)T(\delta'-\delta)}{4}\right)$$

(the formula

$$\Theta\left[{}^{\delta}_{\varepsilon}\right]\left(\vec{z} + \frac{\vec{\varepsilon}' - \vec{\varepsilon}}{2} + \frac{(\vec{\delta}' - \vec{\delta})}{2} T\right) = \exp\left(-\frac{i\pi}{4} (\vec{\delta}' - \vec{\delta}) T (\vec{\delta}' - \vec{\delta})\right) \Theta\left[{}^{\delta'}_{\varepsilon'}\right](\vec{z}) \tag{5.4.27}$$

was used). Two last factors in (5.4.26) are compensated by Quillen's anomaly [10], associated with the transformation (5.4.24). Thus we come to the following answer:

$$\left\langle \prod_{a=1}^{n+u} \hat{b}(z_a) \prod_{b=1}^{n} c(y_b) \right\rangle_{*} = \frac{\prod E(z_a, z_{a'}) \prod_{b<b'} E(y_b, y_{b'})}{\prod_{a,b} E(z_a, y_b)} \left(\frac{\prod \sigma_*(z_a)}{\prod_b \sigma_*(y_b)} \right)^{(2j-1)} \frac{\Theta_*(\sum \vec{z}_a - \sum \vec{y}_b - (2j-1)\Delta)}{(\det \bar{\partial}_0)^{1/2}} \quad (5.4.28)$$

Let us comment also on the case of $j=1$. When $h=0$ and $j=1$ theta-function in (5.4.28) vanishes. This just indicates, that when $j=1$, there is an additional zero mode of the field $b(z)$ and that of $c(z)$. In this case the least possible N is $n_1+1 = p$, and instead of (5.4.28) one may use:

$$\left\langle \prod_{a=1}^{p+u} b(z_a) \prod_{b}^{n+1} c(y_b) \right\rangle_{*} = \frac{\prod E(z_a, z_{a'}) \prod E(y_b, y_{b'})}{\prod E(z_a, y_b)} \frac{\prod \sigma_*(z_a)}{\prod \sigma_*(y_b)} \frac{\Theta_*(\sum \vec{z}_a - \sum \vec{y}_b - \Delta)}{(\det \bar{\partial}_0)^{1/2}} \quad (5.4.29)$$

5.4.3. Consider now eqs. (5.4.14), (5.4.23) from the point of view of bosonization (5.4.3):

$$\begin{aligned} \hat{b} &= \nu_*^{\,2j-1} e^{i\phi(j)} \\ c &= \nu_*^{\,1-2j} e^{-i\phi(j)} . \end{aligned} \quad (5.4.30)$$

We may obtain these formulae directly from (5.4.3) by the following shift of the field $\phi(1/2)$ in (5.4.3):

$$\phi_{(j)} = \phi_{(1/2)} - i(2j-1) \log |\nu_*(\xi)|^2 . \quad (5.4.31)$$

After this shift in the functional integral over ϕ we get:

$$\left| \left\langle \prod_{i=1}^{m} b(z_i) \prod_{\mu=1}^{n} c(y_\mu) \right\rangle \right|^2 = \int \mathcal{D}\phi \, e^{\tilde{S}[\phi]} \prod_{i=1}^{m} e^{i\phi(z_i)} |\nu_*(z_i)|^{2(2j-1)} \cdot \prod_{\mu=1}^{n} e^{-i\phi(y_\mu)} |\nu_*(y_\mu)|^{-2(2j-1)} \quad (5.4.32)$$

where shifted action

$$\tilde{S}[\phi] = \frac{1}{4\pi i} \int d^2\xi \left[|\partial\phi|^2 - i(2j-1)\tfrac{1}{2} R(|\nu_*|^4)\phi + \frac{(2j-1)^2}{4} R(|\nu_*|^4) \frac{1}{\Box} R(|\nu_*|^4) \right) . \quad (5.4.33)$$

Keeping in mind, that integration is over fields, which take values in a circle, we obtain the following answer:

$$\left| \left\langle \prod_{i=1}^{m} b(z_i) \prod_{\mu=1}^{n} c(y_\mu) \right\rangle \right|^2 = \int \mathcal{D}\phi \, e^{S[\phi]} \prod_{a=1}^{p-1} \frac{e^{-i(2j-1)\phi(R_a^*)}}{|\nu_*'(R_a^*)^{(2j-1)/2}|^2} \times \quad (5.4.34)$$

$$\times \prod_{i=1}^{m} e^{i\phi(z_i)} |\nu_*(z_i)|^{2(2j-1)} \prod_{\mu=1}^{n} \frac{e^{-i\phi(y_\mu)}}{|\nu_*(y_\mu)|^{2(2j-1)}} \cdot \delta(m-n-(2j-1)(p-1)) .$$

$$= \sum_{e\text{-even}} \left| \frac{\prod_{i<i'} E(z_i,z_{i'}) \prod_{\mu<\mu'} E(y_\mu,y_{\mu'})}{\prod_{i\mu} E(z_i,y_\mu)} \frac{\prod \sigma_*(z_i)^{(2j-1)}}{\prod \sigma_*(y_\mu)^{(2j-1)}} \left(\frac{\prod_{a<b} E(R_a^*,R_b^*)}{\prod_a \nu_*'(R_a)} \right)^{(2j-1)^2} \times \right.$$

$$\times \left. \frac{\Theta_e\left(\sum \vec{z}_i - \sum \vec{y}_\mu - (2j-1)\Delta_*\right)}{(\det \bar{\partial}_0)^{1/2}} \right|^2 \cdot \exp \frac{2(6j^2-6j+1)}{48\pi i} S_L[|\nu_*|^4]$$

where $S_L[|\nu_*|^4]$ stands for Liouville action, and the coefficient $2(6j^2-6j+1)$ (the central charge for j-differentials) is composed of two pieces:

$$\frac{(2j-1)^2}{16\pi i} - \frac{1}{2}\,\frac{2}{48\pi i} = \frac{2(6j^2-6j+1)}{48\pi i} = \frac{2C_j}{48\pi i}\,. \qquad (5.4.35)$$

The second term on the l.h.s. comes from the general formula

$$\det \Delta_j = |\det \bar{\partial}_j|^2 \exp \frac{2C_j}{48\pi i} S_L[g] \qquad (5.4.36)$$

in the case of j=0.

Taking (5.4.36) into account one sees,that (5.4.34) is in agreement with (5.4.23).

In conclusion it is usefull to stress,that bosonization prescription,discussed in ss.5.4.3 works well with <u>any</u> metric g on Riemann surface (not obligatory singular).

5.5. β-γ-SYSTEMS WITH ARBITRARY SPINS $j \in \frac{1}{2}\mathbb{Z}$

β-γ-systems are the analogues of b-c-systems, but with opposite statistics. They are bosonic fields. Up to now β-γ-systems arised as superghosts in the theory of NSR superstring [11] (in that case $j=3/2$). We believe, however, that free β-γ-systems are important in the study of general conformal theories, and above we demonstrated that they really arise in bosonization of WZWM (in this case $j=1$). The theory of these objects in the case of arbitrary spin is discussed in [12, 13, 14]; in what follows we present a brief extraction of these results.

5.5.1. To begin with let us discuss the general properties of β-γ-systems and their conformal blocks. Because the only difference as compared to b,c-systems is opposite statistics, determinants of β-γ-systems are inverse of those for b,c-systems. To be more precise, the following quantity is unity:

$$\int D\beta\, D\gamma\, Db\, Dc\; e^{\int \beta\bar\partial\gamma + b\bar\partial c}\; \delta(b(z_1))\,\delta(\beta(z_1))\ldots\delta(b(z_{n_j}))\,\delta(\beta(z_{n_j}))\quad (n_j=(2j-1)(g-1))$$

Additional insertions arise because of zero-modes of the fields b and β - which are holomorphic j-differentials. Making use of the simple observation, that

$$\delta(b(z_i)) = \frac{1}{i}\int d\epsilon\, e^{i\epsilon b(z_i)} = b(z_i)$$

we obtain the following answer for determinant of β-γ-system:

$$\int D\beta\, D\gamma\, e^{\int\beta\bar\partial\gamma}\,\delta(\beta(z_1))\ldots\delta(\beta(z_n)) = \frac{1}{\det_{(i,j)} b_i(z_j)}\cdot\frac{1}{(\det\bar\partial_j)} \qquad (5.5.1)$$

where $\{b_i(z)\}$ stands for a basis of holomorphic j-differentials, and $\det \bar{\partial}_j$ - for determinant of b,c-system. From (5.5.1) we see, that central charge of β,γ-system is opposite to that of b,c-system.

Note, that occurrence of zero modes of bosonic fields makes functional integral infinite in contrast with fermionic case, where it became vanishing. Generically, when all z_j in (5.5.1) are different points on Riemann surface, determinant of zero-modes, arising in denominator, is non-vanishing. But if it vanishes, the functional integral diverges. This may be in fact interpreted as appearance of appropriate meromorphic (1-j)-differential, which is a zero-mode of $\gamma(z)$. Sometimes these poles are reffered to as "unphysical" (since they are not implied by local O.P.E., which accounts only for singularities at coincident points). It should be easy to express all functional integrals and correlators of β,γ -fields in terms of b,c-ones, but unfortunately we have nothing in b,c-system, what can be interpreted as β,γ fields themselves. In what follows we present a direct computation of correlators in β,γ -system in the simplest case of j=½. Then by changing variables (as we have already done in the case of b,c-systems), we derive the answers for arbitrary j.

5.5.2. Let us compute correlators in the case of j=½. We shall use the notation $\beta=\tilde{\psi}$, $\gamma=\psi$ in this case. The basic fields of the theory are:

$$\psi;\ \tilde{\psi};\ \delta(\psi)=\int\frac{dp}{2\pi}e^{ip\psi};\ \delta(\tilde{\psi})=\int\frac{d\tilde{p}}{2\pi}e^{i\tilde{p}\tilde{\psi}}.$$

One easily verifies the following O.P.E.:

$$\hat{\Psi}(z)\cdot\delta(\hat{\Psi}(w))\sim(z-w)\partial H(\hat{\Psi}(w))+\dots \tag{5.5.2}$$

$$\delta(\hat{\Psi}(z))\cdot\delta(\Psi(w))\sim(z-w)\cdot 1+\dots \tag{5.5.3}$$

$$H(\hat{\Psi}(z))\cdot\Psi(w)\sim\frac{1}{(z-w)}\delta(\hat{\Psi}(z))+\dots \tag{5.5.4}$$

where additional field, built with the help of Heavyside step function, is introduced:

$$H(\hat{\Psi}(z))=\frac{1}{2\pi}\int\frac{dp}{p+i0}e^{ip\hat{\Psi}(z)}$$

(in the case of superghosts fields of this kind enter the picture changing operator). Combining (5.5.3) and (5.5.2) one obtains: $\hat{\Psi}(z)=\partial_z H(\hat{\Psi}(z))\,\delta(\Psi(z))$.

Thus to find all correlators we need only to know those of $H(\hat{\Psi}(z)),\ \delta(\Psi(z)),\ \Psi(z)$. Let us calculate the correlator

$$\left\langle\prod_{i=1}^{n}\Psi(y_i)\prod_{j=1}^{n}\delta(\Psi(w_j))\prod_{k=1}^{n+1}H(\hat{\Psi}(x_k))\right\rangle_e. \tag{5.5.5}$$

It is easily expressed in terms of Green function for fields $\hat{\Psi},\Psi$, which is absolutely the same as that in the case of fermions, $G_e^{(\lambda_2)}(\xi,\xi')=\frac{\Theta_e(\vec{\xi}-\vec{\xi}')}{\Theta_e(\vec{0})E(\xi,\xi')}$. One should only use integral representation of $\delta(\Psi)$ and $H(\hat{\Psi})$:

$$\left\langle\prod_{i=1}^{n}\frac{\partial}{\partial\eta_i}\Big[\quad e^{i\Psi(y_i)\eta_i}\Big]_{\eta_i=0}\prod_{j=1}^{n}\int\frac{dp_j}{2\pi}e^{i\Psi(w_j)p_j}\prod_{k=0}^{n}\int\frac{dq_k}{q_k+i0}e^{i\hat{\Psi}(x_k)q_k}\right\rangle=$$

$$=\int\frac{dq_k}{q_k+i0}\prod_{i=1}^{n}\Big(\sum q_k G_e^{(\lambda_2)}(y_i,x_k)\Big)\cdot\prod_{j=1}^{n}\delta\Big(\sum q_k G_e^{(\lambda_2)}(w_j;x_k)\Big)\frac{1}{\det\bar{\partial}_{\lambda_2}}. \tag{5.5.6}$$

Let us integrate out all q_k besides q_0. The answer is:

$$\frac{\prod_{i=1}^{n}\Big(G_e^{(\lambda_2)}(y_i;x_0)-\sum_{\substack{k=1\\j=1}}G_e^{(\lambda_2)}(y_i;x_k)[G_e^{(\lambda_2)}(x_k,w_j)]^{-1}G_e^{(\lambda_2)}(w_j;x_0)\Big)}{\prod_{k=1}^{n}\Big(\sum_{j=1}^{n}[G_e^{(\lambda_2)}(x_k,w_j)]^{-1}G_e^{(\lambda_2)}(w_j,x_0)\Big)\Big(\det_{(j,k)}G_e^{(\lambda_2)}(w_j,x_k)\Big)\det\bar{\partial}_{\lambda_2}} \tag{5.5.7}$$

Now the following equation (Cramer rule) may be used:

$$\sum_j \left[G_e^{(\nu_2)}(x_k,w_j)\right]^{-1} G_e^{(\nu_2)}(w_j,x_0)=\det\begin{Vmatrix} G_e^{(\nu_2)}(x_1,w_1)\dots G_e^{(\nu_2)}(x_n,w_1) \\ G_e^{(\nu_2)}(x_1,w_n)\dots G_e^{(\nu_2)}(x_n,w_n)\end{Vmatrix}_{(x_k\to x_0)}\cdot \det\|G_e^{(\nu_2)}(x_i,w_j)\|^{-1} \qquad (5.5.8)$$

Together with the familiar Fay's identity (5.1.9),

$$\det\|G_e^{(\nu_2)}(z_i,w_j)\|=G(z_1\dots z_n|w_1\dots w_n)=\frac{\prod E(z_i,z_{i'})\prod E(w_j,w_{j'})}{\prod E(z_i,w_j)}\,\frac{\Theta_e(\sum\vec z_i-\sum\vec w_j)}{\Theta_e(0)} \qquad (5.5.9)$$

this leads to the following result:

$$\left\langle \prod_{i=1}^n \psi(y_i)\prod_{j=1}^n \delta(\psi(w_j))\cdot\prod_{k=0}^n H(\bar\psi(x_k))\right\rangle=\frac{\prod_{i=1}^n G(x_0\dots x_n|y_i,w_1\dots w_n)}{\prod_{k=0}^n G(x_0\dots \hat{x}_k\dots x_n|w_1\dots w_n)}\,\frac{1}{\det\bar\partial_{\nu_2}} \qquad (5.5.10)$$

It is usefull to express this result in a slightly different form, making use of the relation between determinants,

$$(\det\bar\partial_0)^{1/2}(\det\bar\partial_{\nu_2})_e=\Theta_e(\vec 0) \qquad (5.5.11)$$

The final answer is:

$$\left\langle \prod_{i=1}^n \psi(y_i)\prod_{j=1}^n \delta(\psi(w_j))\cdot\prod_{k=0}^n H(\bar\psi(x_k))\right\rangle=\frac{\prod_{i=1}^n\left\langle b^{(\nu_2)}(x_0)\dots b^{(\nu_2)}(x_n)c^{(\nu_2)}(y_i)c^{(\nu_2)}(w_1)\dots c^{(\nu_2)}(w_n)\right\rangle}{\prod_{k=0}^n\left\langle b^{(\nu_2)}(x_0)\dots \hat{b}^{(\nu_2)}(x_k)\dots b^{(\nu_2)}(x_n)\,c^{(\nu_2)}(w_1)\dots c^{(\nu_2)}(w_n)\right\rangle} \qquad (5.5.12)$$

It is natural to introduce new fields:

$$\xi(z)=H(\bar\psi(z))\;;\quad e^{\varphi(z)}=\delta(\bar\psi(z))\;;\quad \bar\psi(z)=\partial\xi\, e^{-\varphi(z)}$$

$$\eta(z)=\partial_z\psi(z)\delta(\psi(z))\;;\quad e^{-\varphi(z)}=\delta(\psi(z))\;;\quad \psi(z)=\eta\, e^{+\varphi(z)} \qquad (5.5.13)$$

One readilly verifies, that new fields (η,ξ) and Φ from the point of view of O.P.E. are identical to grassmanian b,c-system with spin j=1 and to a free scalar field respectively. Central charge of β,γ -system may be considered as a sum of central charges of (b,c) system and of scalar Φ, provided, that its Lagrangian looks like :

$$\mathcal{L}_{\beta,\gamma} = \frac{1}{2\pi i}\int \beta\bar{\partial}\gamma = \frac{1}{2\pi i}\int\left(\eta\bar{\partial}\xi + \frac{1}{2}|\partial\phi|^2\right). \qquad (5.5.14)$$

Relation between β,γ -systems and those of fields η,ξ,ϕ is known under the name of "bosonization" of bosonic β,γ--systems. In terms of these new fields eq.(5.5.12) has the following form:

$$\left\langle \prod_{i=0}^{n}\xi(x_i)\prod_{j=1}^{n}\eta(y_j)\prod_k \exp[q_k\varphi_k(z_k)]\right\rangle =$$

$$= \frac{\prod_{j}^{n}\Theta_e(\vec{y}_j + \sum\vec{x}_i - \sum\vec{y}_e + \sum q_k\vec{z}_k)}{\prod_{i=0}^{n}\Theta_e(-\vec{x}_i + \sum\vec{x}_e - \sum\vec{y}_j + \sum q_k\vec{z}_k)}\cdot\frac{\prod_{i<i'}E(x_i,x_{i'})\prod_{j<j'}E(y_j,y_{j'})}{\prod E(x_i,y_j)\prod E(z_k,z_{k'})^{q_kq_{k'}}}(\det\bar{\partial}_0)^{1/2} \qquad (5.5.15)$$

5.5.3. Now we proceed to formulae for arbitrary spin j. As in the case of b,c-systems we shall use the change of variables in functional integration:

$$\beta(z) = \tilde{\psi}(z)\,\Omega_{(j-1/2)}(z),\quad \gamma(z) = \psi(z)/\Omega_{(j-1/2)}(z) \qquad (5.5.16)$$

where holomorphic (j-½)-differential $\Omega_{j-1/2}$ posesses zeroes at points $Q_1\dots Q_{n_j}$, $n_j= (2j-1)(p-1)$. The integration measure looks as follows:

$$\mathcal{D}\beta\mathcal{D}\gamma = \mathcal{D}\tilde{\psi}\mathcal{D}\psi\prod_i\delta(\psi(Q_i))\left[\Omega'_{(j-1/2)}(Q_i)\right]^{\frac{1}{2j+1}}. \qquad (5.5.17)$$

Thus we obtain the following expression for correlator for β,γ -system with arbitrary j (we take $\Omega_{j-1/2}=\nu_*^{2j-1}(z)$ as in the case of b,c-system):

$$\left\langle\prod_{i=1}^{n}\gamma(y_i)\prod_{j=1}^{n-(2j-1)(p-1)}\delta(\gamma(w_j))\prod_{k=0}^{n}H(\beta(x_k))\right\rangle = \left\langle\prod_{i=1}^{n}\eta(y_i)e^{\varphi(y_i)}\Big/\nu_*(y_i)^{(2j-1)}\times\right.$$

$$\left.\times\prod_{j=1}^{n-(2j-1)(p-1)}\left(e^{-\varphi(w_j)}\nu_*(w_j)^{(2j-1)}\right)\cdot\prod_{k=0}^{n}\xi(x_k)\prod_a\exp\left(-(2j-1)\varphi(R_a^*)\right)\cdot\left[\nu'_*(R_a^*)\right]^{(2j-1)^2}\right\rangle = \qquad (5.5.18)$$

$$= \frac{\prod_{j=1}^{n} \Theta_e\left(\sum x_\ell - y_j - \sum w_j - (2j-1)\Delta_*\right)}{\prod_{i=0}^{n} \Theta_e\left(\sum x_\ell - x_i - \sum w_j - (2j-1)\Delta_*\right)} \cdot \frac{\prod_{i<i'} E(x_i, x_{i'}) \prod_{i,j} E(y_i, w_j)}{\prod_{i,j} E(x_i, y_j) \prod_{j<j'} E(w_j, w_{j'})} \times$$

$$\times \left[\frac{\prod_j \hat{\Theta}_*(w_j)}{\prod_i \hat{\Theta}_*(y_i)}\right]^{(2j-1)} \cdot \left[\frac{\prod_{a<b} E(R_a^*, R_b^*)}{\prod_a \nu_*'(R_a)}\right]^{(2j-1)^2} (\det \bar{\partial}_0)^{1/2}.$$

Note, that β,γ-systems may be "bosonized" in terms of free Grassmanian fields (η, ξ) with spins $(1,0)$ and free scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2\pi i}\int\left(\eta\bar{\partial}\xi + \frac{1}{2}\int|\partial\phi|^2 + \frac{i}{2}(2j-1)\sqrt{g}R\phi\right) \qquad (5.5.19)$$

in the case of arbitrary j. (In variance with b,c-systems the coefficient before curvature in (5.5.19) is <u>imaginary</u>.) Bosonization rules are:

$$\beta(z) = \partial\xi e^{-\phi}(\nu_*(z))^{2j-1} \qquad \gamma(z) = \eta e^{+\phi}(\nu_*(z))^{1-2j} \qquad (5.5.20)$$

or
$$\xi(z) = H(\beta(z)) \qquad e^{\phi} = \nu_*^{2j-1}(z)\,\delta(\beta(z))$$

$$\eta(z) = \partial\gamma(z)\delta(\gamma(z)) \qquad e^{-\phi} = \nu_*^{1-2j}(z)\,\delta(\gamma(z)). \qquad (5.5.21)$$

It is straightforward to recover (5.5.18), starting from (5.5.19) and (5.5.20).

6. MULTILOOP CORRELATORS IN WZW THEORY

6.1. GENERAL FORM OF CONFORMAL BLOCKS

In this section we shall discuss the implications of bosonization prescription for WZWM in the case of arbitrary closed Riemann surfaces. Note, that bosonized version of a theory contains more irreducible representations of KM algebra, than the WZWM itself. Thus to obtain conformal blocks of WZW theory one should design some linear combinations of conformal blocks of its bosonized version in such a way, that additional fields are projected out. On the sphere (genus 0) these linear combinations are contour integrals of certain dimension one operators, arising after bosonization. In the case of higher genera besides these contour integral insertions one should take linear combinations of conformal blocks, corresponding to different "boundary conditions" (theta-characteristics).

Naive calculation of multiloop correlators of WZWM, relying upon bosonization prescription gives the answer like

$$Z'(z_i) = \int \cdots \int \prod_{\alpha \subset \Delta_+} |\mathcal{F}(\{w_\alpha, \chi_\alpha\}, z_i, u_j)|^2 \left(\sum_e |\mathcal{F}_e(\{\phi_\alpha\}, z_i, u_j)|^2\right) \prod_{i=1}^{N} d^2 u_i \qquad (6.1.1)$$

where $\mathcal{F}(\{w_\alpha, \chi_\alpha\}, z_i, u_j)$ are conformal blocks of β, γ-systems w_α, χ_α with spin $j=1$, and $\mathcal{F}_e(\{\phi_\alpha\}, z_i, u_j)$ are conformal blocks of a multiplet of scalar fields taking values in Cartain torus of the group (it is proportional to theta-function with characteristic e, associated with this torus). Additional operators of dimension one are located at points $\{u_j\}$ and the surface has punctures at points $\{z_i\}$. All these conformal blocks, entering r.h.s. of (6.1.1) were already discussed in Section 5.

Note, that through $\mathcal{F}(W_\alpha, \chi_\alpha)$, Θ-functions naturally arise in denominator of formulae for multiloop characters in WZWM.

In the spirit of usual relation between chiral and non-chiral versions of the theory, we conjecture the following form of chiral conformal blocks in WZW theory:

$$\mathcal{F}^{\lambda}_{WZW}(z_1 \ldots z_k) = \int_{C_1^\lambda} \cdots \int_{C_m^\lambda} \sum_e K_e^\lambda \mathcal{F}_e(\{\varphi_\alpha\}; z_i; u_j) \cdot \prod_{\alpha \in \Delta_+} \mathcal{F}(\{W_\alpha, \chi_\alpha\}; z_i, u_j) \prod_{i=1}^{m} du_i \quad (6.1.2)$$

where K_e^λ are some characteristic-dependent coefficients, and $C_1^\lambda, \ldots, C_m^\lambda$ are some non-contractable cycles on punctured Riemann surface. (Note, that W_α, χ_α are periodic because they are related to single-valued KM currents.) Actually conformal blocks of WZWM arise only for some special choices of K_e^λ and $C_1^\lambda, \ldots, C_m^\lambda$.

In what follows we are going to illustrate this general suggestion in the case of genus 1 (torus). In this case, we have an alternative way to obtain some correlators (including partition functions), using well known characters of Kac-Moody algebras [45]. We shall find a complete agreement with (6.1.2).

6.2. CHARACTERS OF KAC-MOODY ALGEBRAS [15]

Let us consider vacuum conformal block on a torus, which is associated with irreducible representation of KM-algebra with the highest weight λ :

$$\mathcal{J}^{\lambda} = \mathrm{Tr}_{H_\lambda} e^{2\pi i \tau (L_0 - \frac{c}{24})} \tag{6.2.1}$$

where τ is modular parameter of the torus,

H_λ is irreducible representation of KM algebra,

c is central charge of associated Virasoro algebra,

$$c = \frac{k \dim G}{k + C_V} \tag{6.2.2}$$

We shall show, that (6.2.1) is a value of character on the special element $\exp(h)$ of KM group.

To begin with, let us present a brief review of KM algebras and their characters [15] . Let us start with current algebra $L\mathfrak{g}$. Elements of $L\mathfrak{g}$ are Laurent seria with coefficients in $\mathfrak{g}$. There is a bilinear symmetric form on them:

$$(x, y) = \sum_{n \in \mathbb{Z}} x_n y_{-n} \qquad \text{if} \quad x = \sum_{n \in \mathbb{Z}} x_n t^n ; \; y = \sum_{m \in \mathbb{Z}} y_m t^m \tag{6.2.3}$$

In order to get central extension of current algebra, one should add central element c and modify commutation relations:

$$\left[\left(\sum x_n t^n + \lambda c \right) \left(\sum y_m t^m + \lambda' c \right) \right] = \sum [x_n, y_m] t^{n+m} + c \, \mathrm{Res}_t \left(\tfrac{dx}{dt} y \right) \tag{6.2.4}$$

If we add one more element - derivative $d = t\, d/dt$

$$\left[d, \left(\sum x_n t^n + \lambda c \right) \right] = \sum n x_n t^n \tag{6.2.5}$$

we obtain KM algebra $\tilde{\mathfrak{g}}$ with non-degenerate bilinear symmetric form

$$(x + \lambda c + \mu d, \; y + \lambda' c + \mu' d) = (x, y) + \lambda' \mu + \lambda \mu' \tag{6.2.6}$$

Cartan sub-algebra of $\tilde{\mathfrak{g}}$ is

$$\tilde{h} = h \oplus \mathbb{C}c \oplus \mathbb{C}d \qquad (6.2.7)$$

where h stands for Cartan sub-algebra of $\mathfrak{g}$.

Let us introduce the dual space $\tilde{h}^*$,

$$\tilde{h}^* = h^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta \qquad (6.2.8)$$

so, that the following relations hold:

$$\begin{aligned} \Lambda_0(c) &= \delta(d) = 1 \\ \Lambda_0(d) &= \delta(c) = 0 \\ \text{if } \lambda \in h^* &\Rightarrow \lambda(c) = \lambda(d) = 0. \end{aligned} \qquad (6.2.9)$$

Root decomposition for the algebra $\tilde{\mathfrak{g}}$ looks like

$$\tilde{\mathfrak{g}} = \tilde{h} \oplus \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} t^n h_n \oplus \sum_{\substack{m \in \mathbb{Z} \\ \alpha \neq 0}} t^m \mathfrak{g}_{m,\alpha} \qquad (6.2.10)$$

where root subspaces are defined by the conditions:

$$[h_i, \mathfrak{g}_a] = a(h_i)\mathfrak{g}_a \qquad h_i \in \tilde{h},\ a \in \tilde{h}^*. \qquad (6.2.11)$$

Elements $a \in \tilde{h}^*$ are reffered to as roots, and $\dim \mathfrak{g}_a =$ $=$ mult$_a$ are their multiplicities. For algebra $\tilde{\mathfrak{g}}$ we have the following root system:

$$\Delta = \left\{ \begin{array}{lll} (\alpha + n\delta) & (n \in \mathbb{Z}, \alpha \in \dot{\Delta}) & \text{mult}_{\alpha+n\delta} = 1 \\ (n\delta) & (n \neq 0; n \in \mathbb{Z}) & \text{mult}_{n\delta} = r \end{array} \right\} \qquad (6.2.12)$$

where r is rank of G and $\dot{\Delta}$ - the root system of $\mathfrak{g}$. (6.2.12) is a direct consequense of (6.2.10) and the fact, that for finite $\mathfrak{g}$ all mult$_\alpha = 1$.

For a system of simple roots in $\tilde{h}^*$ for basis in the root space) we choose the following roots:

$$a_i = \alpha_i \quad i = 1 \ldots r;\ a_0 = \delta - \alpha_0 \qquad (6.2.13)$$

where α_i are simple roots of $\mathfrak{g}$ and $\alpha_0 = \sum_{i=1}^{r} \alpha_i$ is long root. Thus all positive roots are:

$$\Delta_+ = \left\{ \sum_{k_i \geq 0} k_i a_i \right\} = \left\{ (n-1)\delta + \alpha ;\ n\delta - \alpha ;\ n\delta \,\middle|\, n \geq 1, \alpha \in \mathring{\Delta}_+ \right\}. \qquad (6.2.14)$$

Discuss now the Weyl group of algebra $\tilde{\mathfrak{g}}$. Affine Weyl group is generated by reflections $\tau_0, \tau_1, \ldots \tau_r$:

$$\tau_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i \qquad \left(\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}\right) \qquad (6.2.15)$$

with respect to simple roots $\alpha_i \in \Delta$. Because of the relation $(\delta, \alpha_i^\vee) = 0$ we have $\tau_i(\delta) = \delta$. Therefore W acts on a factor-space $\tilde{h}^*/\mathbb{C}\delta$. It is easy to prove, that on the hyper-plane $E = \{\lambda \mid (\lambda, \delta) = k;\ \lambda \in \tilde{h}^*/\mathbb{C}\delta\}$ the action is affine. Shifts along dual roots $\alpha_j^\vee$ are generated by elements

$$t_{\alpha_j^\vee} = \tau_{\alpha_j} \tau_{\delta - \alpha_j}. \qquad (6.2.16)$$

On the whole space $\tilde{h}^*$ these generators look like

$$t_{\alpha_j^\vee}(\lambda) = \tau_{\alpha_j}\tau_{\delta-\alpha_j}(\lambda) = \bar{\lambda} + m\Lambda_0 + m\alpha_j^\vee + \frac{1}{2m}\left(|\lambda|^2 - |\bar{\lambda} + m\alpha_j^\vee|^2\right)\delta \qquad (6.2.17)$$

where $\bar{\lambda}$ is projection from $\lambda \in \tilde{h}^*$ on h^* and $m = (\lambda, \delta) \neq 0$

Let t_μ be a shift operator acting on $\tilde{h}^*$

$$t_\mu(\lambda) = \bar{\lambda} + m\Lambda_0 + m\mu + \frac{1}{2m}\left(|\lambda|^2 - |\bar{\lambda} + m\mu|^2\right)\delta \qquad (6.2.18)$$

where $\mu \in M = \sum_{i=1}^{r} \mathbb{Z}\alpha_i^\vee$. For simply-laced algebras M coincides with the root lattice. Operators t_μ have the following properties:

$$t_{\mu_1} t_{\mu_2} = t_{\mu_1 + \mu_2} \ ; \quad w t_\mu w^{-1} = t_{w(\mu)} \qquad (6.2.19)$$

where $w \in \overset{\circ}{W}$ and $\overset{\circ}{W}$ is the Weyl group of finite-dimensional algebra $\mathfrak{g}$. Thus $T=\{t_\mu\}$ is a free abelian group, which is a normal subgroup in W. It is not difficult to realize, that W is a semidirect product $W=\overset{\circ}{W}\ltimes T$. In fact $\overset{\circ}{W}\cap T=1$ because $\overset{\circ}{W}$ is a finite group, and T is a free abelian one. $\overset{\circ}{W}$ is generated by $r_1 \dots r_2$ (3.5.1), and W contains an additional generator r_0, which is expressed through the shift $t_{\alpha_0^\vee}$:

$$r_{\alpha_0}=t_{\alpha_0^\vee}\, r_{a_0}$$

Using the properties of Weyl group it is easy to obtain generalized formula for characters of Kac-Moody algebra:

$$\mathrm{Tr}_{H_\lambda} e^{h} = \sum \dim V_\mu e^{\mu(h)} = \sum_{w\in W} e^{(w(\lambda+\rho))(h)} \det(w) \Big/ e^{\rho(h)} \prod_{\alpha\in\Delta_+} \left(1-e^{-\alpha(h)}\right)^{\mathrm{mult}\,\alpha} \qquad (6.2.20)$$

where λ is the highest weight of irreducible representation H_λ and h is some element of Cartan subalgebra. ρ stands for the generalized half-sum of positive roots and is defined by the conditions

$$(\rho, a_i)=1 \quad (0\le i\le r)\ ; \ (\rho,\Lambda_c)=0. \qquad (6.2.21)$$

Let us use the fact, that Weyl group is half-direct product of finite Weyl group and the group of translations, and rewrite the numerator in the following form:

$$\sum_{w\in W}\det(w)\, e^{w(\lambda+\rho)} = \sum_{w\in\overset{\circ}{W}}\det(w) \sum_{\mu\in M} e^{t_\mu(w(\lambda+\rho))} =$$

$$\sum \det(w)\, e^{S\left[\frac{|\bar\lambda+\bar\rho|^2}{2(g+k)}\right]}\, \Theta_{w(\bar\lambda+\bar\rho),\, g+k} \qquad (6.2.22)$$

where theta-functions are introduced through

$$\Theta_{\lambda,m}=\sum_{\mu\in M+\frac{\lambda}{m}} \exp\left(-\frac{S}{2}(\mu,\mu)+m\mu+m\Lambda_c\right) \qquad (6.2.23)$$

(these are in fact lattice theta-functions, corresponding to Cartan torus, of the level, proportional to $(g+k)$).

In (6.2.22) the following notation is used:

$\lambda(c)=k$; $\rho(c)=g$; $\bar\lambda$ and $\bar\rho$ are projections of λ and ρ on h^*, k is the central charge of associated central extension of current algebra, and g is dual Coxeter number, which coincides with C_v, $C_v=g$. For simply-laced algebras dual Coxeter number coincides with Coxeter number h and we may also use the formula

$$\dim G=(h+1)\operatorname{rang} G. \qquad (6.2.24)$$

Coxeter numbers are listed in Table .

Let us choose the following parametrization of Cartan elements:

$$h=-2\pi i\left(d\tau+uc+\sum_{i=1}^{r}z_ih_i\right). \qquad (6.2.25)$$

Then

$$\operatorname{Tr} e^{-2\pi i(d\tau+uc+\sum_{i=1}^{r}z_ih_i)}=\exp\left(-2\pi i\tau\frac{|\bar\lambda+\bar\rho|^2}{2(g+k)}\right)\times$$

$$\times\left(\sum_{w\in\bar W}\det(w)\,\Theta_{w(\bar\lambda+\bar\rho),g+k}(u,\vec z,\tau)\right)\cdot\prod_{n\geq1}\left(1-e^{2\pi i\tau n}\right)^{-\ell}\cdot e^{-2\pi i u g}e^{-2\pi i(\sum z_i)} \qquad (6.2.26)$$

$$\times\prod_{\alpha\in\Delta_+}\left(1-\exp\left(\sum\alpha(h_i)z_i2\pi i\right)\right)^{-1}\cdot\prod_{\substack{\alpha\in\Delta\\ n\geq1}}\left(1+\exp 2\pi i\sum z_i\alpha(h_i)\exp 2\pi i n\tau\right)^{-1}.$$

When z^i are tended to zero, the numerator and denominator of (6.2.26) possess zeroes of order $|\Delta_+|$. Resolving the uncertaintity, one obtains:

$$T_z e^{-2\pi i \tau d} = \exp\left\{-2\pi i\tau\left[\frac{|\bar\lambda+\bar\rho|^2}{2(g+k)} - \frac{|\bar\rho|^2}{2g}\right]\right\} \sum_{w\in\hat W} \det(w) \times$$

$$\times\ \partial_{z_1}^{|\Delta_+|}\Theta_{w(\bar\lambda+\bar\rho),g+k}(0,0,\tau) / \eta(\tau)^{\dim G} \qquad (6.2.27)$$

where $\eta(\tau) = \exp\left(\frac{\pi i\tau}{12}\right)\cdot\prod_{n\geq 1}(1-e^{2\pi i n\tau})$ (6.2.28)

is Dedekind function.

Let us remind that we would like to calculate the following quantity:

$$e^{2\pi i\tau(\Delta_\lambda - \frac{c}{24})} T_z e^{-2\pi i \tau d}. \qquad (6.2.29)$$

Conformal dimension and central charge are given by

$$\Delta_\lambda = \frac{(\lambda+2\rho,\lambda)}{2(g+k)} \qquad c = \frac{\dim G\cdot k}{k + c_v}. \qquad (6.2.30)$$

Taking into account the Freudental's"strange"formula,

$$\frac{|\bar\rho|^2}{2g} = \frac{\dim G}{24} \qquad (6.2.31)$$

we obtain the final answer:

$$T_z e^{2\pi i\tau(L_0 - \frac{c}{24})} = \frac{\sum \det(w)(\partial_{z_1})^{|\Delta_+|}\Theta_{w(\bar\lambda+\bar\rho),g+k}(0,0,\tau)}{\eta(\tau)^{\dim G}} \qquad (6.2.32)$$

It is also easy to calculate conformal blocks of the form of

$$\left\langle \exp \sum_{a=1}^{z} \oint H^a(\xi) z^a d\xi \right\rangle = T_z \exp\left\{2\pi i\tau\left(L_0 - \frac{c}{24}\right) + \sum_{a=1}^{z} H_0^a z^a\right\}. \qquad (6.2.33)$$

Using eq.(6.2.26) with the element

$$g = \exp -2\pi i\left(\tau d - \sum_{a=1}^{z} h_a z^a\right) \qquad (6.2.34)$$

we obtain the following relation:

$$\langle \exp \sum_{a=1}^{r} \oint H^a(\xi) z^a d\xi \rangle = \sum_{w\in \hat{W}} \det(w)\, \Theta_{w(\bar\lambda+\bar\rho)}(z_a, 0, \tau) / e^{\pi i \sum_a \alpha(h^a) z^a} \times$$

$$\times \prod_{n=1} (1-e^{2\pi i \tau})^{-\ell} e^{-\frac{i\pi\tau}{12}\dim G} \prod_{\substack{\alpha\subset\dot\Delta \\ n\geq 1}} (1-e^{2\pi i n\tau} e^{2\pi i \sum_a \alpha(h_a) z^a})^{-1} = \qquad (\underline{6}.2.35)$$

$$= \sum_{w\in\hat{W}} \det(w) \frac{\Theta_{w(\bar\lambda+\bar\rho), k+q}(z_a, 0, \tau)}{\eta(\tau)^{2}} \prod_{\alpha\subset\dot\Delta_+} \left[\frac{\eta(\tau)}{\Theta_*(\sum_a \alpha(h_a) z^a)} \right]$$

In this derivation the product formula for theta-function,

$$\Theta_*[z] = (\sin\pi z) e^{i\pi\tau/4} \prod_{n=1} (1-e^{2\pi i n\tau})(1-e^{2\pi i n\tau} e^{2\pi i z})(1-e^{2\pi i n\tau} e^{-2\pi i z}) \qquad (6.2.36)$$

is applied.

Now let us comment, how these formulae arise in WZW theory. Cartan currents look like (4.3.),

$$H^a(\xi) = -\sum_{\alpha\subset\dot\Delta_+} \alpha(h^a) w_\alpha \chi_\alpha + \quad i q \cdot \partial\varphi^a \qquad (6.2.37)$$

So the l.h.s. of (6.2.33) has the form:

$$\prod_{\alpha\subset\Delta_+} \langle \exp -\sum_a \alpha(h^a) z^a \oint w_\alpha \chi_\alpha \rangle \cdot \langle \exp \sum i q z^a \int \partial\varphi^a \rangle. \qquad (6.2.38)$$

These correlators are easy to calculate (see s.5), and the result is:

$$\langle \exp \sum_{a=1}^{r} \oint H^a(\xi) z^a d\xi \rangle = \prod_{\alpha\subset\dot\Delta_+} \left[\frac{\eta(\tau)}{\Theta_*(\sum_a \alpha(h_a) z^a)} \right] \frac{\Theta_{\rho, q+k}(z_a, 0, \tau)}{\eta(\tau)^2} \qquad (6.2.39)$$

in complete accordance with (6.2.35) and the general expectations about the relation between WZW conformal blocks and their bosonized prototypes (6.1.2).

7. Conclusion

We presented here a rath r detailed discussion of "bosonization" /1/ of Wess-Zumino-Witten model, which represents it in terms of free fields. In variance with other proposals (like /16/) this scheme seems realy self-consistent description of WZWM, since Sugawara's stress tensor and WZW action appear quadratic in these fields.

We demonstrated, that this type of bosonization is applicable for all simple KM algebras with arbitrary central charges K (Sect.4) The number of free fields is equal to dimension D of the group, and this is very natural from the point of view of Lagrangian approach, if one wants to have a unified description for all K, since as $K \to \infty$ WZWM turns into a theory of D free fields. For some low values of K in the strong coupling domain other consistent bosonizations may arise with fewer free fields (as it happens for K=1 or K=2 /17/), but they hardly can be naturally generalized for all K.

We demonstrate, that the bosonization prescription reproduces all known answers for correlators at genus 0, which may be expressed in terms of generalized hypergeometric functions (sects. 2.3, 4.4). Integrals, relating these hypergeometric functions to elementary ones like $\prod_{i\neq j}(\xi_i-\xi_j)^{a_i a_j}$ naturally appear as integrals over insertions of dimension - 1 operators /6/, required to project out the extra degrees of freedom, which arise in the theory of free bosons, - that is to project on irreducible representation of chiral algebra. In the case of WZWM, which possesses explicit Lagrangian formulation, one can interpret new insertions as a result of change of variables, needed to make Lagrangian quadratic, and this allows one to find out the form of relevant dimension-1 operators from the first principles. This should be a proper way to derive an analogue of Felder's prescription [18] from Lagrangian approach. Note that since all non-trivial rational conformal theories are believed to be coset models, related to WZWM [19], these results suggested that all correlators at genus 0 in all RCFT are expressed through generalized hypergeometric functions. We believe, that this suggestion may be verified from the study of monodromy properties on the lines of refs. [20].

Important advantage of free field representation of any conformal theory (leaving aside its more "philosophical" implications) is that it provides one with a constructive technique for calculation of conformal blocks on arbitrary Riemann surfaces with handles and punctures. We have demonstrated this technique in calculations at genus 0 (sect.2.4). We have showed also how one-loop characters of Kac-Moody algebra and WZWM are reproduced and how the multiloop conformal blocks look like (sect.5,6). Of course a more detailed study of Felder's reasoning [18] is necessary in multiloop case.

A new important news in the crucial role of $\beta\gamma$ system of free bosonic fields [11] in bosonization of WZWM. Thus far $\beta\gamma$ systems arised only in the Neveu-Schwarz-Ramond approach to superstrings, but now it seems that they may play a much more important role.

The most trivial explanation of the bosonization prescription [1] comes from the coadjoint orbit approach. The WZW action is nothing but d^{-1} of the Kirillov form on a coadjoint orbit of Kac-Moody group [21]. The Gauss product expansion of group elements diagonalizes the Kirillov form (sect.4.3, 4.5) and a simple change of variables is required to make it quadratic. This choice of the coordinates (Gauss expansion) breaks explicitly G-invariance of Kirillov's form (invariant form is d (WZW action) itself, and it is non-quadratic), but dynamics is of course G-invariant, and this guarantees that the currents have proper Kac-Moody commutational relations. Reduction of Kac-Moody algebra on generic orbits, naturally leads to bosonization of arbitrary coset models. Note that an immediate application of the construction /1/ is description of parafermions, since WZWM is decomposed in free scalar and parafermionic fields /22/.

We are going to return to all these questions in another publication.

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Table

Coxeter numbers h and dual Coxeter numbers g of Lie algebras

G	$A_\ell^{(1)}$	$B_\ell^{(1)}$	$C_\ell^{(1)}$	$D_\ell^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$
h	$\ell+1$	2ℓ	2ℓ	$2\ell-2$	12	18	30	12	6
g	$\ell+1$	$2\ell-1$	$\ell+1$	$2\ell-2$	12	18	30	9	4

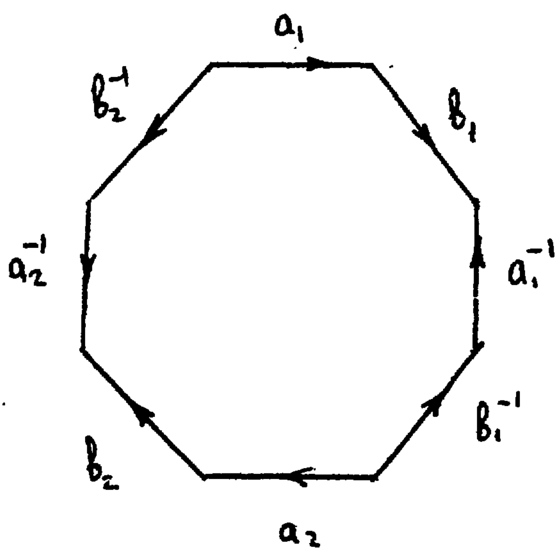

Fig.1.

The cut of the Riemann surface S_p $(p=2)$

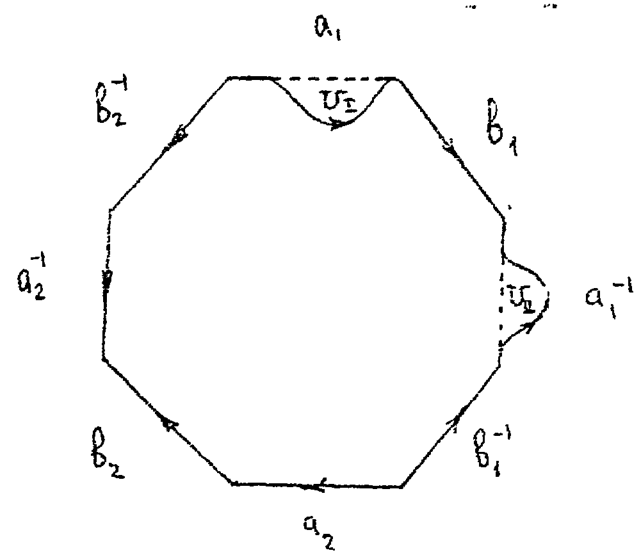

Fig.2.

Deformation of the cut in the case p=2

R E F E R E N C E S

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