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TWO-DIMENSIONAL QUANTUM GAS

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Abstract

Identical impenetrable particles in a 2-dimensional configuration space obey braid statistics, intermediate between bosons and fermions. This statistics, based on braid groups, is introduced as a generalization of the usual statistics founded on the symmetric groups. The main properties of an ideal gas of such particles are presented. They do interpolate the properties of bosons and fermions but include classical particles as a special case. Restriction to 2 dimensions precludes lambda points but originates a pecul[illegible] symmetry, responsible in particular for the iden[illegible] of boson and fermion specific heats.

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1. Introduction

Quantum mechanics of a given system is controlled by the group of classes of closed loops (the fundamental group) of its configuration space [1] and, of course, statistical mechanics will follow suit. The configuration space of a system of N identical particles will have a non-trivial fundamental group [2], that is, will be multiply connected. Let M be the configuration space of each particle and consider the set $M^N = \{x = (x_1, x_2, \ldots, x_N)\}$, the cartesian product of manifold M by itself N times. M^N will be the configuration space of a gas of N distinguishable particles contained in M. M is usually a 3-dimensional Euclidean box whose volume V is taken to infinity at the thermodynamical limit. When the particles are indistinguishable, two points x and x' of M^N are equivalent if the sets $(x_1, x_2, \ldots, x_N)$ and $(x'_1, x'_2, \ldots, x'_N)$ differ only by a permutation, a transformation belonging to the symmetric group S_N, and the configuration space becomes the quotient M^N/S_N. When M is indeed Euclidean, and so topologically trivial, the fundamental group, indicated $\pi_1(M^N/S_N)$, is just S_N. Space M^N is then the universal covering of the configuration space and the N-particle wavefunction will belong in the carrier space of some representation of S_N. When the particles are also impenetrable, coincident positions for them must be excluded. This is done by defining the set

$$D_N = \{(x_1, x_2, \ldots, x_N) \text{ such that } x_i = x_j \text{ for some } i,j\}$$

and taking its complement in M^N,

$$F_N M = M^N \setminus D_N . \tag{1.1}$$

This would be the configuration space for N distinct impenetrable particles. The fundamental group of this space, $P_N = \pi_1(F_N M)$, is the *pure braid group* [3]. It is in particular the fundamental group for the configuration space of a classical hard-sphere gas.

Consider further the particles to be indistinguishable. Let $B_N M$ be the space obtained by identifying all equivalent points, the quotient by S_N:

$$B_N M = F_N M / S_N = [M^N \setminus D_N]/S_N \quad . \tag{1.2}$$

The fundamental group of this configuration space, $B_N = \pi_1[B_N M]$, is the *full braid group*, or simply *braid group*.

Summing up, $F_N M$ is the configuration space for a gas of N impenetrable particles, $B_N M$ is the configuration space for a gas of N impenetrable *and* identical particles and the pure and full braid groups are the respective fundamental groups of such spaces. Quantization is to be performed on such highly complicated, multiply-connected space [4]. We shall see below that braid groups concern real, usual braids and that, because braids can be unwoven in spaces of three or more dimensions, all this is of possible practical interest only for two-dimensional systems. The quotient of the full braid group by the pure braid group is the symmetric group:

$$B_N/P_N = S_N \, . \tag{1.3}$$

When $\dim M > 2$, the braid groups are extensions by S_N of $[\pi_1(M)]^N$. When M is itself topologically non-trivial, $\pi_1(M)$ will be some discrete group. When M is trivial, it reduces to the identity. In the Euclidean case $M = E^3$, $\pi_1(M) = 1$, $B_N = S_N$, $\pi_1[E^{3N}/S_N] = S_N$. The classes of loops, highly non-trivial on a punctured E^2, become trivial in punctured E^3, where each loop may be continuously deformed to a point. Consequently, for $\dim M \geq 3$, statistical mechanics will be just as usual, regulated by the symmetric group, with bosons and fermions.

The whole point is that this is not forcibly the case for $\dim M = 2$. A gas at 2 dimensions may have a different statistics, governed by braid groups, a *braid statistics*. Such statistics is expected to play a role in 2-dimensional condensed matter physics phenomena such as the fractional quantum Hall effect and high temperature superconductivity [5]. Braid statistics has been the object of highly sophisticated analysis [6], in comparison to which our approach is undoubtedly naive. Most treatments are concerned with "particles" whose exotic statistics is due to their special structure, such as Wilczek's anyons [7]. In reality, manifestations of such exotic statistics are to be expected as soon as impenetrable particles are considered in two dimensions [8], whatever their internal structure [9].

It is our contention that the first step in the understanding of a new statistics is to examine the corresponding ideal gas. That is what we intend to do here, taking into account the fact that impenetrability is enough to ensure the emergence of braid statistics. Such a free gas might come to model, up to interactions, the electron gas in superconductors, in which the electrons are confined to the surface. It is anyhow the starting point, to which dynamical effects are to be added in a deeper analysis. Dynamics has been extensively studied, mainly in the line of Landau-Ginzburg models [10]. A complete knowledge of the ideal case will help to separate effects of dynamical nature from those of purely statistical origin.

As the best way to introduce braid groups is through the symmetric groups, and also because these would anyhow regulate the case of interpenetrable particles, we start by recalling in §2 the ideal gas statistical mechanics as ruled by S_N. The canonical partition function of a real gas of N particles is a certain S_N-invariant polynomial, the cycle indicator polynomial. Furthermore, the symmetric group maintains a bookkeeping role even for braid statistics, as it counts the classical configurations in terms of which the quantum partition function is ultimately written. The adopted pedagogical tune seems unavoidable because we want to stress some aspects to be later adapted to the braid case. In particular, permutations are introduced in a pictorial way specially convenient to generalization. The 2-dimensional case exhibits some characteristic symmetries, leading in particular to the identity of boson and fermion specific heat. In the bosonic case, although ground state crowding of particles do occur at low enough temperatures, no lambda point is found, so that transition must proceed smoothly. A more mathematical interpretation of the usual realization of wavefunctions in terms of distinguishable particles is given in S3. It helps clarifying what happens when exchange symmetries distinct from S_N are at work. We give a (necessarily very incomplete) introduction to braids and their groups in S4 and proceed to study the corresponding statistical mechanics in S5. There is one braid statistics for each value of an angular parameter φ, with $\varphi = 0$ corresponding to bosons and $\varphi = \pi$ to fermions. As it might be expected, intermediate values of φ leads to intermediate behavior of the physical quantities, but this "interpolation" goes through the classical case at $\varphi = \pi/2$, where a Boltzmmann gas comes out. Some final comments, including a comparison to parastatistics, are made in S 6.

2. Symmetric group statistics

The role of the symmetric group [11] in Statistical Mechanics is best seen in the cluster decomposition [12] of the canonical partition function, which happens to be an invariant polynomial of S_N. In order to see it, let us start by shortly reviewing some well known facts about permutations.

A general permutation P of particles labelled $x_1, x_2, \ldots, x_{N-1}, x_N$ is usually indicated by $P = \begin{pmatrix} x_1 & x_2 & \cdots & x_{N-1} & x_N \\ x_{p_1} & x_{p_2} & \cdots & x_{p_{N-1}} & x_{p_N} \end{pmatrix}$ and can always be decomposed into the product of disjoint cycles, particular tail-biting permutations of type $\begin{pmatrix} x_1 & x_2 & \dots & x_{r-1} & x_r \\ x_2 & x_3 & \dots & x_r & x_1 \end{pmatrix}$. This is a cycle of length r, or an r-cycle. It is convenient [13] to attribute a variable t_r to a cycle of length "r" and indicate the cycle structure of a permutation by the monomial $t_1^{v_1} t_2^{v_2} t_3^{v_3} \dots t_r^{v_r}$, meaning that there are v_1 1-cycles, v_2 2-cycles, etc. Permutations of the same cycle type, that is, with the same set $\{v_j\}$, go into each other under the action of any element of S_N: they constitute conjugate classes. To all permutations of a fixed class will be attributed the same monomial above. In this sense, such monomials are invariants of the group S_N. The total number of permutations with such a fixed cycle configuration is $\frac{N!}{\prod_{j=1}^{N} v_j! j^{v_j}}$. The N-variable generating function for these numbers is the cycle indicator polynomial [14]

$$C_N(t_j) = C_N(t_1, t_2, t_3, \dots t_N) = \sum_{\{\nu_j\}}^{*} \frac{N!}{\prod_{j=1}^{N} \nu_j! j^{\nu_j}} t_1^{v_1} t_2^{v_2} t_3^{v_3} \dots t_r^{v_r} \qquad (2.1)$$

where the star (*) recalls that the summation takes place over the sets $\{\nu_i\}$ of non-negative integers for which

$$\sum_{i=1}^{N} i\nu_i = N.$$

Cycle indicator polynomials satisfy the relation [15]

$$C_N(t_j) = \sum_{m=0}^{N} \frac{N!}{(N-m)!} t_{m+1} C_{N-m}(t_j) , \qquad (2.2)$$

which may lead to recursion formulae when the t_r's are known and will be useful later on.

The canonical partition function of a real non-relativistic gas of N particles contained in a d-dimensional volume V is a cycle indicator polynomial for the symmetric group S_N, with the (combinatorially meaningless) variables $t_j = jb_j \frac{V}{\lambda^d}$ giving the contribution of the j-th order cluster integral:

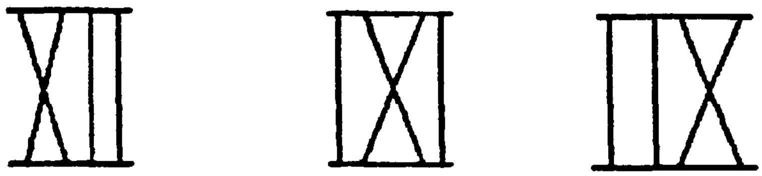
$$Q_N(\beta,V) = \frac{1}{N!} C_N(jb_j \frac{V}{\lambda^d}) \tag{2.3}$$

where $\beta = 1/kT$; λ is the mean thermal wavelength; b_j is the j-th cluster integral [16]. For 3-dimensional ideal quantum gases, statistics is simulated by an effective interaction for which the j-th cluster integral is $b_j = (\pm)^{j-1}/(j)^{5/2}$ for bosons (upper sign) and fermions (lower sign). As seen below, $b_j = (\pm)^{j-1}/j^2$ in the 2-dimensional case. The above result might lead to some academic speculations. Any discrete finite group is isomorphic to some permutation group [17] and to any subgroup G of S_N corresponds a cycle indicator polynomial [18], which would provide the canonical partition function for an imaginary gas whose "statistics" is governed by G. For $G = Z_N$ (the cyclic group of order N), for example, $Q_N(\beta,V) = b_N \frac{V}{\lambda^d}$ [19]. In reality, S_N will remain in the background also in the case of braid statistics and the indicator polynomial will retain its role, with only the "combinatorially meaningless" variables t_j modified.

Every cycle is a composition of elementary transpositions, cycles of length 2, so that ultimately every permutation may be written as a product of transpositions. That is to say that transpositions *generate* the symmetric group. Permutations with an even (odd) number of transpositions are called even (odd) permutations. We shall use as generator basis for S_N the set $\{s_i\}$ of elementary transpositions: s_i exchange the i-th and the (i+1)-th entry:

$$s_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_i & x_{i+1} & \cdots & x_{N-1} & x_N \\ x_1 & x_2 & \cdots & x_{i+1} & x_i & \cdots & x_{N-1} & x_N \end{pmatrix} . \qquad (2.4)$$

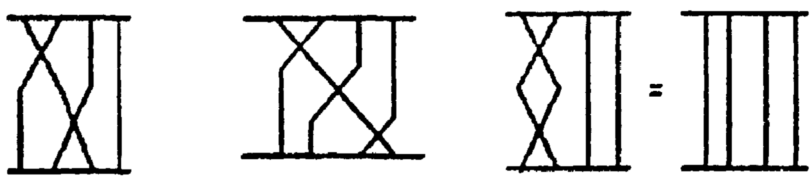
Each generator s_i may be represented by a diagram of evident conception, indicated in fig. 1 for $N = 4$



s_1 s_2 s_3

Fig. 1: Diagrams of the S_4 generators.

Elements of S_N are pictured as compositions of such graphs, product being represented by downward concatenation, as in the examples in fig. 2.



$s_1 s_2$ $s_1 s_2 s_3$ $s_1^2 = 1$

Fig. 2: Elements of S_4 are obtained by concatenation.

It is noteworthy that the s_i's obey the algebraic relations

$$s_j \, s_{j+1} \, s_j = s_{j+1} \, s_j \, s_{j+1}$$
$$s_i \, s_j = s_j \, s_i \qquad \text{for } |i-j| \geq 2, \tag{2.5}$$

as can be easily verified by composing diagrams. Also, as in the example at the right of fig. 2, squaring a generator corresponds to no exchange at all, refleting the property

$$(s_i)^2 = 1, \tag{2.6}$$

typical of transpositions. Actually, (2.5) and (2.6) completely characterize the symmetric group, in the sense that any group with generators satisfying them will be isomorphic to S_N. A group introduced in this way, by specifying generators and the equations they satisfy, is said to be given by a *presentation*.

To recall the import of all that in elementary Quantum Mechanics, let us indicate all coordinates, spins, etc, of the N particles by the collective variable x. Exchanges of particles are given by the action of elements of S_N on x. General permutations are products of elementary transpositions of two particles, each one given by $s_i x$, for some i. For instance, if $N = 2$ and $x = (r_1, r_2)$, the only exchange will be to $(r_2, r_1) = s_1[(r_1, r_2)]$. States correspond to rays in the Hilbert space of wavefunctions: the same state is represented by $\psi(x)$ and $e^{i\eta}\,\psi(x)$, with η an arbitrary phase. Symmetry under a group will imply that wavefunctions respond to transformations according to a unitary (or anti-unitary) representation of the group. Under a permutation P the wavefunction will undergo a unitary transformation U(P) in Hilbert space. The set $\{U(s_i)\}$ of unitary operators will constitute a basis for such a representation and will have to satisfy conditions (2.5) and (2.6). There is, however, an additional condition: as wavefunctions ψ are supposed to have values in a 1-dimensional complex space, this representation must be a 1-dimensional unitary representation and consequently only phase factors appear: $\psi(s_j x) = U(s_j)\psi(x) = e^{i\varphi_j}\psi(x)$. It is immediate to see that (2.5) imposes the equality of all the phases, so that $U(s_j)\psi(x) = e^{i\varphi}\psi(x)$, with the same φ for all s_j. And that (2.6) imposes $U^2(s_j)\psi(x) = U(s_j{}^2)\psi(x) = e^{i2\varphi}\psi(x) = \psi(x)$, so that $e^{i\varphi} = \pm 1$. There are only two 1-dimensional representations of S_N: the totally symmetric one, related to bosons and with $U(P) = +1$ for every permutation P; and the totally antisymmetric one, in which fermions find their place, with $U(P) = +1$ when P is even and $U(P) = -1$ when P is odd. The 1-dimensional nature of S_N representations is responsible for the lack of parastatistics observation. Notice that, once unitarity is supposed, (2.6) is equivalent to hermiticity.

Let us now proceed to Statistical Mechanics. As the statistical aspects are very similar, we shall examine directly the 2-dimensional case and only recall from time to time the usual 3-dimensional results for comparison. Momenta being 2-dimensional vectors, we have now, instead of an energy sphere, an energy circle: the number of microstates for particles on a surface of area S with energy less than or equal to E is $\Sigma(E) = 4\pi m S E/h^2$, and the corresponding number of microstates with energy between E and E + dE will be $g(E)\,dE = \frac{d\Sigma}{dE}\,dE = \frac{4\pi m S}{h^2}\,dE$. For example, if we want to characterize a Fermi temperature by $kT_F = E_F$, we impose $\int_0^{E_F} g(E)\,dE = N$ and find $E_F = \sigma\frac{h^2}{4\pi m}$ (with $\sigma = S/N$) or, in terms of a "critical" thermal wavelength, $\lambda_F^2 = \frac{h^2}{2\pi m k T_F} = 2\sigma$. Unlike the 3-dimensional case, the state density is here constant This may be a reason for the specific heat symmetry which will be found below.

The pressure and the particle number density expressions in terms of the fugacity z are

$$\frac{P}{kT} = \mp \sum_{E} \ln[1 \mp ze^{-\beta E}] = \mp \frac{4\pi m}{h^2} \int_0^\infty dE \ln[1 \mp ze^{-\beta E}] \qquad (2.7)$$

$$n = \sum_{E} \frac{1}{z^{-1}e^{\beta E} \mp 1} = \frac{1}{\lambda^2} \ln(1 \mp z) \qquad (2.8)$$

To see how (2.3) comes out in the present case, let us examine the partition function for N particles:

$$Q_N(\beta,V) = \frac{S^N}{N!} \int d^2p_1 d^2p_2 d^2p_3 \ldots d^2p_N$$

$$e^{-\beta \sum_{i=1}^{N} \frac{p_i^2}{2m}} \langle p_1, p_2, \ldots, p_N | p_1, p_2, \ldots, p_N \rangle \quad . \qquad (2.9)$$

All the statistical content lies in the normalization amplitude $\langle p_1, p_2, \ldots, p_N | p_1, p_2, \ldots, p_N \rangle$. The ket $|p_1, p_2, \ldots, p_N\rangle$ is written as a sum of *ordered* products of one-particle kets $|p_j\rangle$ normalized to delta, $\langle p_i | p_j \rangle = \delta^2(p_i - p_j)$, and the resulting amplitude is a certain sum of deltas. We may look at the first, second, third, etc terms in each ordered product as corresponding to the first, second, third, etc particles, so that ultimately the physical ket is given as a sum of contributions of distinct particles, with coefficients fixed by statistics. Consider the usual ket for case $N = 2$:

$$|p_1, p_2\rangle = \frac{1}{2}\left[\,|p_1\rangle|p_2\rangle \pm |p_2\rangle|p_1\rangle\,\right] \qquad (2.10)$$

(upper sign for bosons, lower for fermions), from which

$$\langle p_1, p_2|p_1, p_2\rangle = \frac{1}{2}\left[\,\delta^2(p_1 - p_1)\delta^2(p_2 - p_2) \pm \delta^2(p_1 - p_2)\delta^2(p_2 - p_1)\,\right].$$

This amplitude is actually a decomposition into cycles: the first factor gives the contributions of the two possible 1-cycles, the second the contribution of the only 2-cycle. Because of the integrations in $Q_N(\beta,V)$, it does not matter which momentum is in each place: only the number of cycles of each type is important. We can introduce the notation $\hat{\delta}_1 = \delta^2(p_i - p_i)$ for a 1-cycle contribution, $\hat{\delta}_2 = \delta^2(p_i - p_j)\delta^2(p_j - p_i)$ for a 2-cycle contribution, $\hat{\delta}_3 = \delta^2(p_i - p_j)\delta^2(p_j - p_k)\,\delta^2(p_k - p_i)$ for a 3-cycle contribution, etc, so that

$$\langle p_1, p_2|p_1, p_2\rangle = \frac{1}{2}\left[\,\hat{\delta}_1^{\,2} \pm \hat{\delta}_2\right]. \qquad (2.11)$$

For $N = 3$, we take

$$|p_1, p_2, p_3\rangle = \frac{1}{3!}\left[\,|p_1\rangle|p_2\rangle|p_3\rangle \pm |p_2\rangle|p_1\rangle|p_3\rangle \pm |p_1\rangle|p_3\rangle|p_2\rangle \right.$$
$$\left. \pm |p_3\rangle|p_2\rangle|p_1\rangle + |p_2\rangle|p_3\rangle|p_1\rangle + |p_3\rangle|p_1\rangle|p_2\rangle\,\right] \qquad (2.12)$$

and find

$$\langle p_1, p_2, p_3|p_1, p_2, p_3\rangle = \frac{1}{3!}\left[\,\hat{\delta}_1^{\,3} \pm 3\,\hat{\delta}_1\hat{\delta}_2 + 2\,\hat{\delta}_3\right]. \qquad (2.13)$$

In reality, the numerical factors just count how many permutations there are of the corresponding cycle configuration and the amplitudes behave as cycle indicator polynomials:

$$\langle p_1, p_2, \ldots, p_N | p_1, p_2, \ldots, p_N \rangle = \frac{1}{N!} C_N [(\pm)^{j-1} \hat{\delta}_j] \qquad (2.14)$$

Notice that an amplitude is not really a cycle indicator polynomial: it only behaves like one under the multiple integration sign, which puts all momenta on an equal footing. Taking (2.14) into (2.9), the final expression of the partition function is easily obtained:

$$Q_N(\beta,V) = \frac{1}{N!} C_N \left[\frac{(\pm)^{j-1}}{j} \frac{S}{\lambda^2} \right] \qquad (2.15)$$

In last resort, the partition function is written as the sum of contributions of all configurations of distinct particles, with coefficients fixed by statistics. In the present case, there is a one-to-one correspondence between such configurations and the elements of S_N. We may start from the configuration $|p_1\rangle |p_2\rangle \ldots |p_N\rangle$, corresponding to the identity element and then get the remaining configurations by apllying all the group elements. For this reason, the partition function for an ideal quantum gas will have a form analogous to that of a "real" classical gas, the statistical effects being simulated by an effective interaction represented by non-zero "configuration integrals".

Actually, the amplitude normalization for the two-particle case fixes the higher cases: as the t_j's ($= (\pm)^{j-1} \hat{\delta}_j$) are known in the present case, (2.2) leads here to a recursion,

$$\langle p_1, p_2, \ldots, p_N | p_1, p_2, \ldots, p_N \rangle =$$

$$\frac{1}{N} \left\{ (\pm)^{N-1} \hat{\delta}_N + \sum_{s=1}^{N-1} (\pm)^{N-s-1} \hat{\delta}_{N-s} \langle p_1, p_2, \ldots, p_s | p_1, p_2, \ldots, p_s \rangle \right\}. \quad (2.16)$$

In terms of the fugacity, the expressions for the density and pressure will involve, instead of the familiar Bose and Fermi functions $g_{5/2}(z)$, $g_{3/2}(z)$, $f_{5/2}(z)$ and $f_{3/2}(z)$ of the 3-dimensional case, the dilogarithm $g_2(z) = Li_2(z)$ and the logarithm $g_1(z) = -\ln(1-z)$, which are specially simple. The formal "configuration integrals" are

$$b_j = \frac{(\pm)^{j-1}}{j^2} \quad \text{for} \quad \left\{ \begin{matrix} \text{bosons} \\ \text{fermions} \end{matrix} \right\}, \quad (2.17)$$

and the grand-canonical partition function will be $\Xi = \sum_{N=0}^{\infty} Q_N z^N$.

The pressure is then obtained from $\frac{p}{kT} = \ln \Xi$ and the density as $z \frac{\partial}{\partial z} \ln \Xi$:

$$\frac{p\lambda^2}{kT} = \sum_{j=1}^{\infty} b_j z^j = \sum_{j=1}^{\infty} \frac{(\pm)^{j-1}}{j^2} z^j = \left\{ \begin{matrix} g_2(z) \\ f_2(z) = -g_2(-z) \end{matrix} \right\},$$

$$\frac{N\lambda^2}{S} = \sum_{j=1}^{\infty} j\, b_j z^j = \sum_{j=1}^{\infty} \frac{(\pm)^{j-1}}{j} z^j = \left\{ \begin{matrix} g_1(z) \\ f_1(z) = -g_1(-z) \end{matrix} \right\}.$$

Notice that $n\lambda^2 = \frac{N\lambda^2}{S} = \frac{1}{2}\frac{\lambda^2}{\lambda_F^2} = \frac{1}{2}\frac{T_F}{T}$. Compactly written,

$$n\lambda^2 = \pm\, g_1(\pm z) = \mp \ln(1 \mp z) \tag{2.18}$$

$$P\lambda^2/kT = \pm\, g_2(\pm z) = \pm\, \mathrm{Li}_2(\pm z)\ . \tag{2.19}$$

Quantum effects are to be expected, of course, for large $n\lambda^2$. As a consequence of (2.18) the fugacity for bosons is directly given by $z_b = 1-e^{-n\lambda^2}$ and the equation of state is

$$\left(\frac{P}{nkT}\right)_{\text{bosons}} = \frac{1}{n\lambda^2}\, g_2(1-e^{-n\lambda^2})\ . \tag{2.20}$$

For the fermion gas, the fugacity is $z_f = e^{n\lambda^2} - 1$ and the equation of state is

$$\left(\frac{P}{nkT}\right)_{\text{fermions}} = -\frac{1}{n\lambda^2}\ g_2(-z) = -\frac{1}{n\lambda^2}\ g_2(1-e^{n\lambda^2})\ . \tag{2.21}$$

These equations of state are qualitatively similar to those appearing in the 3-dimensional case and are shown in fig. 9 as particular cases labelled ± 1. For boson and fermion gases of the same temperature, mass and density (same value of $n\lambda^2$) it follows from (2.18) that

$$z_f = \frac{z_b}{1-z_b}\ ,\ z_b = \frac{z_f}{1+z_f}\ . \tag{2.22}$$

Notice also that, both for bosons and fermions, the internal energy is $U = PS$. The constant-surface specific heat is

$$\frac{C_S}{Nk} = \frac{1}{Nk}\left[\frac{\partial U}{\partial T}\right]_{N,S} = \frac{1}{Nk}\left[\frac{\partial\ P\ S}{\partial T}\right]_{N,S} .$$

For bosons, we find

$$\left(\frac{C_S}{Nk}\right)_{bosons} = 2\ \frac{1}{n\lambda^2}\ g_2(1-e^{-n\lambda^2}) - n\lambda^2\,(1/(e^{n\lambda^2}-1)). \qquad (2.23)$$

One verifies that no derivative singularity occurs in this 2-dimensional case (curve labelled [± 1] in fig. 10). The low temperature trend is linear with coefficient $2\ \zeta(2) = \pi^2/3$. The ground state occupancy is

$$(n_0)_{bosons} = \frac{z_b}{1-z_b} = z_f = e^{n\lambda^2} - 1 \qquad (2.24)$$

For lower and lower temperatures the ground state gets more and more crowded, but with no singularity.

For fermions,

$$\left(\frac{C_S}{Nk}\right)_{fermions} = -2\ \frac{1}{n\lambda^2}\ g_2(1-e^{n\lambda^2}) + n\lambda^2\,(1/(e^{-n\lambda^2}-1)) \qquad (2.25)$$

The occupancy of the ground state is now

$$(n_0)_{fermions} = \frac{z_f}{1+z_f} = z_b = 1-e^{-n\lambda^2} . \qquad (2.26)$$

A peculiar symmetry appears in the 2-dimensional case, due to (2.22) plus a very special property of the dilogarithm [20],

$$g_2(1-x) + g_2(1-1/x) = -(1/2)(\ln x)^2. \qquad (2.27)$$

Expressions (2.23) and (2.25) are of the form

$$f(u) = \frac{2}{u}\, g_2(1-e^{-u}) - \frac{u}{e^u - 1}\,, \qquad (2.28)$$

with $u = \pm n\lambda^2$. It comes immediately from (2.27) with $x = e^{-u}$ that $f(u) = f(-u)$. We find then, for each fixed value of $n\lambda^2$,

$$\left(\frac{p\,\sigma}{kT}\right)_{bosons} - \left(\frac{p\,\sigma}{kT}\right)_{fermions} = -\frac{1}{2}\,n\lambda^2 \qquad (2.29)$$

and the rather surprising result

$$\left(\frac{C_S}{Nk}\right)_{bosons} = \left(\frac{C_S}{Nk}\right)_{fermions}\,. \qquad (2.30)$$

Bosons and fermions have consequently the same response to local energy concentrations, as the energy fluctuations are $\langle E^2\rangle - \langle E\rangle^2 = kT^2C_S$. Only to check how general is this property, we may examine another fluctuation, for example the isothermal compressibility:

$$(\chi_T)_{bosons} = \frac{e^{n\lambda^2} - 1}{n^2\lambda^2 kT} = \frac{z_f}{n^2\lambda^2 kT} \qquad (2.31)$$

$$(\chi_T)_{fermions} = \frac{1-e^{-n\lambda^2}}{n^2\lambda^2 kT} = \frac{z_b}{n^2\lambda^2 kT} = \frac{1}{2nkT_F}\left[1 - e^{-2(T_F/T)}\right] \qquad (2.32)$$

Consequently, the number fluctuations are different,

$$\left[\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle}\right]_{bosons} = nkT\chi_T = \frac{z_f}{n\lambda^2} \tag{2.33}$$

$$\left[\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle}\right]_{fermions} = nkT\chi_T = \frac{z_b}{n\lambda^2}\,. \tag{2.34}$$

We see that, although the symmetry still shows itself, the equality for bosons and fermions is a special characteristic of energy fluctuations.

3. Covering spaces

In order to see what happens when exchange groups distinct from S_N are involved, we shall need a more detailed understanding of the meaning of decompositions like (2.10) and (2.12). The configuration space for N identical interpenetrable particles is E^{2N}/S_N. E^{2N} is the universal covering, as the fundamental group π_1 is S_N. A covering space of a space X is another space which is locally homeomorphic to X, an unfolding of X breaking some equivalence between its points. Every space has a unique universal covering, which is simply-connected (has π_1 = {identity}) and whose folds, or sheets, are in one-to-one relationship with the elements of π_1. The different values of a multivalued function Ψ on a multiply-connected space are obtained through a representation of a group, the "monodromy group" of Ψ, in general a subgroup of π_1. A function becomes single-valued on a covering whose sheets are in one-to-one relationship with the elements of its monodromy group. All functions become single-valued on the universal covering.

Consider again, to fix the ideas, the case $N = 2$. Suppose that positions are sufficient to describe the particles and call x_1 and x_2 the position vectors of the first and the second particles. The covering space E^4 is the set $\{(x_1, x_2)\}$. The physical configuration space X would be the same, but with points (x_1, x_2) and (x_2, x_1) identified. Point (x_2, x_1) is obtained from (x_1, x_2) by the action of the transposition s_1: $(x_2, x_1) = s_1(x_1, x_2)$. A complex function $\Psi(x_1, x_2)$ (say, the wavefunction of the 2-particle system) will be single-valued on the covering space, but 2-valued on the configuration space. E^4/S_2 involves a cone [21] and is rather difficult to picture. To make a drawing easier to look at, we consider instead the covering related to the function $\sqrt{z}$, whose group, the cyclic group Z_2, is isomorphic to S_2. The scheme in fig. 3 shows how $\Psi(x_1, x_2)$ is single-valued on E^4,

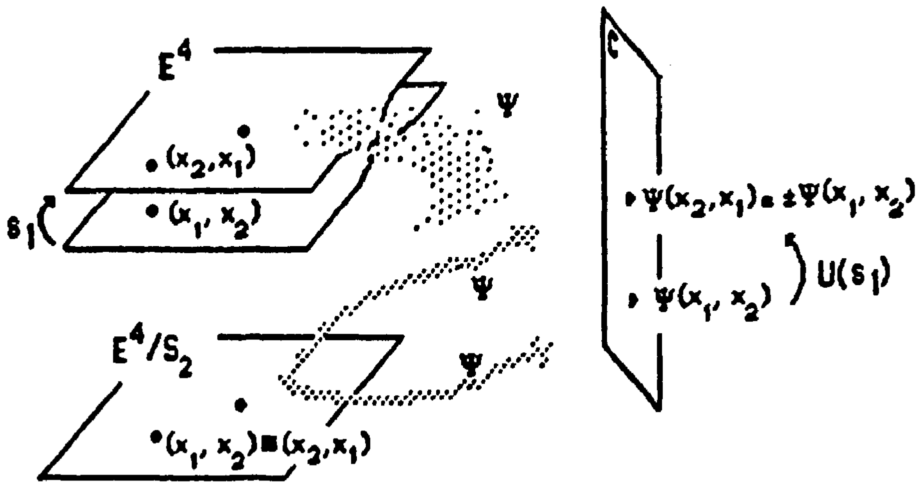


Fig. 3: Scheme of the covering space for $N = 2$.

where $(x_1, x_2) = (x_2, x_1)$, and double-valued on X, where the two values $\Psi(x_1, x_2)$ and $\Psi(x_2, x_1)$ correspond to the same point $(x_1, x_2) = (x_2, x_1)$. $\Psi(x_2, x_1) = \Psi[s_1(x_1, x_2)] = U(s_1)\Psi(x_1, x_2) = \pm\Psi(x_1, x_2)$ is obtained from $\Psi(x_1, x_2)$ by the action of an operator $U(s_1)$ representing s_1 on the Hilbert space of wavefunctions. There are two sheets because s_1 applied twice is the identity. Commonly used wavefunctions are taken on the covering space, where they are single-valued, and on which, in our scheme, particles are supposed to be distinguishable. Here the monodromy group is the whole group S_2 and two sheets appear, one for each distinct group element, 1 and s_1. This means that in (2.10) we sum over all distinct sheet contributions. An analogous treatment may be aplied to the N = 3 case (2.12), in which 6 (= 3! = number of elements of S_3) contributions come out, one for each sheet. The academic example of the "Z_N-gas" mentioned below equation (2.3) would show no difference in the N = 2 case because groups Z_2 and S_2 are isomorphic, but would exhibit quite a different covering for N = 3, as the monodromy group Z_3 would require only 3 sheets. We learn in this way what is really done when "physical" kets or wavefunctions are written in terms of distinct-particle contributions: a superposition of all the values is taken. In the analysis of the Aharonov-Bohm effect, superposition of contributions coming from two sheets are usually considered, although only at that point where the interference is supposed to be detected [22]. In reality, the (ideal) configuration space is infinitely connected and the usual treatment is to be seen as an approximation, as contributions from infinite paths belonging to all distinct homotopy classes should be taken into account. Infinite-connected cases are in general fairly complicated. We shall see below that braid statistics does require infinitely-folded covering spaces, yet normalization eliminates all but two of the infinite contributions.

4. Braid groups

Mathematicians have several definitions for braid groups [23], althought they seem to prefer that sketched in the Introduction because it holds for any manifold M and consequently lends itself more easily to generalizations. In the most suggestive of such definitions, real, usual braids are concerned and their theory is included in the (still in progress) study of general weaving patterns, which also encompasses knots and links. It is customary to call braids, when introduced in this way, *geometrical braids* and the corresponding groups, *Artin's groups* after their creator. A braid is seen [24] as a family of N non-intersecting curves $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ on the cartesian product $E^2 \times [0,1]$ with

$$\gamma_j(0) = (P_j, 0) \text{ for } j = 1, 2, \ldots, N$$
$$\gamma_j(1) = (P_{\sigma(j)}, 1) \text{ for } j = 1, 2, \ldots, N$$

where σ is an index permutation. By historical convention, the strings are to be considered as going from top to bottom. Braids are multiplied by concatenation: given two braids A and B, AB is obtained by drawing B below A. The braid group B_N consists now of all such compositions of path meshes. Fig. 4 depicts some simple braids of four

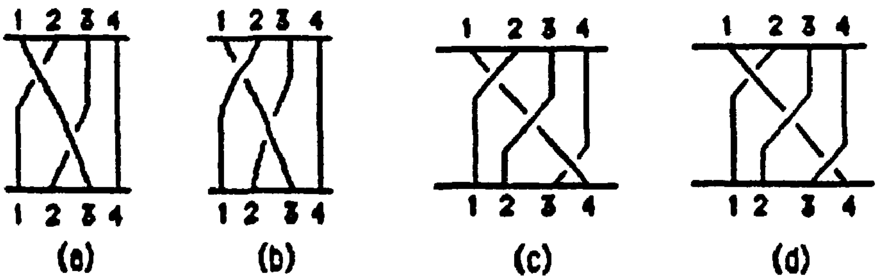


Fig. 4: Some simple examples of braids with 4 strings.

strands. Choosing on the plane E^2 four "distinguished" points and taking two copies of such punctured plane, a braid will result by linking two by two the distinguished points of the two copies with strings. Notice that in the drawings the plane E^2 is represented by a line only for the sake of facility. In 4a, the line from 2 to 1 goes down *behind* that from 1 to 3. The opposite occurs in 4b. These braids, like those of 4c and 4d, are different because they are thought to be drawn between two planes, so that the extra dimension needed to make strings go behind or before each other is available. Fig. 5b shows the trivial 4-braid, with no interlacing of strands at all.

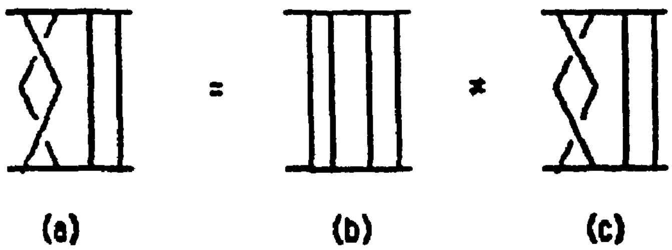
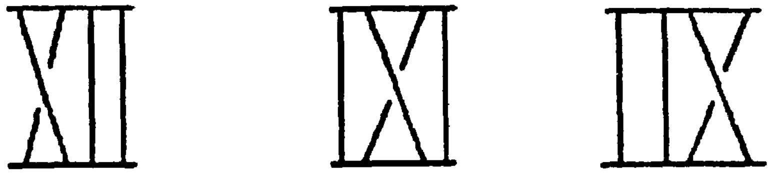


Fig. 5: Colored braids: (a) is equivalent to the identity (b), while (c) is not.

It is identical to 5a (which can be unwoven, continuously deformed into it) but quite distinct from 5c. The latter cannot be unwoven or, more formally, reduced to the trivial braid by any continuous family of deformations (isotopies) of E^2. Experiment shows that it would be possible to disentangle it if the space were E^3. Actually, any braid on E^3 may be unbraided ... as witnessed by millenia of practice with hair braids. Hair braids reduced to E^2 can be simulated by gluing together their extremities, thereby eliminating one degree of freedom.

Braids not leading to real exchange of end-points, such as those of fig.5, are called *colored* braids. The strings may be seen as time-trajectories of particles in E^2, on which "passing behind" and "passing before" correspond to distinct motions. Colored braids will correspond to closed paths on E^2, wherefrom the role of P_4 as the fundamental group of the punctured space $F_4(E^2 x E^2)$ When further the distinguished points are identified by supposing that their exchanges have no consequence, B_4 appears as the fundamental group of the identification space $F_4(E^2 x E^2)/S_4$. In the multiplication of the identity into infinite possibilities lays the essential difference between braid groups and symmetric groups.

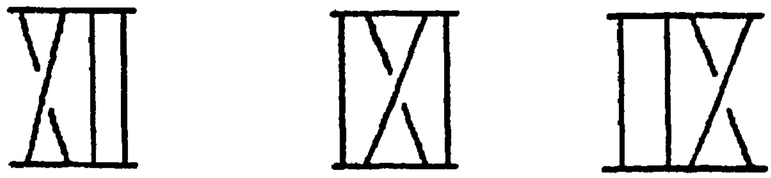
Fig 6 shows the basic, elementary steps of weaving for $N = 4$, the simplest nontrivial braids. Their inverses are shown if fig. 7. Their



σ_1 σ_2 σ_3

Fig. 6: B_4 generators corresponding to the S_4 generators of fig. 1.

respective composition yield the trivial braid, which is the neutral identity element, which changes nothing when multiplied by any braid. The product is clearly non-commutative. Any braid of 4 strands may be obtained by successive multiplications of the elementary braids σ_1, σ_2, σ_3 and their inverses. Such elementary braids are consequently said to *generate* the 4th braid group, B_4. The procedure of building



σ_1^{-1} σ_2^{-1} σ_3^{-1}

Fig. 7: Inverses to the generators, also corresponding to those of fig. 1.

general braids by products from elementary braids may be used indefinitely. The braid group is consequently of infinite order.

Impenetrability is obviously essential. If strands could traverse each other at will, all colored braids reduce to the trivial braid,

Figure 6 would be the same as Figure 1, 4a would be identical to 4b, 4c to 4d and the group would be simply S_4, with $\sigma_i = s_i$.

All this can be easily generalized to the N-th braid group B_N, whose elements are braids with N strands. Real experiments with a few strings are very helpful to give the feeling of it.

In the passage of S_N to B_N, any permutation of points becomes multiform. A colored geometrical braid is a representative of a class of loops on $F_N E^2$, that is, an element of $\pi_1[F_N E^2]$. Including exchanges of points, a general geometrical braid is a representative of a class of loops on $B_N E^2$, an element of $\pi_1[B_N E^2]$. In more precise language, the correspondence between B_N and S_N is a homomorphism of the braid group into the symmetric group,

$$h: B_N \rightarrow S_N \quad . \qquad (4.1)$$

The center of this homomorphism (that is, the elements of B_N going into the identity of S_N) is composed by the colored braids. This homomorphism "erases" the differences coming from strings going behind or before each other. Compare figs. 6 and 7 with fig. 1: both σ_i and σ_i^{-1} correspond to s_i. For the N-strand group, we may use as basis the set $\{\sigma_i\}$ of (N-1) generators which generalize the above N = 4 case. Such generators are led by this homomorphism into the elementary transpositions: $h(\sigma_j) = s_j$. They obey relations (2.5),

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad , i = 1, 2, \ldots, N-2;$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2 \; . \qquad (4.2)$$

These relations provide a presentation of the braid group and can be used as an alternative definition of B_N.

Notice the absence of a condition corresponding to (2.6): unlike the elementary exchanges of the symmetric group, the square of an elementary braid is not the identity, as illustrated by $\sigma_1{}^2$ in Fig. 5c. Going back to Quantum Mechanics, a basis for a unitary 1-dimensional representation of a braid group will be given by operators $U(\sigma_j)$ acting on wavefunctions according to $U(\sigma_j)\,\psi(x) = e^{i\varphi_j}\,\psi(x)$. Conditions (4.2) enforce the identity of all the phases. Now there is no constraint enforcing $U(\sigma_j{}^2) = 1$, so that $U^2(\sigma_j)\,\psi(x) = U(\sigma_j{}^2)\,\psi(x) = e^{i2\varphi}\,\psi(x)$, $U(\sigma_j{}^3)\,\psi(x) = e^{i3\varphi}\,\psi(x)$, etc. The representation is now, like the group, infinite. The boson and fermion cases are attained when $\varphi = 0$ and π, respectively.

5. Braid statistics

As said below (2.9), all the statistical content lies in the amplitude $\langle p_1, p_2, \ldots, p_N \mid p_1, p_2, \ldots, p_N \rangle$. In order to obtain the convenient representations for the physical kets in terms of ordered products of one-particle kets, analogous to (2.10) and (2.12), we must proceed to an analysis similar to that of fig. 3. Take again the case $N = 2$. The covering space has now infinite sheets (see fig. 8). The physical ket will have the general form $|p_1, p_2\rangle = f(\varphi)\,|p_1\rangle\,|p_2\rangle + g(\varphi)\,|p_2\rangle\,|p_1\rangle$. Contributions along $|p_1\rangle|p_2\rangle$ will come from all colored elements, those which ultimately do not exchange the particles, such as $1, \sigma_1^2, \sigma_1^{-2}, \sigma_1^4, \sigma_1^6, \ldots, \sigma_1^{2n}, \ldots, \sigma_1^{-2m}$. For instance, we may take $f(\varphi) = 1 + e^{i2\varphi} + e^{-i2\varphi} + e^{i4\varphi} + e^{-i4\varphi} + \ldots$. As to $g(\varphi)$, it will receive all particle exchanging contributions, those coming from

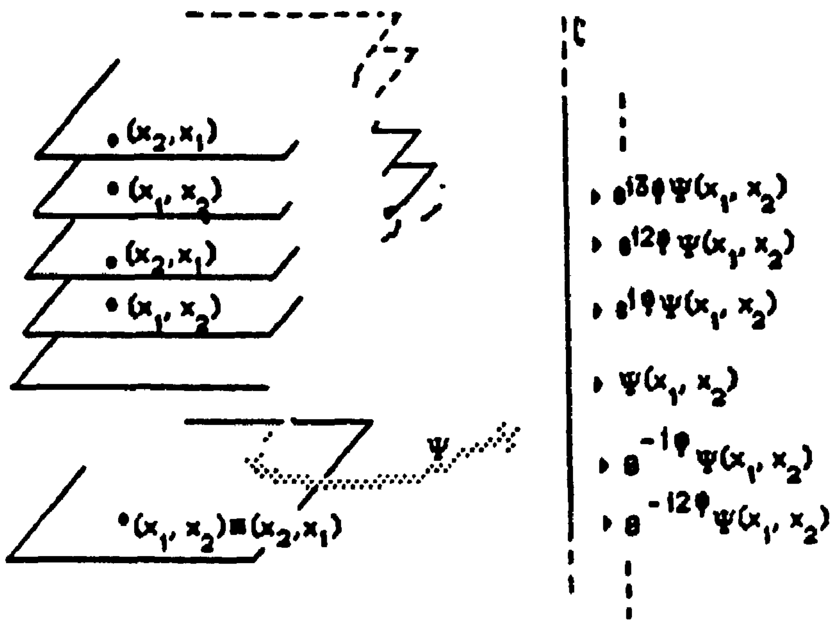


Fig. 8. The infinite unfolding of the 2-particle configuration space for braid statistics.

the odd powers of σ_1: $g(\varphi) = e^{i\varphi} + e^{-i\varphi} + e^{i3\varphi} + e^{-i3\varphi} + \ldots = e^{i\varphi} f(\varphi)$. Actually, there is an arbitrariness in the choice of starting sheet, that corresponding to the identity element in the infinite foliation. This arbitrariness is reflected in the indeterminacy of the series $f(\varphi)$ and $g(\varphi)$. We may, for instance, recollect the terms in such a way that also $f(\varphi) = e^{i\varphi} g(\varphi)$. The important fact remains that we can always choose for $f(\varphi)$ some real though indeterminate series keeping with $g(\varphi)$ the relation $g(\varphi) = e^{i\varphi} f(\varphi)$. As a consequence,

$$|p_1, p_2\rangle = f(\varphi)\,|p_1\rangle\,|p_2\rangle + e^{i\varphi} f(\varphi)\,|p_2\rangle\,|p_1\rangle,$$

which can be normalized to become

$$|p_1, p_2\rangle = \frac{1}{2}\left[\,|p_1\rangle|p_2\rangle + e^{i\varphi}|p_2\rangle|p_1\rangle\,\right] . \qquad (5.1)$$

The indeterminacy has been eliminated. Of course, this reduces to (2.10) when $\varphi = 0$ and π and we fall back into the penetrability case $\sigma_i^2 = 1$. An analogous though far more involved analysis leads to

$$|p_1, p_2, p_3\rangle = \frac{1}{3!}\left[\,|p_1\rangle|p_2\rangle|p_3\rangle + e^{i\varphi}|p_2\rangle|p_1\rangle|p_3\rangle + e^{i\varphi}|p_1\rangle|p_3\rangle|p_2\rangle + \right.$$
$$+ e^{i\varphi}|p_3\rangle|p_2\rangle|p_1\rangle + e^{i2\varphi}|p_2\rangle|p_3\rangle|p_1\rangle +$$
$$\left. + e^{i2\varphi}|p_3\rangle|p_1\rangle|p_2\rangle\,\right] , \qquad (5.2)$$

generalizing (2.12). In this way a realization of the physical kets in terms of products of distinguished-particle kets is obtained, with the symmetric group still selecting the terms. The coefficients are, after normalization, simply products of terms $e^{i\varphi}$ corresponding to the number of transpositions. Because such a realization is still feasible, the symmetric group will keep a fundamental role and the general lines of S 2, with cycle decompositions and indicator polynomials, will remain valid.

As a consequence of (5.1),

$$\langle p_1, p_2|p_1, p_2\rangle = \frac{1}{2}\left[\,\hat{\delta}_1^{\,2} + \cos\varphi\,\hat{\delta}_2\right], \qquad (5.3)$$

and of (5.2),

$$\langle p_1, p_2, p_3 | p_1, p_2, p_3 \rangle = \frac{1}{3!}[\hat{\delta}_1{}^3 + 3\cos\varphi\, \hat{\delta}_1\hat{\delta}_2 + 2\cos^2\varphi\, \hat{\delta}_3]. \tag{5.4}$$

The sum over distinct classical configurations is restored, with a generalization of the "effective statistical interaction": with respect to the boson and fermion cases, the signals are replaced by $\cos\varphi$. We see in this way that the purely combinatorial aspects remain the same as for S_N, the amplitudes keeping their cycle decomposition character. The canonical partition function will follow easily and only the b_j in (2.3, 15) will change. We may obtain $\langle p_1, p_2, \ldots, p_N | p_1, p_2, \ldots, p_N\rangle$ by using the recursion relation (2.2). The general result is

$$\langle p_1, p_2, \ldots, p_N | p_1, p_2, \ldots, p_N \rangle =$$

$$\frac{1}{N}\left[(\cos\varphi)^{N-1}\hat{\delta}_N + \sum_{s=1}^{N-1} (\cos\varphi)^{N-s-1}\hat{\delta}_{N-s}\langle p_1, p_2, \ldots, p_s | p_1, p_2, \ldots, p_s\rangle\right] \tag{5.5}$$

and a simple rule results: in order to obtain the formal configuration integrals starting from those of the symmetric group, it is enough to make the substitution $(\pm)^{j-1} \to \cos^{j-1}\varphi$

$$Q_N(\beta, V) = \frac{1}{N!}\ C_N\left\{\frac{\cos^{j-1}\varphi}{j}\ \frac{S}{\lambda^2}\right\}, \tag{5.6}$$

a simple indeed generalization of (2.15). The pressure, calculated through the grand canonical partition function Ξ will have the following equivalent expressions:

$$\frac{P}{kT} = \ln \Xi = \sec\varphi \sum_E \ln[1 - \cos\varphi \; z \; e^{-\beta E}] =$$

$$= \sec\varphi \; \frac{4\pi m}{h^2} \int_0^\infty dE \ln[1 - \cos\varphi \; z \, e^{-\beta E}] =$$

$$= \frac{S}{\lambda^2} \sec\varphi \sum_{j=1}^{\infty} \frac{1}{j^2} (z \cos\varphi)^j = \frac{S}{\lambda^2} \sec\varphi \; g_2(z \cos\varphi) \quad . \qquad (5.7)$$

The number concentration will be

$$n = \sum_E \frac{1}{z^{-1} e^{\beta E} - \cos\varphi} = \frac{1}{\lambda^2} \sec\varphi \; g_1(z \cos\varphi), \qquad (5.8)$$

from which

$$n\lambda^2 \cos\varphi = g_1(z \cos\varphi) = -\ln(1 - z\cos\varphi),$$

or

$$z \cos\varphi = 1 - e^{-n\lambda^2 \cos\varphi} \qquad (5.9)$$

Notice the ground state occupancy:

$$N_0 = \frac{z}{1 - z\cos\varphi} \qquad (5.10)$$

Strictly speaking, condensation only appears in the bosonic case, but a high ground-state concentration of particles can be attained whenever $\cos\varphi$ approaches the value +1.

The equation of state becomes

$$\frac{p}{nkT} = \frac{1}{n\lambda^2 \cos\varphi}\, g_2\left[1 - e^{-n\lambda^2 \cos\varphi}\right]. \qquad (5.11)$$

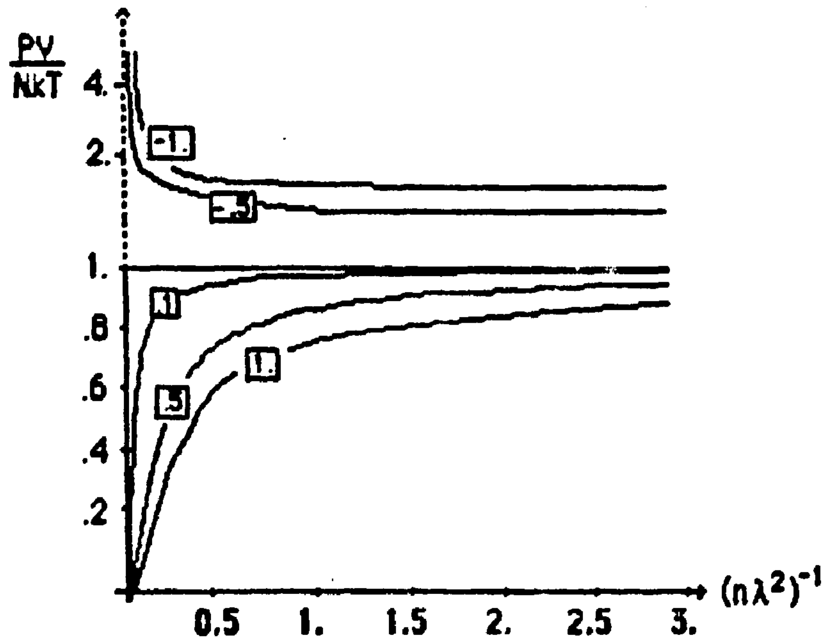


Fig. 9. General trend of the equations of state for different values of $\cos\varphi$ (notice variable scale).

This includes as extreme cases both the bosonic ($\cos\varphi = 1$) and the fermionic case ($\cos\varphi = -1$). In reality, also the Boltzmann case is included ($\cos\varphi = 0$). Fig. 9 shows how the equation of state changes progressively with the value of $\cos\varphi$, with the remarkable intermediate classical case $\frac{P}{nkT} = 1$. Actually, all the physical quantities interpolate in a very simple way those of § 2. The specific heat is of special interest, as the eventual presence of a lamba structure could signal condensation and the onset of superfluidity. The internal energy is

$$U = -\left[\frac{\partial}{\partial\beta} \ln \Xi\right]_{z,S} = P S = \frac{S\,kT}{\lambda^2 \cos\varphi}\, g_2\left[1 - e^{-n\lambda^2 \cos\varphi}\right]. \qquad (5.12)$$

By calculating $C_S = \left(\frac{\partial U}{\partial T}\right)_{N,S} = \left(\frac{\partial P\,S}{\partial T}\right)_{N,S}$, we find

$$\left(\frac{C_S}{Nk}\right)_\varphi = \frac{2}{n\lambda^2 \cos\varphi}\, g_2\left[1 - e^{-n\lambda^2 \cos\varphi}\right] - n\lambda^2 \cos\varphi \,\frac{1}{(e^{n\lambda^2 \cos\varphi} - 1)} .$$

(5.13)

This expression is of the form (2.28), now with $u = n\lambda^2 \cos\varphi$. As a consequence, the symmetries found in § 2 reappear here: $\left(\frac{C_S}{Nk}\right)_\varphi$ is the same for $\cos\varphi$ with opposite signs. As $|\cos\varphi|$ tends to zero, the 2-dimensional Dulong-Petit limit $\left(\frac{C_S}{Nk}\right) = 1$ is approached. Figure 10 depicts for $\cos\varphi = \pm 1, \pm 0.5, \pm 0.2$ and 0. There is no sign of lamba point even in the bosonic case. Consequently, even if some condensation come to take place for $\cos\varphi \approx -1$, no abrupt transition is to be expected. Detailed numerical analysis confirms the trend shown in the figure: starting from the fermionic case, the specific heat curve is continuously deformed as $\cos\varphi$ tends to zero, reaches

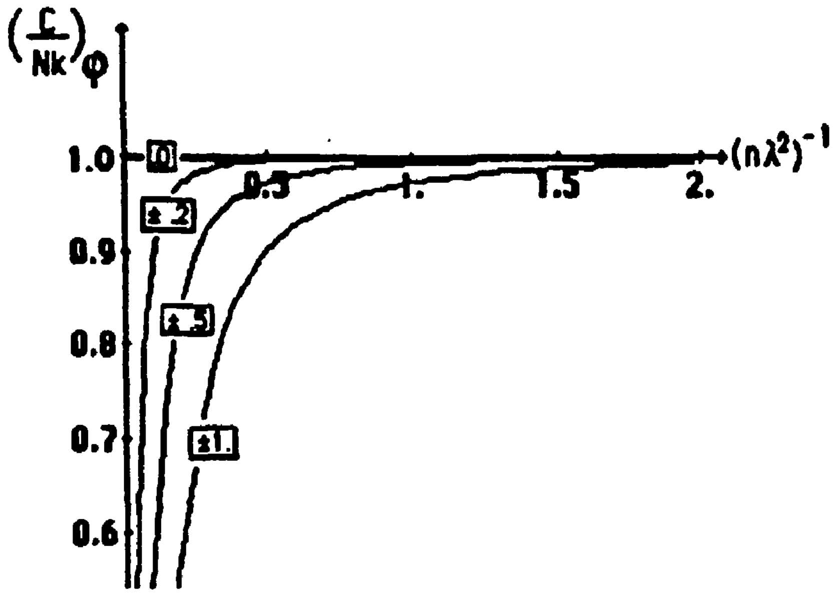


Fig. 10: Specific heat curves, identical for opposite values of $\cos\varphi$.

the straight horizontal line at this limit and then retraces back its way down to the bosonic case, identical to the starting point.

6. Final comments

There is a different statistics for each value of the angular parameter φ, which is in principle totally arbitrary. Notice however that $\varphi = 0$ and $\varphi = \pi$ correspond to penetrable particles. On the other hand, braid gases interpolate between bosons and fermions in such a way that Boltzmann particles stay in the middle, $\varphi = \pi/2$, a curious intermediate case with distinguishable, classical particles. Taking some risk in forwarding an interpretation, we might take φ as a measure of "penetrability" and, consequently, of topological "puncturedness": the more φ departs from the extreme values, the less are the particles allowed to penetrate each other, utmost impenetrability standing in the middle. Quantum effects (like, say, degeneracy) come precisely from forced superposition of individual particle wavefunctions. Highest impenetrability forbidding such superposition, it would be natural to find it related to classical behavior.

We might inquire into possible relationship to that other well-known case of exotic statistics, parastatistics. Parastatistics is also characterized by a parameter, its order. The grand canonical partition function for a parafermion ideal gas of order r is given by [25]

$$\ln \Xi = \sum_{E} \ln \left[\sum_{n=0}^{r} z^{n} e^{-\beta E n} \right] . \qquad (5.14)$$

The order r is the maximum occupation number per state. Fermions correspond to r = 1, bosons to r = ∞. In this way, parastatistics interpolates between fermions and bosons for r integer in the interval [1, ∞]. Nevertheless, the gases remain quantal at all intermediate values of r, while the interpolation given by φ includes a Boltzmmann case. Parastatistics involves well-defined representations of the symmetric group, which is not the case for braid statistics. The two interpolations are so of different characters.

As a final point, let us recall that, in the case of usual superconductivity, it was London's remark about the lambda structure in ideal boson gas which triggered the idea that some kind "bosonization" played a fundamental role in the phenomenon. However, the absence of a lambda point by no means excludes the possibility of phase transitions in real cases when dynamics become dominant. In reality, there seems to be a good theoretical evidence [26] in favor of its presence.

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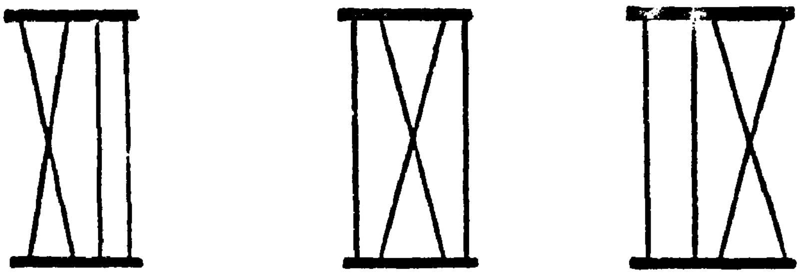
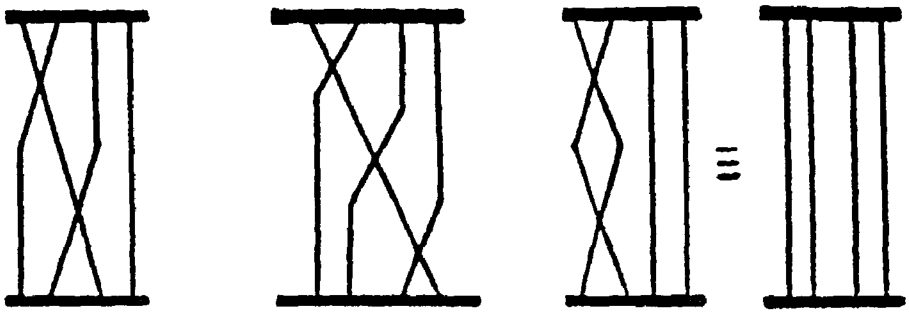
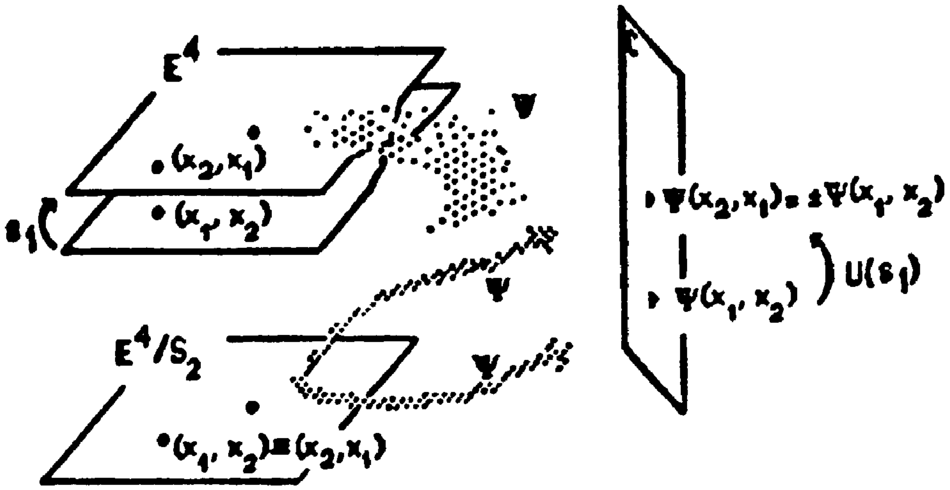
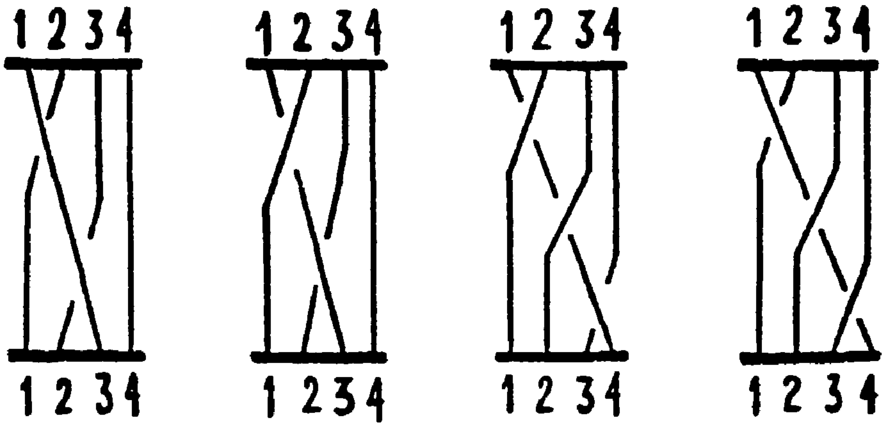


S_1 S_2 S_3

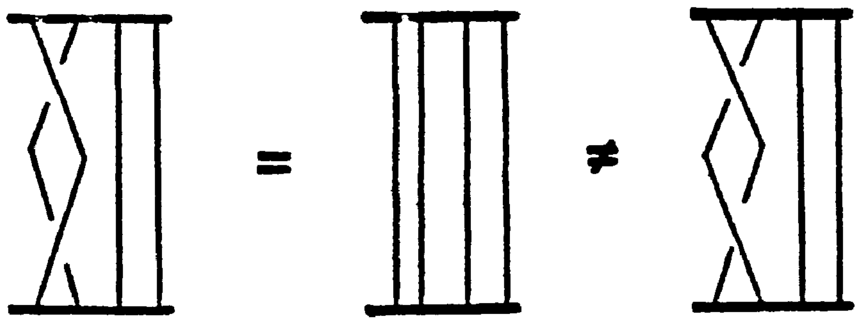


$S_1 S_2$ $S_1 S_2 S_3$ $S_1^2 = 1$

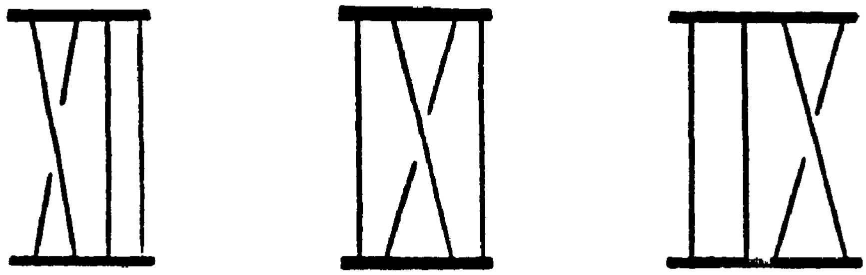

E4
(x2, x1)
s1
(x1, x2)
Ψ
Ψ
E4/S2
Ψ
(x1, x2)≡(x2, x1)
Ψ(x2, x1)= ±Ψ(x1, x2)
Ψ(x1, x2)
U(s1)



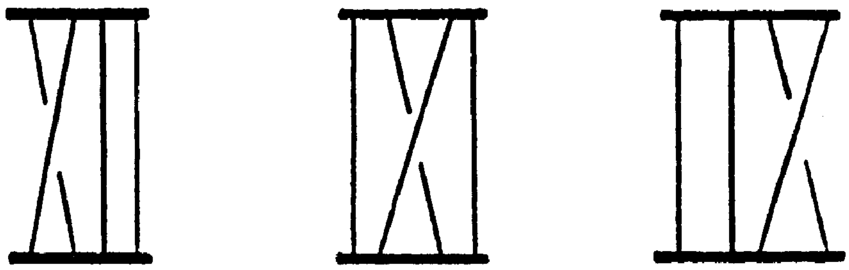
(a) (b) (c) (d)



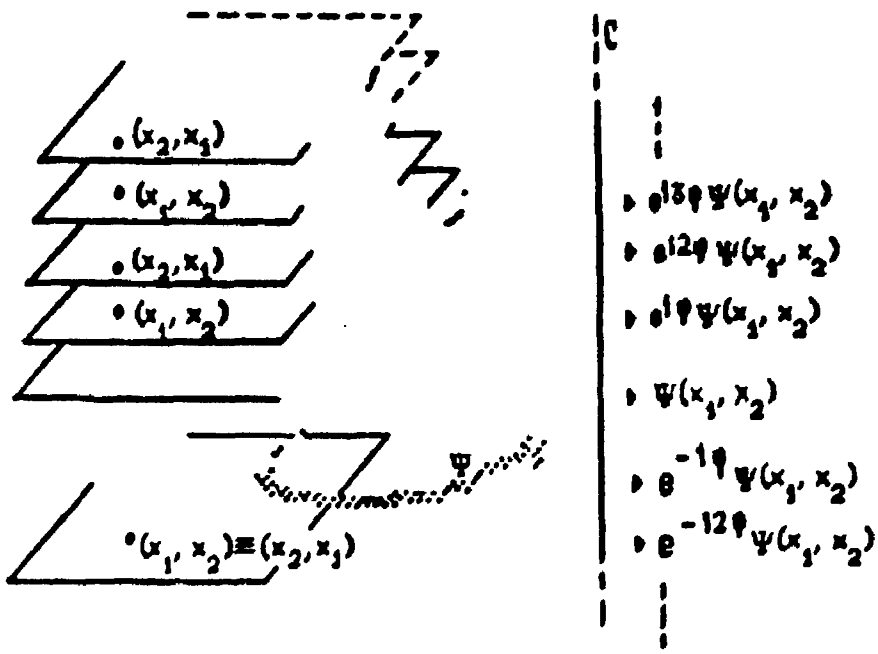
(a) (b) (c)



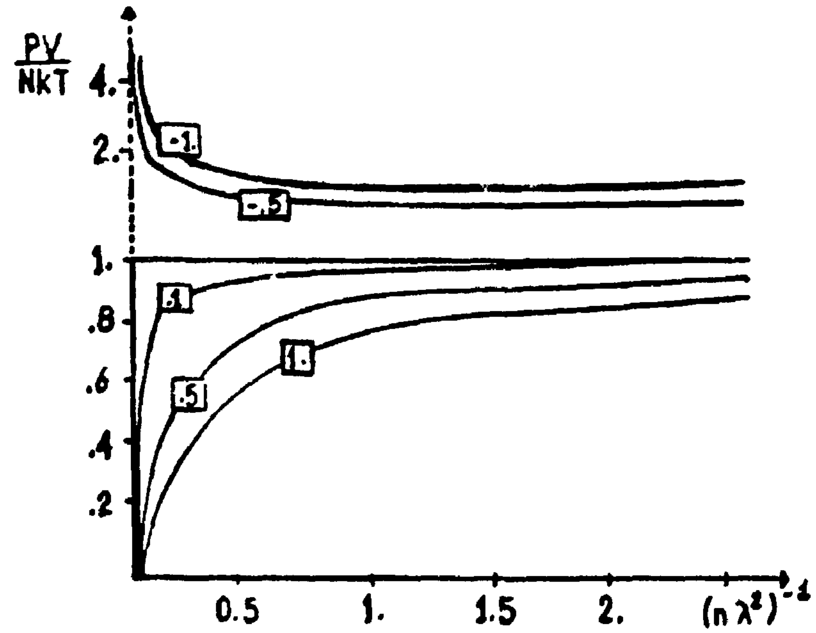
σ_1 σ_2 σ_3



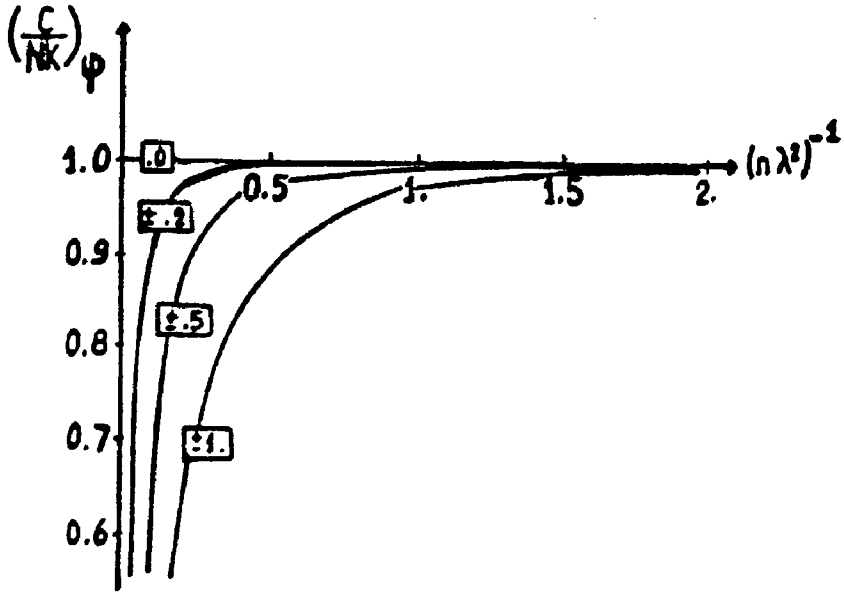
σ_1^{-1} σ_2^{-1} σ_3^{-1}



C
(x2, x1)
(x1, x2)
(x2, x1)
(x1, x2)
Ψ
(x1, x2) ≡ (x2, x1)
e^{i3φ} Ψ(x1, x2)
e^{i2φ} Ψ(x1, x2)
e^{iφ} Ψ(x1, x2)
Ψ(x1, x2)
e^{-iφ} Ψ(x1, x2)
e^{-i2φ} Ψ(x1, x2)



PV/NKT
4.
2.
-1.
-.5
1.
.1
.8
.6
.5
1.
.4
.2
0.5
1.
1.5
2.
$(n\lambda^2)^{-1}$



$\left(\frac{C}{Nk}\right)_\psi$
$(n\lambda^2)^{-1}$
1.0
0.9
0.8
0.7
0.6
0.5
1.
1.5
2.
.0
±.2
±.5
±1.

LEGENDS

Fig. 1: Diagrams of the S_4 generators.

Fig. 2: Elements of S_4 are obtained by concatenation.

Fig. 3: Scheme of the covering space for $N = 2$.

Fig. 4: Some simple examples of braids with 4 strings.

Fig. 5: Colored braids: (a) is equivalent to the identity (b), while (c) is not.

Fig. 6: B_4 generators corresponding to the S_4 generators of fig. 1.

Fig. 7: Inverses to the generators, also corresponding to those of fig. 1.

Fig. 8: The infinite unfolding of the 2-particle configuration space for braid statistics.

Fig. 9: General trend of the equations of state for different values of $\cos\varphi$ (notice variable scale).

Fig. 10: Specific heat curves, identical for opposite values of $\cos\varphi$.