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SPIN STRUCTURES ON ALGEBRAIC CURVES AND THEIR APPLICATIONS IN STRING THEORIES

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ABSTRACT

The free fields on a Riemann surface carrying spin structures live on an unramified r-covering of the surface itself. When the surface is represented as an algebraic curve related to the vanishing of a Weierstrass polynomial, its r-coverings are algebraic curves as well. We construct explicitly the Weierstrass polynomial associated to the r-coverings of an algebraic curve. Using standard techniques of algebraic geometry it is then possible to solve the inverse Jacobi problem for the odd spin structures. As an application we derive the partition functions of bosonic string theories in many examples, including two general curves of genus three and four. The partition functions are explicitly expressed in terms of branch points apart from a factor which is essentially a theta constant.

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1. INTRODUCTION

In recent times we have witnessed a large number of applications of algebraic curves with abelian group of symmetry Z_n in the framework of string theories, integrable models and conformal field theories on higher genus surfaces. In the case of string theories, the use of Z_n-symmetric curves as an explicit representation of Riemann surfaces has led to some interesting physical results. Let us mention here the proof that the cosmological constant vanishes up to genus two in superstring theory [1-3] and the method of [4] for the study of strings at very high energies. Moreover the first explicit derivation of four graviton amplitudes in perturbative superstring theories implementing the modular invariance and checking the factorization properties, has been performed using the hyperelliptic curves [5].

In the sector of conformal field theories it is possible to express directly in terms of branch points the correlation functions for the minimal models and their supersymmetric generalizations on Riemann surfaces of genus one and two [6-7]. The problem of finding the correct paths of integration for the screening charges is in fact easily solved on algebraic curves once a basis of branch cuts and nontrivial homology cycles is known.

Another interesting result consists in the proof done on hyperelliptic curves that the Beilinson Manin formula [8] is independent of the chosen odd spin structure [9,10]. Let us finally remember the Knizhnik's conjecture [11,12] that the sum over all higher loops in string theory is equivalent to a conformal field theory with the insertions of conformal operators, the socalled twist fields, which simulate the behavior of the Green functions near the branch points.

As we have already stressed, all the wisdom about field theories on algebraic curves was mainly based on certain classes of curves with abelian symmetry group. Already for $n > 2$ the problem of the spin structures was untouched apart from the Z_n symmetric characteristics of refs. [13,14] which are however very peculiar. Only recently the explicit form of the Green functions for the free scalar fields and the b-c systems on general algebraic curves has been obtained [15,16] as a first step toward the non abelianess. A first application was the confirmation of the Knizhnik's conjecture also for "general" curves of genus 3 and 4. With "general" we mean here a curve depending of a number $3g-3$ of branch points spanning a portion of the moduli space which is not of measure *zero*. The procedure followed in [15,16] has the advantage that it is easily generalizable to any curve. Another proof of the Knizhnik's conjecture for a non abelian curve of genus 1 appeared in ref. [17] where a different approach was adopted.

The second natural step in the scheme outlined above is represented by the construction of the Green functions for the fermionic fields of superstring theory. The aim is to detect the presence of twist fields in the amplitudes also in the supersymmetric case. We succeeded in doing this just for the Riemann surfaces of [4] due to their intrinsic relationships with hyperelliptic curves [18]. Despite of their simplicity, these curves are very significative because they can be imagined as the world sheet of strings at any genus g when very high energies are involved. The extension of the results contained in ref. [18] to any curve is a challenging task since one has to cope with the inverse Jacobi problem (IJP) [19] emerging also in the language of theta functions. To solve the IJP on a algebraic curve is almost impossible because it implies a set of complicated nonlinear algebraic equations [20].

Another topic which is still not very much developed is the construction of the partition function of string theories on algebraic curves. This is an old problem of string theory which was solved in [21,22] on Riemann surfaces of genus two and three using an explicit parameterization of the moduli space by the period matrix. However the form of the integration measure when $p = 4$ is still a conjecture due to the Schottky problem [11]. On a "general" algebraic curve the difficulty arises when computing the chiral determinants. The stress energy tensor method [23], which is successful in the case of Z_n symmetric curves, leads in fact to a system of partial differential equations which is in practice impossible to solve [15]. One way in order to overcome the problem, consists in the use of the Beilinson Manin formula [8] as in [24]. In doing this we need however many ingredients. The first ingredient is a holomorphic half differential $\nu_*(z)dz^{1/2}$ corresponding to an odd spin structure $*$. The second are the positions on the curve of its zeros which solve the IJP for the spin structure $*$. Finally we require a basis for the holomorphic differentials and the holomorphic quadratic differentials normalized as explained in [25]. Regarding the abelian differentials we can apply the algorithm of [15] provided we know a basis of homology cycles. In the case of quadratic differentials the situation is more complicated. We have to rely heavily on the work of [26], choosing the branch points to parametrize the moduli space. After that, exploiting the variational formulas of [26], it is possible to repeat the procedure developed on [9] for the hyperelliptic curves and to derive the normalized holomorphic quadratic differentials.

All the above motivations compel to the study of spin structures and their generalizations living on a r-sheeted unramified covering of a Riemann surface. Particular care is devoted to the solution of the IJP for the odd spin structures. We realize that the r-coverings of algebraic curves defined by the spin structures are again algebraic curves determined by two groups. The first is the monodromy group of the underlying Riemann surface. The monodromy properties of the Green functions of free field theories in presence of spin structures seem therefore very likely to be simulated by the twist fields as in Knizhnik's conjecture. Nearby the monodromy group, we have however also an abelian group of symmetry since the r-covering of a curve related to a Z_r spin structure has an intrinsic cyclic symmetry Z_r. The g points z_i, which solve the IJP for a given Z_r spin structure, are degenerate branch points of the r-covering. Due to the fact that an r-covering should be realized in an unramified way, the points z_i do not introduce any new ramification. This characteristic of the degenerate branch points is well explained in [27]. What matters here is that the behavior of Green functions near the points z_i is very difficult to realize in terms of conformal operators and probably the Knizhnik conjecture fails in the case of field theories carrying spin structures. Also in the simple case of Z_n symmetric spin structures, in which the points z_i coincide with the branch points of the curve, the Green functions show a very complicated structure which seems impossible to reproduce using the twist fields [28].

As an application of the spin structures we discuss in this paper the partition functions of bosonic string theory in the language of algebraic curves. In many examples, including two "general" surfaces of genus 3 and 4, we derive the partition function in terms of branch points apart from a factor which is essentially a theta constant. This fact does not represent a problem in our context. In principle the theta constants can be expressed through the elements of the period matrix. The latter has a direct expression in terms of branch points once the abelian differentials and the non trivial homology cycles are known. Therefore the Schottky problem is absent on algebraic curves. Of course it would be important to have a more compact form for the theta constants. Unfortunately the evaluation of the theta constants, especially for the odd spin structures which are complicated by the presence of zero modes, is an old problem of algebraic geometry which was never solved completely [29].

Let us notice however that the motivation for studying free fields on algebraic curve does not merely consist in constructing the amplitudes of string theories loop after loop. The goal is instead to provide an explicit meaning to the rather abstract geomeric formulation of string theory, involving for example quantities likke the chiral determinants. When it is possible to put these quantities in a relatively simple form as in the case of hyperelliptic curves or of the curves discussed here, one can give a glance inside the structure of conformal field theories on higher genus surfaces to understand them better. Otherwise there is always the support of numerical methods as it happens in many other physical theories. Between the various possible applications of algebraic curves beyond strings, we remember the Riemann problem, whose solution is tied to the solution of the IJP [30,31], and the connections with the multistring amplitudes on non-abelian orbifolds [32-35]. Moreover the conformal field theories on the complex plane are intimately related with the b-c systems on Riemann surfaces [36,37]. Finally the Coulomb gas representation method allows the construction of more complicated conformal field theories on algebraic curves [[6,7].

The material presented in this paper is divided as follows: In section 2 we show how to compute the non trivial homology cycles for two relevant classes of Riemann surfaces. We discuss the way in which the modular group acts on algebraic curves. In section 3 we develope the general formulation of the Z_n spin structures on algebraic curves, showing that the related r-coverings are again algebraic curves. We also discuss a generalization to non abelian coverings in analogy with [38]. An algorithm to solve the IJP on algebraic curves is derived is section 4 and applied to the Z_n symmetric characteristics and to the odd spin structures. In section 5 we exploit the Beilinson Manin formula [8] and the variational formulas of [26] in order to derive the partition function of bosonic string theory on generic algebraic curves of any genus. The example of a Z_3 symmetric curve is fully worked out and the expression of the theta constants for an odd spin structure derived through a set of linear equations. Finally section 6 is dedicated to the evaluation of the partition functions on general algebraic curves of genus 3,4 and 5. Unfortunately the curve of genus 5 depends only of $3g-4$ moduli and therefore is not "general". A nice method due to Baker to find the Weierstrass polynomial for an algebraic curve with given genus and number of sheets is also introduced.

2. EXPLICIT CONSTRUCTION OF RIEMANN SURFACES

In this paper we represent the closed and orientable Riemann surfaces in terms of algebraic curves associated to a Weierstrass polynomial of the kind:

$$F(z,y) = y^n + P_{n-1}(z)y^{n-1} + P_{n-2}(z)y^{n-2} + \ldots + P_0(z) = 0 \tag{2.1}$$

where y and z are complex variables. A particular interesting and simple case [10-14,37,39] is provided by the Z_n symmetric algebraic curves associated with the equation:

$$y^n = -P_0(z) \tag{2.2}$$

The $P_i(z)$ are rational functions of the variable $z \in CP_1$:

$$P_i(z) = \frac{\sum_{k=1}^{m} \beta_{k,i} z^k}{\sum_{k'=1}^{m'} \gamma_{k',i} z^{k'}} \tag{2.3}$$

The branch points $\alpha_i, i = 1, \dots, n_{tot}$ are defined as the solutions of the system of equations:

$$\begin{cases} F(z,y) = 0 \\ \partial_y F(z,y) = 0 \end{cases} \tag{2.4}$$

The number of branch points at finite values of z can be derived eliminating the variable y in eqs. (2.4). To this purpose we can use for example the algorithm of Sylvester [40]. It yields the polynomial equation $D(z) = 0$, where $D(z)$ is the resultant of the two eqs. (2.4). The roots of $D(z)$ specify the positions of the branch points when $z < \infty$. The degree of $D(z)$ provides the number of finite branch points. The branch points at $z = \infty$ can be detected using the technique explained in [15]. When eq. (2.4) is fulfilled for a given value of z together with the relation:

$$\partial_z F(z,y) = 0$$

then z is called a singular branch point. We will suppose throughout this paper that all branch points are regular. At some branch point $F(z,y)$ can vanish together with its derivatives of order $\nu - 1$ in the variable y. In this case ν is called the multiplicity of the branch point.

Solving eq. (2.1) in y, we get a multivalued function $y(z)$. The branches of $y(z)$ are denoted by $y^{(l)}(z), l = 0, \dots, 1$ and are the roots of the polynomial (2.1):

$$F(y,z) = \prod_{i=0}^{n-1} (y - y^{(l)}(z))$$

When $y(z)$ is transported along a closed small cycle surrounding a branch point α of multiplicity ν, the branches of y undergo a permutation of order ν. If we form a vector out of the branches of y: $\vec{y}^{(l)}(z) = (y^{(0)}(z), .., y^{(n-1)}(z))$, then we can express such permutations by means of a matrix M_α, the so called monodromy matrix (see [15] and references therein). A Riemann surface Σ [(1)] can be constructed in terms of sheets and branch points demanding that the function $y(z)$ is one-valued on Σ. The details on how the Riemann surface is actually constructed are contained in classical textbooks on the subject such as [27,40-42] etc.

The explicit construction of an algebraic curve in terms of sheets and branch points is a difficult technical problem. The usual strategy [27,40] consists in drawing a set of branch lines joining the branch points and in cutting the sheets along these lines. After that the sheets are joined at the edges of the cuts. It is clear that the choice of the branch lines is not arbitrary. It should respect the monodromy properties of $y(z)$ providing the exact sequence with which the branches are interchanged at the branch points. In a general case we do not know exactly such a sequence either because $y(z)$ has a too complicated analytic continuation or simply because eq. (2.1) is not analytically solvable in y. Fortunately, most of the quantities entering in the amplitudes of free fields on a Riemann surface can be constructed out from the Weierstrass polynomial of eq. (2.1). An example is provided by the Green functions of the b-c systems [15]. However the two point functions of the free scalar fields, also entering in string theories, and moreover the discussion on the spin structures, require the knowledge of a canonical basis of homology cycles [15] and therefore of a system of branch lines. Only in this case, in fact, we can use the method of regularization of [14,41] which reduces the algebraic curve to the canonical polygon with 4g sides of fig.1. The canonical polygon is of course intersected by the branch lines. It

(1) The phrases Riemann surface, algebraic curve or simply curve will be used interchangeably from now on.

is therefore simple to draw the homology cycles on the polygon and to see in which way they cross the branch lines in the original algebraic curve. An algorithm to construct a canonical system of branch lines exists just for the following two classes of algebraic curves:

a) the function $y(z)$ generating the curve is realized in terms of radicals:

$$y(z) = \sqrt[\nu_1]{p_1(z) + \sqrt[\nu_2]{p_2(z) + \ldots} + \ldots} \pm \sqrt[\nu_k]{p_k(z) + \ldots} + \ldots$$

and all the points at which the roots of order ν_k vanish are branch points. In this case the branch points of multiplicity ν_k associated to each root are connected by the branch lines as on a Z_n-symmetric algebraic curve. It is easy to proof that for such Riemann surfaces the ratios $\frac{n}{\nu_i}$, $(i = 1, \ldots, n_{tot})$, of the number of sheets divided by the multiplicity of the branch points is always an integer.

An example of a multivalued function which does not fit in group a) is:

$$y(z) = \sqrt[3]{P_0(z) + \sqrt[2]{P_0(z)^2 + P_1(z)^3}} + \sqrt[3]{P_0(z) - \sqrt[2]{P_0(z)^2 + P_1(z)^3}}$$

satisfying the algebraic equation:

$$y^3 + 3P_1(z)y - 2P_0(z) = 0$$

The points in which the two cubic roots vanish are zeros of order three and therefore cannot be branch points. In this case the analytic continuation of $y(z)$ on the sheets becomes difficult and it is not very easy to find a system of branch lines.

b) General algebraic curves in which all the branch points are simple: All the branch points have multiplicity two and we can apply the Lüroth-Clebsch theorem [40]. Consequently the surface can be taken in such a form that there is a single branch line between consecutive sheets connecting just two branch points and between the last two sheets there are $g + 1$ branch lines joining the remaining $2g - 2$ branch points pairwise.

In figs. 2,3,4 we show a canonical system of nontrivial homology cycles for the hyperelliptic curves, the Z_3 symmetric curves $y^3 = \prod_{i=1}^{n} \frac{(z-\alpha_i)}{(z-\alpha_i')}$ [43] and for the surfaces of class b) [27] respectively. The relevance of the curves of class b) lies in the fact that a general algebraic curve and therefore each abstract Riemann surface is conformally equivalent to one in which the branch points are simple.

We discuss now the modular transformations on algebraic curves. On Σ the integrals over the nontrivial homology cycles of differentials can be reduced to definite integrals having the branch points as extrema (see also figs. 2,3,4):

$$\oint_{C_i} \omega_i(z)dz = \sum_{i,j} \int_{\alpha_i}^{\alpha_j} \omega_i(z)dz$$

As a matter of fact, all the nontrivial homology cycles are closed paths surrounding two or more branch points. If we interchange the branch points in the equation above, then we get another homology cycle. This is clear on the surfaces belonging to group b) because they strongly resemble the hyperelliptic curves. It is possible to use now the free parameters

$\beta_{k,i}$, $\gamma_{k',i}$ in eq. (2.3) to specify the positions of some of those branch points. In fact we can reduce eq.(2.1) to the following form:

$$F(z,y) = \sum_{r,s} c_{r,s} y^r z^s \tag{2.5}$$

The coefficients $c_{r,s}$ are computed from the determinant:

$$F(z,y) = det \begin{vmatrix} y^n & zy^{n-1} & y^{n-1} & \dots & z & 1 \\ y^n_{\alpha_1} & \alpha_1 y^{n-1}_{\alpha_1} & y^{n-1}_{\alpha 1} & \dots & \alpha_1 & 1 \\ ny^{n-1}_{\alpha_1} & (n-1)\alpha_1 y^{n-2}_{\alpha_1} & (n-1)y^{n-2}_{\alpha_1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \end{vmatrix} \tag{2.6}$$

Obviously on a general algebraic curve some terms of the kind $y^s z^r$ may not appear. It is nevertheless clear that using the determinant form of $F(z,y)$ it is possible to fix a number k of predetermined branch points $\alpha_1, \dots, \alpha_k$ satisfying eq. (2.4). k depends of the number of different monomials $y^s z^r$ appearing in eq. (2.5) which, for a curve of genus g, is given by the Baker's method. Unfortunately this way of specifying the positions of the branch points requires two parameters of the curve for each branch point. One parameter specifies the branch point α_i and the other the value of y at α_i. Thus in general the k branch points obtained will be just a subset of the whole set of branch points.

3. THE GENERAL FORMULATION OF THE INVERSE JACOBI PROBLEM ON ALGEBRAIC CURVES

Let us consider a flat line bundle $L^0_{\bar{u}}$ on a algebraic curve Σ defined by the characteristic:

$$\bar{u} = \begin{bmatrix} \varphi(A_1)\dots\varphi(A_g) \\ \varphi(B_1)\dots\varphi(B_g) \end{bmatrix}$$

The sections of $L^0_{\bar{u}}$ are the functions $f_{\bar{u}}(z)$ which, when transported along the non trivial homology cycles A_i, B_i, $(i = 1, \dots, g)$, pick up phases of the kind: $e^{\varphi(A_i)}$, $e^{\varphi(B_i)}$ respectively. Here we restrict to the case in which:

$$\bar{u} = \bar{u}_r = \begin{bmatrix} \frac{k_1}{r}, \dots, \frac{k_g}{r} \\ \frac{q_1}{r}, \dots, \frac{k_g}{r} \end{bmatrix}$$

where r, k_i and q_i are integers. The characteristics $\bar{u}_r$ will be called the Z_r characteristics or alternatively the Z_r spin structures. The usual spin structures occur when $r = 2$. The function $f_{\bar{u}_r}(z) = f^{(l,\omega)}_{\bar{u}_r}$ carries two indices and it is onevalued on a r-cyclic covering of Σ $\Delta_r(\Sigma)$. The index $l = 0, \dots, n-1$ is referred to the branches of $f_{\bar{u}_r}$ which interchange when the function is locally transported around a closed path surrounding a branch point. The index $\omega = 0, \dots, r-1$ takes instead into account the multivaluedness of $f_{\bar{u}_r}$, realized in a unramified way, along the nontrivial homology cycles. Let us remember that the latter can be never shrunk to small cycles surrounding a single branch point. Together with the monodromy matrices M_{α_i} we have therefore also the matrices M_{A_i}, M_{B_i} corresponding to the operators which transport $f_{\bar{u}_r}$ along the canonical cycles A_i and B_i. The commutators between these matrices applied to $f_{\bar{u}_r}^{(l,\omega)}$ generate the finite group of symmetries of $\Delta_r(\Sigma)$ which is in general not abelian apart from a few cases [13,14]. Multiplying the fields of the b-c systems with conformal weight λ with $f_{\bar{u}_r}(z)$, we get sections of the line bundle $L_{\bar{u}_r} = K^\lambda \cdot L^0_{\bar{u}_r}$ where K is the canonical line bundle [44]. These are the socalled twisted b-c systems coupled with an external abelian gauge field via an holomorphic connection [45].

We start now to construct explicitly the covering $\Delta_r(\Sigma)$ of a Riemann surface of genus g. As we will see it is associated to an algebraic equation of degree nr similar to eq. (2.1) but of a special form. The aim is to set up an algorithm which is able to compute the zeros of the function $f_{\bar{u}_r}(z)$. Since $f_{\bar{u}_r}$ can be expressed in terms of theta functions [19]:

$$f_{\bar{u}_r} = \frac{\theta[\bar{u}_r](z)}{\theta[\bar{0}](z)} \tag{3.1}$$

its zeros $\xi_1, \ldots, \xi_g$ solve the inverse Jacobi problem:

$$\sum_{j=1}^{g} \int_{p_0}^{\xi_j} \tilde{\omega}_\nu(\xi_j) = \{\bar{u}_r\}_\nu - K_\nu^{p_0} \tag{3.2}$$

In eq. (3.2) $K_\nu^{p_0}, \nu = 1, \ldots, g$ is the vector of Riemann constants calculated at the basepoint p_0, $\{\bar{u}_r\}$ is the period in the Jacobian variety of Σ corresponding to the characteristic $\bar{u}_r$ and $\tilde{\omega}_\nu$ denotes a basis of holomorphic differentials in the canonical normalization.

Instead of directly computing an algebraic equation for $f_{\bar{u}_r}(z)$, we do it for its r-th power: $g_r(z) = [f_{\bar{u}_r}(z)]^r$. The reason is that now $g_r(z)$ is onevalued on Σ and therefore it can be expressed as a rational function of z and $y(z)$:

$$g_r(z) = g_r(z, y(z)) = \frac{\sum\limits_{i,j} a_{ij} y^i(z) z^j}{\sum\limits_{k,l} b_{kl} y(z)^l z^k} \tag{3.3}$$

Eq. (3.3) is derived from a theorem of algebraic geometry [27,40], stating that the most general onevalued function on an algebraic curve is always a rational function. The theorem tells us moreover that $g_r(z)$ satisfies an algebraic equation of the kind of eq. (2.1) and has the same number of branches of y. As a consequence:

$$g_r^n + \tilde{P}_{n-1}(z) g_r^{n-1} + \ldots + \tilde{P}_1(z) g_r + \tilde{P}_0(z) = 0 \tag{3.4}$$

In particular the number of branch points of $g_r(z)$ and of $y(z)$ coincide and the branches are interchanged according to the same monodromy properties. One can notice from eq. (3.4) that the zeros of g_r are those of $\tilde{P}_0(z)$ because near a zero z_0 of $g_r(z)$ the equation above is approximated by: $\tilde{P}_1(z_0) g_r \sim -\tilde{P}_0(z_0)$. The poles of $g_r(z)$ are instead given by the singularities of the functions $\tilde{P}_i(z)$, $i = 0, \ldots, n-1$. Eq. (3.4) can be derived explicitly from eq. (3.3) computing the various powers $g_r(z), g_r^2(z), \ldots, g_r^n(z)$ and summing them up with the appropriate coefficients $\tilde{P}_i(z)$, $i = 1, \ldots, n$. The $\tilde{P}_i(z)$ are rational functions of z and should be chosen in such a way that in the sum all the terms containing $y(z)$ disappear. The result of the sum will be at the end $\tilde{P}_0$. Let us remember now that the zeros and poles of $g_r(z)$ are of order r or multiples of r, conversely $f_{\bar{u}_r}(z) = \sqrt[r]{g_r(z)}$ becomes branched at these points and the covering $\Delta_r(\Sigma)$ is no longer realized in a unramified way. Hence the coefficient a_{ij}, b_{kl} in equation (3.3) cannot be arbitrary and to find an algorithm expressing them in terms of the parameters of the curve appearing in eq. (2.3) in an explicit form is a very difficult issue (see [20] for an example when $r = 2$). For the zeros of $g_r(z)$, to which we are interested, the coefficients $\tilde{\beta}_{0,k} = \tilde{\beta}_{0,k}(a_i, a_{jm}, b_{kl})$ of $\tilde{P}_0(z)$ corresponding to the various powers in z should be such that $\tilde{P}_0$ has all roots of order r. An analogous condition has to be put on the singularities of the $\tilde{P}_i(z)$ to ensure that $f_{\bar{u}_r}(z)$ has poles of order r. Both these conditions entails $2rg$ equations in the $\tilde{\beta}_{0,k}$, $\tilde{\gamma}_{i,k}$ which are algebraic relations between the parameters a and b and the branch points. Solving this system of algebraic equations in the $a_{i,j}$, $b_{k,l}$ amounts to solve the IJP on a algebraic curve. Once we have the correct coefficients in eq. (3.3), we can take the r-th root of $g_r(z)$ in eq. (3.4) getting the algebraic equation for $f_{\bar{u}_r}(z)$:

$$f_{\bar{u}_r}^{rn} + \tilde{P}_{n-1}(z) f_{\bar{u}_r}^{r(n-1)} + \ldots + \tilde{P}_1(z) f_{\bar{u}_r}^{r} + \tilde{P}_0(z) = 0 \tag{3.5}$$

This equation can be well considered as the algebraic equation of the unramified covering $\Delta_r(\Sigma)$. Let us notice that eq. (3.5) leads to an algebraic curve with the same monodromy group of eq. (3.4). As a matter of fact $g_r(z)$ and $f_{\bar{u}_r}(z)$ are related through a root of order r which does not introduce any new branch point. Therefore the monodromy groups acting on $f_{\bar{u}_r}(z)$ and $y(z)$ are equal. Moreover eq. (3.5) has an additional Z_r symmetry $f_{\bar{u}_r} \rightarrow e^{\frac{2\pi i k}{r}} f_{\bar{u}_r}$, $k \in \mathbf{Z}$. The two groups are combined in such a way to give a non abelian group of symmetry of $\Delta_r(\Sigma)$ even in the case in which the monodromy group of the algebraic curve Σ is abelian. An exception is provided by the Z_n symmetric characteristics discussed in [13,14].

Before to give some applications of the above results, we will introduce some possible generalizations which will be useful in the following. Instead of a function $f_{\bar{u}_r}$ we can consider a λ differential $\psi_{\bar{u}_r}(z)dz^\lambda$ with poles and zeros of order r. Taking its r-th root, two different cases are distinguished:

i) λ is an integer multiple of r. This is the case of the twisted b-c systems or twisted bosons described above (see also [46-47]). They are sections of the line bundle $L_{\bar{u}_r} = K^\lambda \cdot L^0_{\bar{u}_r}$, λ integer

ii) $\frac{\lambda}{r}$ is a rational number [(2)]. These are the λ-differentials with rational conformal weight discussed in [48] in the case in which $L^0_{\bar{u}_r} = 1$. When $\frac{\lambda}{r} = \frac{1}{2}$ we have the Z_r symmetric fermions of refs. [13,14].

For the λ-differentials it is not possible to write an algebraic equation as eq. (3.5) directly. However a general meromorphic λ-differentials is always of the form [27,40]:

$$\psi_{\bar{u}_r}(z)dz^\lambda = \tilde{y}(z, y(z)) \left[\frac{dz}{F_y(z, y(z))}\right]^\lambda \tag{3.6}$$

where $\tilde{y}(z, y(z))$ is a rational function in z and y. The zeros and poles of the λ-differential $\left[\frac{dz}{F_y(z,y(z))}\right]^\lambda$ occur only at the singularities of the polynomials $P_i(z)$ of eq. (2.1). We will consider for simplicity only the case in which the $P_i(z)$ are just polynomials, i.e. $\gamma_{i,k} = 0$

(2) Since for a λ-differential the difference between the number of its zeros and poles is always: $\delta = \lambda(2g-2)$, it is clear that the ratio δ/r has to be an integer number. Conversely the covering of Σ generated by $\psi_{\bar{u}_r}(z)dz^\lambda$ is branched.

in eq. (2.3). Therefore all the singularities of $y(z)$ occur at $z = \infty$ and $\frac{dz}{F_y(z,y(z))}$ has its zeros at $z = \infty$ [3]. Consequently the degrees of $\tilde{y}$ in y and z has to be such that the poles at $z = \infty$ cancel the corresponding zeros of $\left[\frac{dz}{F_y(z,y(z))}\right]^\lambda$ at infinity and generate a λ-differential with poles and zeros of order r. It is simple to find on a algebraic curve the most general form that $\tilde{y}$ should have in y and z in order to fulfill this condition apart from some constant coefficients as we did for $g_r(z)$. At this point we can write an equation analogous to eq. (3.5) in the variable $\tilde{y}$ and proceed as before. The problem is again to choose the constants a_{ij}, b_{kl} through which $\tilde{y}(z,y(z))$ is defined in order that its zeros and poles are all of order r, of course apart from the $2g-2$ poles which are dedicated to the elimination of the unwanted zeros of the λ-differential $\left[\frac{dz}{F_y(z,y)}\right]^\lambda$. This problem, which is the analog of the IJP for the λ-differentials, leads always to systems of algebraic equations in the parameters a_{ij} and b_{kl}. As we will see in the next section the possibility to write an algebraic equation for $\tilde{y}$ suggests an algorithm to derive such parameters for many important algebraic curves in the case of the odd spin structures.

Another possible generalization of eq. (3.5) consists in the construction of unramified coverings of Σ with non-abelian group of symmetries. On this point see for example [38] and references therein. For example one could imagine to extend the equation $f_{\tilde{u}_r}(z)^r = g_r(z)$ defining a cyclic Z_r-unramified covering to the equation:

$$f_{D_r}^{2r} - 2g(z,y(z))f_{D_r}^r + 1 = g'(z,y(z)) \tag{3.7}$$

where $g(z,y)$ and $g'(z,y)$ are rational functions of z and y. Eq. (3.7) has a D_r symmetry:
Z_r symmetry:

$$(f_{D_r}, g') \rightarrow (\epsilon f_{D_r}, g') \quad \epsilon^r = 1$$

Z_2 symmetry:

$$(f_{D_r}, g') \rightarrow (\frac{1}{f_{D_r}}, f_{D_r}^{-2} g')$$

The solution of eq. (3.7) is:

$$f_{D_r} = \sqrt[r]{g \pm \sqrt{g(z)^2 - 1 + g'(z)}}$$

Using the same procedure applied before for the case of Z_r unramified coverings, a non abelian D_r covering of Σ is realized in a unramified way requiring that the functions $h_1(z) = g^2(z) - 1 + g'(z)$ and $h_2(z) = g'(z) - 1$ have zeros and poles of order 2 and n respectively. The zeros and poles of h_1 and h_2 are in fact the branch points, with multiplicity two and r resp., of the algebraic equation (3.7).

[3] In ref. [15] we have computed explicitly the divisor of dz and $F_y(z,y(z))$ for curves in which the functions $P_i(z)$ have singularities just at $z = \infty$.

4. THE EXPLICIT SOLUTION OF THE INVERSE JACOBI PROBLEM ON ALGEBRAIC CURVES

1) Z_n symmetric characteristics.

Let us rewrite eq. (2.2) in the following convenient way:

$$y^n = \frac{\prod_{i=1}^{M}(z-\alpha_i)}{\prod_{j=0}^{M'}(z-\alpha'_i)} \tag{4.1}$$

Let us also suppose for simplicity that the difference $M - M'$ is a multiple of n. In this way the point at $z = \infty$ is not a branch point in eq. (4.1). The monodromy matrices in eq. (4.1) at the branch points α_i, α'_j are $M_{\alpha_i} = M$ and $M_{\alpha'_j} = M^{-1}$ respectively, where M is the usual matrix generating the Z_n cyclic group of permutations. It is clear that the functions $(z-\alpha_i)$ and $(z-\alpha_j)$ have zeros of order n in α_i, α_j. For example $(z-\alpha_i) \sim yn$ near the point α_i as it descends from eq. (4.1). Therefore a general combination of the kind:

$$g_r(z) = \prod_i (z-\alpha_i)^{q_i} \prod_j (z-\alpha'_j)^{q'_j}$$

has zeros and poles of order n if $\sum_i q_i + \sum q'_j = mn$, $m \in \mathbf{Z}$. The n-th root of $g_r(z)$ gives in this case:

$$f^n_{\bar{u}_n}(z) = \prod_i (z-\alpha_i)^{q_i} \prod_j (z-\alpha'_j)^{q'_j} \tag{4.2}$$

Clearly the covering originated by the above equation, which is the analog of eq. (3.5) in this simple case, is again Z_n symmetric as the surface Σ of eq. (4.1). The monodromy at the branch points are now changed by the non local effects due to these Z_n symmetric spin structures and become:

$$M_{\alpha_i} = M^{q_i} \qquad M_{\alpha'_j} = M^{q'_j}$$

These matrices generate the group of symmetries of the covering $\Delta(\Sigma)$ which is again abelian since it is the group Z_n. The computation of the Green functions and of the chiral determinants for the b-c systems carrying these Z_n symmetric spin structures is quite complicated since they have zero modes and will be treated elsewhere [28].

2) Odd spin structures.

On a Riemann surface there are $2^{2g} - 2^g$ odd spin structures for which the vectors $\{\bar{u}_{r=2}\}_\nu$, $\nu = 1,\ldots,g$ of eq. (3.2) correspond to the zeros of a theta function: $\theta[\bar{0}](\{\bar{u}_{r=2}\}_\nu) = 0$. Therefore due to the Riemann theorem [19]:

$$\{\bar{u}_{r=2}\}_\nu = \sum_{i=1}^{g-1} \int_{q_0}^{q_i} \bar{\omega}_\nu + K^{q_0}_\nu, \qquad \nu = 1,\ldots,g$$

Moreover the divisor $\xi = (q_1, \ldots, q_{g-1})$ of degree $g-1$ has multiplicity $i(\xi) = 1$ [(4)]. The divisor of the zeros of an odd theta function of the kind: $\theta[\bar{u}_{r=2}](\int_{q_0}^{x} \bar{\omega}_\nu - \int_{q_0}^{a} \bar{\omega}_\nu + K_\nu^{q_0}) = \theta[\bar{u}_{r=2}](x-a)$ is $div(\theta[\bar{u}_{r=2}](x-a)) = (a, q_1, \ldots, q_{g-1})$ and is determined by the $g-1$ points $q_1, \ldots, q_{g-1}$. Consequently in this case the rational function $P_0(z)$ of eq. (3.4) and (3.5) will have $g-1$ quadratic zeros.

In the mathematical literature the set of $g-1$ points corresponding to the odd spin structures are known just for the hyperelliptic curves [19] and for a "general" algebraic curve [(5)] of genus 3 [49]. We do not treat here the hyperelliptic case, which can be considered as a particular case of the Z_n symmetric spin structures with $n=2$ [1,3,9,10]. We wish instead to study the Z_n symmetric curves with $n>2$. This is an example which is easy to generalize to more complicated Riemann surfaces.

For simplicity we consider just an algebraic equation of the kind:

$$y^n = -P_0(z) = z^{nm} + \beta_{0,nm-1} z^{nm-1} + \ldots + \beta_{0.1} z + \beta_{0,0}, \quad m \in \mathbf{Z} \qquad (4.3)$$

The genus of such curves is given by $g = 1 - m + \frac{mn(n-1)}{2}$. We seek for an holomorphic differential $\omega_*(z)dz$ with $g-1$ zeros of order two. Its square root is in fact an half differential $\nu_*(z)dz^{\frac{1}{2}}$ corresponding to the odd spin structure $*$. In the language of theta functions it is possible to construct the abelian differential

$$\tilde{\omega}_*(z)dz = \sum_i \theta_i[*]\tilde{\omega}_i(z)dz \qquad (4.4)$$

satisfying the condition to have $g-1$ quadratic zeros. In eq. (4.4)

$$\theta_i[*] = lim \frac{\partial}{\partial_{u_\nu}} \theta_i[*](u_\nu)|_{u_\nu=0}$$

and $\tilde{\omega}_i(z)dz$ is the basis of abelian differentials in the canonical normalation [19]. On an algebraic curve we can build a differential $\omega_*(z)dz$ proportional to that of eq. (4.4) following the methods of the previous section. We start with a general abelian differential for eq. (4.3) given by [37]:

$$\omega(z)dz = \sum_{l=1+\delta_{1,m}}^{n-1} \sum_{\lambda=0}^{ml-2} A_{\lambda,l} z^\lambda y^{n-1-l} \frac{dz}{y^{n-1}}$$

Since we are seeking for the zeros of $\omega(z)$, one of the coefficients, let say $A_{n-1,m(m-1)-2}$ can be put equal to 1. Looking at the divisors of z, y and dz we see that the function

$$\tilde{y} = \sum_{l=1+\delta_{1,m}}^{n-1} \sum_{\lambda=0}^{ml-2} A_{\lambda,l} z^\lambda y^{n-1-l}$$

(4) let us remember that if a divisor ξ has multiplicity $i(\xi) = s$, then $\theta[\bar{0}](\{\bar{u}_{r=2}\}_\nu)$ vanishes with its derivatives of order $s-1$.

(5) Let us remember that a curve is said to be "general" if it covers a subset of the moduli space M_g of dimension $3g-3$. An algebraic curve can never cover the entire moduli space. In fact the branch points are just local coordinates in M_g and M_g is a complicated variety with singularities [26].

has exactly $2g-2$ zeros and its poles eliminate the zeros of the differential dz/y^{n-1} at $z=\infty$. We have still to fix the coefficients $A_{\lambda,l}$ appearing in the definition of $\tilde{y}$ in such a way that these zeros are all of order two. To this purpose, we write the equation $\tilde{F}(z,\tilde{y})=0$ defining $\tilde{y}$. This is very easy to do when $n=3$ since $\tilde{y}$ is linear in y. After making this further restriction we get:

$$\tilde{y}^3-3\tilde{y}^2q_{2m-2}(z)+3\tilde{y}q_{2m-2}^2(z)-q_{2m-2}^3(z)=P_0(z)q_{m-2}(z) \tag{4.5}$$

Here the $q_k(z)$ are polynomials of degree k in z given by:

$$q_{m-2}(z)=\sum_{\lambda=0}^{m(1+\delta_{1,m})-2}A_{\lambda,1}z^{\lambda}y(z)$$

and

$$q_{2m-2}(z)=\sum_{\lambda=0}^{2m-2}A_{\lambda,2}z^{\lambda}$$

In the notation of eq. (3.4) we have:

$$\tilde{P}_2(z)=q_{2m-2}(z)\qquad \tilde{P}_1(z)=q_{2m-2}^2(z)\qquad \tilde{P}_0(z)=P_0(z)q_{2m-2}(z)+q_{2m-2}^3(z)$$

As it is possible to check, $\tilde{P}_0(z)$ is a polynomial of degree $2g-2=6m-6$ as expected. If we write $\tilde{P}_0(z)=\sum_{i=0}^{6m-6}\tilde{\beta}_{0,6m-6-i}z^{6m-6-i}$, it is then clear that the coefficients $\tilde{\beta}_{0,6m-6-i}$ will be functions of the parameters $\beta_{0,l}$ appearing in eq. (4.3) and of the $A_{\lambda,l}$. Following section 3, we impose now that $\tilde{P}_0(z)=q^2(z)$, where $q(z)=\prod_{i=0}^{g-2}(z-z_i)$. If this condition is fulfilled then the differential $\omega_*(z)dz=\tilde{y}(z)\frac{dz}{y^2}$ becomes a differential with $g-1$ quadratic zeros since the zeros of $\tilde{y}$ coincide with those of $\tilde{P}_0(z)$. The first idea in order to solve the equation $\tilde{P}_0(z)=q^2(z)$ in the $\tilde{\beta}_{0,6m-6-i}$, is to write down the conditions for which the polynomial $\tilde{P}_0(z)$ has quadratic zeros. Unfortunately, already when $m=2$, i.e. $g=4$, we have three complicated equations involving the $\tilde{\beta}_{0,6m-6-i}$:

$$\left(\tilde{\beta}_{0,4}-\frac{\tilde{\beta}_{0,5}^2}{4}\right)^2=\frac{\tilde{\beta}_{0,1}^2}{\tilde{\beta}_{0,0}}\qquad \left(\tilde{\beta}_{0,2}-\frac{\tilde{\beta}_{0,1}^2}{4}\tilde{\beta}_{0,0}\right)^2=\tilde{\beta}_{0,5}^2\tilde{\beta}_{0,0}$$

$$\left(2\tilde{\beta}_{0,0}+\frac{\tilde{\beta}_{0,5}\tilde{\beta}_{0,1}}{2}\right)^2=\tilde{\beta}_{0,0}\tilde{\beta}_{0,3}^2$$

These equations represent the algebraic equations in the variables $A_{k,\lambda}$ and it is impossible to solve them analytically. However the polynomial $\tilde{P}_0(z)$ in eq. (4.5) depends linearly of the coefficients $\beta_{0,k}$ of $P_0(z)$. We can therefore use the $\beta_{0,k}$ to solve the equation $\tilde{P}_0(z)=q^2(z)$ considering the parameters $A_{\lambda,l}$ and the zeros $z_0,\dots,z_{g-2}$ appearing n $q(z)$ as independent. The advantage is that we have to solve in this way a set of linear equations in the constant coefficients $\beta_{0,k}$ instead of the highly nonlinear system written above in the $\tilde{\beta}_{0,6m-6-i}$. This strategy is of course working on if the number of the coefficients

$\beta_{0,i}$ is greater or equal than the number of the zeros z_i and of the $A_{\lambda,l}$. Otherwise just a part of the quadratic zeros can be easily found with this method. This is a limitation in solving analytically the IJP on Z_n symmetric curves and the situation when $n > 3$ is even worse because in this case the coefficients $\beta_{0,i}$ do not appear any longer in $\tilde{P}_0(z)$ in a linear way. The troubles arise because the IJP is very complicated to solve explicitly either in the formalism of theta functions (eq 3.2) or in the language of algebraic curves. Nevertheless the method explained above can be applied when $n = 3, m = 1,2,3$ and when $n = 4, m = 1,2$. These examples are very useful in order to understand how the spin structures are realized in terms of twist fields. Once this problem is understood, then it is possible to extend the solution to any curve, apart of course from the difficulties with the nonlinear algebraic equations.

The main motivation for the study of the odd spin structures lies here in the fact that once we know the zeros $z_0, \ldots, z_{g-2}$ which solve the inverse Jacobi problem, it is then possible to express the chiral determinants using the Beilinson Manin formulas [8] as in [11,24]. Luckily enough the procedure outlined above is able to get these zeros for the odd spin structures of two "general" Riemann surfaces of genus $g = 3, 4$.

5. THE CHIRAL DETERMINANTS ON ALGEBRAIC CURVES

We start here with the formulas of the chiral determinants given in [24]. Actually we consider the following combinations of chiral determinants from which it is possible to construct the partition function of the bosonic string theory:

$$\Lambda_j = (det\bar{\partial}_0)^{\frac{1}{2}}(det\bar{\partial}_j)$$

In the case of string theories we are interested just to Λ_0 and Λ_2 which are given by:

$$\Lambda_0 = \frac{\tilde{\omega}''_*(z_0)}{det\,|\tilde{\omega}''_i(z_0)\tilde{\omega}_i(z_0)\ldots\tilde{\omega}_i(z_{g-2})|} \tag{5.1}$$

and

$$\Lambda_2 = \frac{\tilde{\omega}''_*(z_0)}{det\,|\varphi_r(z_0)\varphi''_r(z_0)\varphi''''_r(z_0)\varphi_r(z_1)\varphi'_r(z_1)\varphi''_r(z_1)\ldots\varphi_r(z_{g-2})\varphi'_r(z_{g-2})\varphi''_r(z_{g-2})|} \tag{5.2}$$

In eqs. (5.1) and (5.2) the prime denotes a derivative in z and the $\varphi_r(z)dz^2, r = 1, \ldots, 3g - 3$, form a basis for the holomorphic quadratic differentials. It is possible to derive an expression of the partition function of bosonic string theory Z_g through Λ_0 and Λ_2 in the following way:

$$Z_g = \int_{M_g} \prod_{r=1}^{3g-3} d^2\alpha_r |F(\alpha)|^2 det^{-13} Im(T_{ij}) \tag{5.3}$$

where

$$F(\alpha) = det\bar{\partial}_2(det\bar{\partial}_0)^{-13} = \frac{\Lambda_0^{-9}\Lambda_2}{[\tilde{\omega}''(z_0)]}$$

and $T_{i,j}$ is the period matrix characterising the Riemann surface. The derivation of this matrix requires the knowledge of a system of nontrivial homology cycles which can be constructed as explained in section 2.

The parameters α_r spanning the moduli space are the branch points of the algebraic curve. This is a perfectly legitimate choice [40]. The most simple choice to parametrize the moduli space would be the coefficients $\beta_{i,k}, \gamma_{i,k'}$ of eq. (2.3). However they do not fulfill the definition of moduli given in [40]. Moreover the variational formulas of [26], which will play a key role in the construction of the partition functions, do not exist for the parameters of the curve. In eq. (5.3) all the dependence of the parameters $\beta_{i,k}$, $\gamma_{i,k'}$ is transformed in a dependence of the branch points in the following way. For certain curves, in particular those of group a), the number of branch points is smaller then $3g^{(6)}$. In this case varying the branch points we obtain just a subset of all the possible Riemann surfaces of genus g. In some other cases, for example the curves in group b), the number of branch points can be bigger than $3g$. In both situations we rely on the theorems established in ref. [26], stating that the number N of free parameters on a algebraic curve is never more than $3g$. Thus we fix three branch points $\alpha_{N-2}, \alpha_{N-1}, \alpha_N$ keeping free the remaining $N-3$: $\alpha_1, \ldots, \alpha_{N-3}$. At this point we impose the relation $D(z) = \prod_{i=1}^{N}(z-\alpha_i)P_{3g-N}(z)$ for the resultant of the two eqs. (2.4). The roots of the polynomial $P_{3g-N}(z)$ take into account of those branch points which cannot be chosen arbitrarily. It is now possible to solve the above relation term by term in the various powers of z with respect to the N free coefficients $\beta_{i,k}$ and $\gamma_{i,k'}$. In this way they become functions of the branch points as promised. The branch points appearing in $P_{3g-N}(z)$ represent instead the remaining roots of the equation $D(z) = 0$ and are again functions of the $\alpha_1, \ldots, \alpha_N$.

Finally eqs. (5.1), (5.2) depend of the normalization of the zero modes. In eq. (5.3) one has to insert the canonically normalized basis for the abelian differentials which can be obtained as in ref. [15] once we have a basis of homology cycles. Regarding the quadratic differentials, we need a linear combination of the $\varphi_r(z)dz^2$ satisfying the condition to be orthonormal with respect to the Beltrami differentials:

$$< \tilde{\varphi}_{r,zz}\mu^z_{s,\bar{z}} >= \int_\Sigma d^2z\tilde{\varphi}_{r,zz}\mu^z_{s,\bar{z}} = \delta_{rs} \tag{5.4}$$

Using the variational formulas of [26] we can obtain the normalized quadratic differentials $\tilde{\varphi}_r(z)dz^2$ as in the hyperelliptic case [9]. We perform the derivation just for "general" and Z_n symmetric curves, but the method can be extended also to other curves with little modifications.

Let us start with the "general" case. It is well known that for such curves a basis of holomorphic differentials is provided multiplying the abelian differentials pairwise in all the possible ways:

$$\varphi_r(z)dz^2 \in \{\omega_i(z)\omega_j(z)dz^2 | i < j \in 1, \ldots, g\}$$

Now we use the fact [26] that under the variation $\alpha_r \to \alpha_r + \delta\alpha_r$ of a branch point α_r with multiplicity ν_r, the period matrix changes according to the following rule:

$$\frac{\partial T_{ij}}{\partial \alpha_r} = \frac{2\pi i}{\nu_r}\{\tilde{\omega}_i(z)\tilde{\omega}_j(z)\}^{(\nu_r-1)}_{z=\alpha_r} \tag{5.5}$$

In eq. (5.5) $\{f(z)\}^{(\nu_r-1)}_{z=\alpha_r}$ means that we have to take the residue of order $\nu_r - 1$ of the function $f(z)$ at $z = \alpha_r$. Following [9] we start now from the general relation:

$$\frac{\partial T_{ij}}{\partial \alpha_r} = \int_\Sigma \eta_r\tilde{\omega}_i\tilde{\omega}_j \tag{5.6}$$

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(6) Let us remember that algebraic curves are coverings of CP_1 and that 3 branch points can be arbitrarily fixed using the group of automorphisms $SU(2, \mathbf{C})$ of CP_1. On this point see also [11].

Comparing eq. (5.5) with eq. (5.6) we see that the Beltrami differentials associated with a branch point with multiplicity ν is, on a generic curve, proportional to the derivative of order $\nu - 2$ of the delta function $\delta^{(2)}(z, \alpha_i)$. Now we substitute in eq. (5.6) the following expression for the Beltrami differentials η_r orthonormal to the $\bar{\varphi}_s(z)$:

$$\eta_r = (N_2^{-1})_{rs}\bar{\bar{\varphi}}_s(\bar{z})g^{-1}$$

where g is the metric on the Riemann surface. Summing eq. (5.6) over the $3g - 3$ branch points generating the independent variations in the moduli space we get:

$$\sum_{r=1}^{3g-3} \frac{\partial T_{ij}}{\partial \alpha_r} \int \bar{\bar{\varphi}}_s(\bar{z})\bar{\varphi}_r(z)g^{-1} = \int \bar{\bar{\varphi}}_s(\bar{z})g^{-1}\bar{\omega}_i(z)\bar{\omega}_i(z)d^2z$$

Equating the different integrands in the above equation and using eq. (5.5) we have:

$$\sum_{r=1}^{3g-3} \frac{\partial T_{ij}}{\partial \alpha_r}\bar{\varphi}_r(z) = \sum_{r=1}^{3g-3} \frac{2\pi i}{\nu_r} \{\bar{\omega}_i(z)\bar{\omega}_j(z)\}_{z=\alpha_r}^{\nu_r-1} \bar{\varphi}_r(z) = \bar{\omega}_i(z)\bar{\omega}_j(z) \tag{5.7}$$

It is clear that the above equation provides a linear system to construct the holomorphic quadratic differentials $\bar{\varphi}_r(z)$. Moreover it is possible to use a linear combination of formula (5.7) in such a way that we can use an arbitrary basis of holomorphic differentials $\omega_i(z)$.

A brief discussion of the Z_n symmetric curves of eq. (2.2) is in order here. The number of branch points is now less than $3g$ unless $n = 2$, $g = 1, 2$. On such Riemann surfaces the branch points are the moduli related to the pseudo-differential transformations which preserve the Z_n symmetry $y \to \epsilon y$, $\epsilon = e^{\frac{2\pi i k}{n}}$, $k = 1, \ldots, n-1$ [50]. The quadratic differentials $\varphi_r(z)$ associated with the branch points are therefore symmetric under the above transformations and have the form:

$$\varphi_r(z)d^2z = R_r(z)\frac{d^2z}{y^n}$$

where $R_r(z)$ is a singlevalued rational function in z. The number of the Z_n-symmetric quadratic differentials is, as a matter of fact, equal to the number of branch points. Therefore in the case of Z_n symmetric curves we have to insert the Z_n-symmetric differentials in eq. (5.7). We generalize in this way the treatment of Beilinson Manin formulas developed in ref. [9] which was restricted to the case $n = 2$.

As we see from eqs. (5.1), (5.2) and (5.7) the explicit construction of the partition function for the bosonic string theory on an algebraic curve requires the knowledge of four things: the nontrivial homology cycles, the abelian differentials, the abelian differential $\omega_*(z)dz$ with $g - 1$ quadratic zeros and the positions of such zeros. Unfortunately the methods of sections 3-4 allow the derivation of an abelian differential which is only proportional to the true normalized abelian differential $\bar{\omega}_*(z)dz$ of eq. (4.4). The constant of proportionality depends of the parameters of the curve which ultimately are functions of the branch points and this is unsatisfactory. In order to solve partially this extra problem we normalize our differential $\omega_*(z)dz$ in such a way that its value is 1 at some point $z = a$ which is not a zero of $\omega_*(z)$. In this way it is clear for example that the differential:

$$\omega_*(z) = \frac{\theta_{*,i}\bar{\omega}_i(z)}{\theta_{*,i}\bar{\omega}_i(0)}$$

is proportional to $\tilde{\omega}_*(z)$ apart from the factor $\theta_{*,i}\tilde{\omega}_i(0)$. Hence, since the normalized abelian differentials $\tilde{\omega}_i(z)$ are known from ref. [15], the formulas (5.1) and (5.2) for the chiral determinants are determined up to the theta constants $\theta_{*,i}$.

The first example we treat is provided by the Z_n symmetric curve of eq. (4.3) with $n = 3$, $m = 2$. In this case $q_{m-2}(z)$ in eq. (4.5) is a constant which can be set equal to one and the condition $\tilde{P}_0(z) = q(z)^2$ is satisfied if $P_0(z) = q^2(z) - q_2^3(z)$. Here $q(z) = (z-z_0)(z-z_1)(z-z_2)$ is a polynomial of degree three and $q_2(z)$ is like in eq. (4.5). The original six free parameters $\beta_{0,i}$ of eq. (4.3) have been substituted by the six parameters $z_0, z_1, z_2, A_{0,0}, A_{0,1}, A_{0,2}$. The algebraic equation

$$y^3 = -q^2(z) + q_2^3(z) \tag{5.8}$$

has therefore an abelian differential with $g-1$ quadratic zeros given by:

$$\omega_*(z)dz = \frac{y + q_2(z)}{y^2}dz \tag{5.9}$$

In eq. (5.9) we have $\tilde{y} = y + q_2(z)$. Let us notice that in eq. (5.8) we loose the position of the branch points because they are given by the relation $q^2(z) - q_2^3(z) = 0$, which is of degree six and it is impossible to solve explicitly. Nevertheless we have the bonus that we have solved the inverse Jacobi problem for one odd spin structure on the curve with genus $g = 4$ of eq.(5.8). Moreover we have found more than one odd spin structure: just looking at eq. (5.8) it is easy to proof that also the following functions have $g-1$ quadratic zeros and correspond to different odd spin structures:

$$\tilde{y} = y - \epsilon^k q_2(z) \qquad k = 0,1,2 \qquad \epsilon = e^{\frac{2\pi i}{3}}$$

If we choose the spin structure represented by the differential of eq. (5.9), the zeros z_0, z_1, z_2 occur in the first sheet, i.e. when $y = \sqrt[3]{P_0}$. We have now all the ingredient to construct Λ_0 and Λ_2 in eqs. (5.1) and (5.2). The abelian differentials are in fact given by:

$$\omega_j(z)dz = \frac{z^{j-1}dz}{y^2} \quad j = 1,2,3 \qquad \omega_4(z)dz = \frac{dz}{y}$$

while the quadratic differentials invariant under the Z_n transformations to be inserted in the RHS of eq. (5.7) are of the form: $\varphi_r = \omega_r\omega_4$, $r = 1,2,3$. In the particular case of the Z_n symmetric curves the twist fields are already known [10,37,39] and it is possible to derive the chiral determinants using the stress energy tensor method. The result is:

$$(det\bar{\partial}_0)^{\frac{3}{2}} = [det(K)]^{\frac{3}{2}} \prod_{i<j}(\alpha_i - \alpha_j)^{\frac{5}{8}}$$

where $K = \oint_{A_j} \omega_i dz$. Equating the above expression with eq. (5.1) we get:

$$\theta_{*,i}\tilde{\omega}_i(0) = \frac{det|\tilde{\omega}_i''(z_0)\tilde{\omega}_i(z_1)\tilde{\omega}_i(z_2)|}{\omega_*''(z_0)} det(K)^{\frac{3}{2}} \prod_{i<j}(\alpha_i - \alpha_j)^{(5/6)} \tag{5.10}$$

We are now able to evaluate in a easy way the theta constants $\theta_{*,i}$ since the above equation is a linear system in these variables. Let us remember that in general such theta constants are defined through complicated differential equations [29] and are known only in the hyperelliptic case.

6. THE CASE OF ALGEBRAIC CURVES OF HIGHER GENUS

In this section we apply the method discussed previously in order to provide all the ingredients needed for the construction of the partition functions on "general" algebraic curves of any genus. The first task consist in constructing an algebraic equation with the biggest number of free parameters in order to cover the moduli space as much as possible. This problem can be solved using the Baker's method [27]. On a lattice on the plane whose units are squares of side 1, we trace two cartesian axis. The vertical axis is associated to the variable r and the horizontal to the variable s, where r and s are the two indices appearing in eq. (2.5). Now we draw on such a plane all the points (r,s) for which the coefficients $c_{r,s}$ in eq. (2.5) are different from zero. Finally all the external points are connected by straight lines in such a way that none of the points (r,s) is kept outside of the close polygon we get. The polygon should be convex, otherwise one can perform the transformations $y \rightarrow y+p(z)$ or $z \rightarrow z+p'(y)$ with p,p' polynomials in z and y respectively, adding new terms of the kind $c_{r,s}y^r z^s$ in the final curve. Let us remember that after a rational transformation the new curve is equivalent to the old one as explained in section three because it has the same number of branch points and the monodromy group does not change. p and p' are arbitrary polynomials with a constraint on their degrees due to the fact that we want to keep in the transformed curve the same maximum degree in z and y of the original one. Conversely, the former curve will have a different number of sheets than the latter. At the end, the Baker procedure applied to the final curve yields a convex polygon. After the polygon is convex, the above transformations are no more effective and do not change its shape. Returning to the Baker's method, the number of free parameters of the curve is exactly the number of points contained inside the convex polygon together with those lying on the boundary, while the genus is given by the number of the points which are inside the polygon but do not belong to the boundary. In this way we can construct the algebraic equation for a curve of genus g with the biggest possible number of free parameters $c_{r,s}$ and with a determined number of sheets just drawing a suitable convex polygon on the plane. Since the branch points are functions of these parameters, we have in this way the possibility of constructing a "general" curve of genus g or at least the best approximation of it. Unfortunately the Baker's method does not provide the most general form of an algebraic curve. For example when $g=5$ we start with a n-sheeted Riemann surface associated to a reference multivalued function y. According to section 3, there is a function $\tilde{y}$, corresponding to an odd spin structure, with n branches and with $2g-2=8$ zeros. Therefore $\tilde{y}$ satisfies eq. 3.4 with a $\tilde{P}_0(z)$ of degree 8. As we already explained, the function $\tilde{y}$ and its Weierstrass polynomial $\tilde{F}(z,\tilde{y})=0$ describe the Riemann surface under consideration as well as y. However any attempt to construct a convex polygon for

$$\tilde{F}(z,\tilde{y}) = \sum_{r,s} z^r \tilde{y}^s = 0$$

with a side of eight units along the r axis and containing just five points is impossible for any n. The apparent contradiction is solved constructing the Weierstrass polynomial for $\tilde{y}$ explicitly as we did for the 4-sheeted surface of genus 5 of eq. (6.10). In this case the coefficients $\tilde{P}_i(z)$ are of a very special form. Therefore, despite of the fact that the Baker's method applied to $\tilde{F}(z,\tilde{y})$ gives a genus which is much higher than five, it is possible to see that there are many cancellation in the leading terms in z of the resultant $D(z)$ between the two equations $\tilde{F}(z,\tilde{y})=0$ and $\tilde{F}_{\tilde{y}}(z,\tilde{y})=0$. At the end the number of branch points, which is equal to the degree of $D(z)$ in z, and their multiplicities, is in agreement with the Riemann Hurwitz formula. Let us recall here that the number of branch points and their multiplicities for a n-sheeted curve are related to the genus by the Riemann Hurwitz equation:

$$2g-2=-2n+\sum_{i=1}^{n_{tot}}(\nu_i-1) \tag{6.1}$$

We are now ready to construct the partition functions on "general" algebraic curves of genus g.

i) "General" curves of genus three. Firstly we study the Riemann surface of ref. [15] associated to the equation:

$$y^3+P_1(z)y+P_0(z)=0 \tag{6.2}$$

$P_1(z)$ and $P_0(z)$ are polynomials of degree 3 and 4 respectively. They provide exactly 9 parameters $\beta_{i,k}$, $i=0,1$. This is in agreement with the fact that a "general" algebraic curve should depend on $3g$ free parameters. Using the Sylvester method to solve eq. (2.4), we find that the resultant $P_1^3(z)+P_0^2(z)=0$ has 9 roots corresponding to 9 branch points. Another branch point of multiplicity two occur at $z=\infty$. The Riemann Hurwitz formula is then satisfied if all ten branch points have multiplicity two. Therefore the curve described by eq. (6.2) belongs to the group b) discussed in section 2. A non canonical basis of abelian differentials is given in [15]:

$$\omega_1(z)=\frac{dz}{F_y(z,y)}\qquad \omega_2(z)=\frac{zdz}{F_y(z,y)}\qquad \omega_3=\frac{ydz}{F_y(z,y)}$$

The canonical abelian differentials are obtained as in ref. [15] using the basis of homology cycles given in fig. 4. The six normalized quadratic differentials

$$\tilde{\varphi}_1(z),\ldots,\tilde{\varphi}_6(z)$$

are computed from the products of two abelian differentials by means of eq. (5.7). The result is too long to be rewritten here. The differential $\omega_*(z)dz$ with $2g-2$ quadratic zeros is of the form:

$$\omega_*(z)=\frac{y+A_{0,1}z+A_{0,0}}{F_y(z,y)}dz$$

As we can see the function $\tilde{y}=y+A_{0,1}z+A_{0,0}$ has exactly four zeros. In fact in this case $2g-2=4$. The position of such zeros is given by the polynomial $\tilde{P}_0(z)$ appearing in the equation $\tilde{F}(z,\tilde{y})=0$ which defines $\tilde{y}$. It is clear that $\tilde{P}_0(z)$ is proportional to the resultant of the system of equations:

$$\begin{cases} F(z,y)=0 \\ \tilde{y}=y+A_{0,1}z+A_{0,0}=0 \end{cases}$$

Using the method of algebraic elimination of Sylvester, the position of the zeros of $\tilde{y}$ is specified by the resultant of the above equation:

$$D(z)=R_1^3(z)+P_1(z)R_1(z)-P_0(z)=0$$

where $R_1(z)=A_{0,0}+A_{0,1}z$. The degree of $D(z)$ is four as expected. If we require that $\tilde{y}(z)$ has two zeros of order two at $z=z_0,z_1$, then we have to solve the equation $D(z)=q^2(z)=\gamma(z-z_0)^2(z-z_1)^2$. In order to fulfill such relation, it is sufficient to reparametrize the initial curve as we did in the example of eq. (5.8) putting: $P_0(z) = R_1^3(z) + P_1(z)R_1(z) - q^2(z)$. In this way the "reparametrized" equation of genus 3:

$$y^3 + P_1(z)y + R_1^3(z) + P_1(z)R_1(z) - q^2(z) = 0 \qquad (6.3)$$

has a natural odd spin structure associated with the holomorphic differential:

$$\omega_*(z) = \frac{(y + R_1(z))}{F_y(z,y)} \qquad (6.4)$$

$\omega_*(z)$ has two zeros of order 2 at $z = z_0, z_1$ located in the sheet where

$$y(z) = \sqrt[3]{P_0(z) + \sqrt[2]{P_0(z)^2 + P_1(z)^3}} + \sqrt[3]{P_0(z) - \sqrt[2]{P_0(z)^2 + P_1(z)^3}}$$

The square root $\nu_*(z) = \sqrt{\omega_*(z)dz}$ defines a natural metric on the surface of eq. (6.2): $g_{z\bar{z}} = |\nu_*(z)|^2$. The five parameters $\beta_{0,k}$ of $P_0(z)$ are now functions of five new independent parameters represented by $z_0, z_1, \gamma, A_{0,0}$ and $A_{0,1}$. At this point we have all the ingredients for the construction of the partition function of string theory on a algebraic curve of genus 3 which covers all the moduli space apart from the subset related to the hyperelliptic surfaces. Let us notice however that this set is of zero measure in the moduli space M_3. We list below the two formulas for Λ_0 and Λ_2:

$$\Lambda_0 = det|K|(\theta_{*,i}\tilde{\omega}_i)\frac{\omega_*''(z_0)/\omega_*(0)}{det|\omega_i''(z_0)\omega_i(z_0)\omega_i(z_1)|} \qquad (6.5)$$

$$\Lambda_2 = det|\tilde{K}|(\theta_{*,i}\tilde{\omega}_i)\frac{\omega_*''(z_0)/\omega_*(0)}{det|\tilde{\varphi}_r(z_0)\tilde{\varphi}_r''(z_0)\tilde{\varphi}_r''''(z_0)\tilde{\varphi}_r(z_1)\tilde{\varphi}_r'(z_1)\tilde{\varphi}_r''(z_1)|} \qquad (6.6)$$

Here K is as in eq. (5.10) while

$$\tilde{K}_{rs} = \frac{2\pi i}{\nu_r}\{\varphi(z)_s\}_{z=\alpha_r}^{\nu_r - 1}$$

Let us notice here that the quartic of genus three is the most well studied case after hyperelliptic curves in the mathematical literature. Since the very early times [49] it was found the following remarkable curve with three natural odd spin structures.

$$yzt - f^2(z,y) = 0 \qquad (6.7)$$

with $t = c_1 z + c_2 y + c_3$ and

$$f = a_{00}z^2 + a_{11}y^2 + a_{22} + a_{10}zy + a_{20}z + a_{21}y$$

Using again the Sylvester method for eqs. (2.4), we get a polynomial of degree 12 whose solution are the branch points. Two of the branch points occur at $z = 0$. At these point the four sheets are connected pairwise at the two projections 0_1 and 0_2 of the point $z = 0$ on the algebraic curve. Therefore the point $z = 0$ has multiplicity 4 and the algebraic curve of eq. (6.7) does not belong to group b). The relevant divisors [15] for the construction of the abelian differentials are:

$$div(dz) = \frac{0_1 0_2 \alpha_3 \ldots \alpha_{12}}{\infty_{(0)}^2 \cdots \infty_{(3)}^2} \qquad div(F_y(z,y)) = \frac{0_1 0_2 \alpha_3 \ldots \alpha_{12}}{\infty_{(0)}^3 \cdots \infty_{(3)}^3}$$

$$div(z) = \frac{0_1^2 0_2^2}{\infty_{(0)} \cdots \infty_{(3)}} \qquad div(y) = \frac{\gamma_+^2 \gamma_-^2}{\infty_{(0)} \cdots \infty_{(3)}}$$

As usual, $a_{(i)}$ denotes the projection of the point $z = a$ on the sheet i. Moreover in the above formula $0_1, 0_2, \alpha_3 \ldots \alpha_{12}$ denote the 12 branch points with multiplicity two of eq. (6.7) and:

$$\gamma_\pm = \frac{-a_{20} \pm \sqrt{a_{20}^2 - 4a_{00}a_{22}}}{2a_{00}}$$

The three independent holomorphic differentials are:

$$\omega_1(z) = \frac{dz}{F_y(z,y)} \qquad \omega_2(z) = \frac{zdz}{F_y(z,y)} \qquad \omega_3 = \frac{ydz}{F_y(z,y)}$$

It is now clear that we have three possible odd spin structures associated to the differentials:

$$\omega_*(z) = \frac{zdz}{F_y(z,y)} \qquad \omega_{*'} = \frac{ydz}{F_y(z,y)} \qquad \omega_{*''}(z) = \frac{t(z,y)dz}{F_y(z,y)}$$

Proceeding as for the curve in eq. (6.2) we can construct the Beilinson Manin expression of the partition function using all three spin structures $*, *'$ and $*''$. In this way it is probably possible to check that the Beilinson Manin formula is not affected also at genus 3 when the reference odd spin structure $*$ in eq. (5.1) is changed.

ii) "General" curves of genus 4. A "general " Riemann surface of genus four can be represented by the algebraic equation:

$$y^3 + P_2(z)y^2 + P_1(z)y + P_0(z) \tag{6.8}$$

The polynomials $P_2(z), P_1(z)$ and $P_0(z)$ have degrees 2,4 and 6 in z respectively. Again the Sylvester method tells us that there are 12 branch points $\alpha_1, \ldots \alpha_{12}$. The point $z = \infty$ has no ramifications. Therefore the Riemann Hurwitz formula (6.1) assures that all branch points have multiplicity two and the curve (5.8) belongs to the group b) analyzed in section 2. The number of parameters $\beta_{i,k}$, $i = 0,1,2$ in eq. (6.8) is 15. However a simple transformation $y \to y - P_2(z)/3$ eliminates the term $P_2(z)y^2$ from eq. (6.8) containing exactly three parameters. Hence the number of free parameters is equal to the number of branch points which is $3g = 12$ as expected on a "general" algebraic curve of genus four. The relevant divisors for the construction of the abelian differentials are:

$$div(dz) = \frac{\alpha_1 \ldots \alpha_{12}}{\infty_{(0)}^2 \infty_{(1)}^2 \infty_{(2)}^2} \qquad div(F_y(z,y) = \frac{\alpha_1 \ldots \alpha_{12}}{\infty_{(0)}^4 \infty_{(1)}^4 \infty_{(2)}^4}$$

$$div(y) = \frac{\delta_1 \ldots \delta_6}{\infty_{(0)}^2 \infty_{(1)}^2 \infty_{(2)}^2}$$

$\delta_1 \ldots \delta_6$ denote here the zeros of $P_0(z)$. The four independent abelian differentials are:

$$\omega_i(z) = \frac{z^{i+1}dz}{F_y(z,y)} \quad i = 0,1,2 \qquad \omega_4(z)dz = \frac{ydz}{F_y(z,y)}$$

Therefore an holomorphic differential with $g-1$ quadratic zeros should be of the form:

$$\omega_*(z) = \frac{(y + q_2(z))dz}{F_y(z,y)} \tag{6.9}$$

with $q_2(z) = A_{0,2}z^2 + A_{0,1}z + A_{0,0}$. It is easy to verify that $\tilde{y} = y + q_2(z)$ is a meromorphic function on the curve with $2g-2$ zeros. The position of these zeros is provided by the following set of equations:

$$\begin{cases} y^3 + P_2(z)y^2 + P_1(z)y + P_0(z) = 0 \\ y + q_2(z) = 0 \end{cases}$$

With the method of Sylvester, the resultant of these two equations is given by:

$$\tilde{P}_0(z) = q_2^3(z) - P_2(z)q_2^2(z) + P_1(z)q_2(z) - P_0(z)$$

Now we require that $\tilde{P}_0(z)$, i.e. $\tilde{y}$, has $g-1=3$ quadratic zeros z_0, z_1, z_2. This implies:

$$q_2^3(z) - P_2(z)q_2^2(z) + P_1(z)q_2(z) - P_0(z) = q^2(z) = \gamma \prod_{i=0}^{2}(z - z_i)$$

Therefore, if we choose the reparametrization of the curve (6.8) in such a way that

$$P_0(z) = q_2^3(z) - P_2(z)q_2^2(z) + P_1(z)q_2(z) - q^2(z)$$

then the algebraic curve has a natural spin structure $*$ represented by the abelian differential $\omega_*(z)dz$ of eq. (6.9). The zeros which solve the IJP for $*$ are z_0, z_1 and z_2 and are located all on the same sheet. Again the 7 parameters $\beta_{0,k}$ of $P_0(z)$ becomes functions of the new independent parameters: γ, z_i, and $A_{0,i}$, $i = 0,1,2$. A natural metric on the curve is provided by $g_{z\bar{z}} = |\omega_*|$. We can proceed as for the case $g = 3$ and construct the partition function of bosonic string theory in a similar way.

iii) The curves of genus 5. Applying the Baker's method the most "general" curve of genus five is associated with the following algebraic equation:

$$P_4(z)y^4 + P_3(z)y^3 + P_2(z)y^2 + P_1(z)y + P_0(z) = 0 \tag{6.10}$$

where P_3, P_2, P_1 and P_4 are polynomials of degrees 2,3,3,3 and 1 respectively. The branch points are 16 and the Riemann Hurwitz formula is satisfied only if they are all simple. Therefore also the curve of eq. (6.10) belongs to group b). Let us put $P_4(z) = z - a$. It turns out that y is singular both in $z = a$ and $z = \infty$ but not branched at these points. Despite the appearances, the algebraic curve associated to eq. (6.10) has only $3g-1$ free parameters. In fact if we project the curve on the z axis, then $z = z(y)$ is a three sheeted function which can at most have 14 branch points due to Riemann Hurwitz. Instead a "general" curve of genus 5 requires at list 15 branch points generating independent pseudo-conformal transformations. We follow now the procedure outlined in ref. [15] to determine the degree of divergence of y in $z = a$ and $z = \infty$. It is relatively simple to realize that just a branch of y has a singularity of the first order in $z = a$. It is not important in the context to know exactly what is the branch. Let us denote with a_1 the projection of the point $z = a$ on the sheet where the singularity occurs. At $z = \infty$ there are singularities of y with a first order pole just on two of the four sheets. Again we denote the projection of the point $z = \infty$ on these sheets with ∞_1 and ∞_2. The relevant divisors for the construction of the holomorphic differentials are:

$$div(dz) = \frac{\alpha_1 \ldots \alpha_{16}}{\infty_{(0)}^2 \infty_{(1)}^2 \infty_{(2)}^2 \infty_{(3)}^2} \qquad div(F_y(z,y)) = \frac{\alpha_1 \ldots \alpha_{16}}{\infty_{(0)}^4 \infty_{(1)}^4 \infty_{(2)}^3 \infty_{(3)}^3 a_1^2}$$

$$div(y) = \frac{\delta_1 \delta_2 \delta_3}{\infty_1 \infty_2 \alpha_1}$$

Here $\delta_1, \delta_2, \delta_3$ denote the zeros of $P_0(z)$ and $\alpha_1, \ldots, \alpha_{16}$ are the 16 branch points of the algebraic curve (6.10). A basis for the abelian differentials is provided by:

$$\omega_1(z)dz = \frac{dz}{F_y(z,y)} \qquad \omega_2(z)dz = \frac{zdz}{F_y(z,y)} \qquad \omega_3(z)dz = \frac{yzdz}{F_y(z,y)}$$

$$\omega_4(z)dz = \frac{ydz}{F_y(z,y)} \qquad \omega_1(z)dz = \frac{y^2dz}{F_y(z,y)}$$

A general abelian differential with $g-1$ quadratic zeros is therefore of the form:

$$\omega_*(z)dz = \frac{(y^2 + q_1(z)y + R_1(z))}{F_y(z,y)}dz$$

where $q_1(z)$ and $p_1(z)$ have both degree 1. In this case $\tilde{y} = y^2 + q_1(z)y + R_1(z)$ has exactly $2g-2$ zeros. In order to have $g-1$ quadratic zeros, the resultant $D(z)$ between $\tilde{y}$ and eq. (6.10) should be zero. We simplify the calculation putting $P_0(z) = \gamma(z-z_0)^2(z-z_1)$ and $R_1(z) = 0$. Therefore the resultant $D(z)$ becomes:

$$D(z) = P_0(z)[P_4(z)q_1^4(z) - P_3(z)q_1^3(z) + P_2(z)q_1(z)^2 - P_1(z)q_1(z) + P_0(z)] = 0$$

The equation above depends linearly of the coefficients $\beta_{i,k}$, $i = 0,1,2,3$ of eq. (6.10) and has already a quadratic zero in $z = z_0$ and a simple zero in $z = z_1$. Five of these coefficients can be used to rewrite $D(z)$ in the form:

$$D(z) = \gamma(z-z_0)^2(z-z_1)^2(z-z_2)^2(z-z_3)^2$$

At the end we finish with 10 of the 17 parameters appearing in eq. (6.10) becoming dependent of the 5 parameters γ, z_i, $i = 0, \ldots, 3$ and of the two parameters $A_{1,0}, A_{1,1}$ in $q_1(z)$. There is no contradiction with the fact that the algebraic curve depends effectively of a set of 14 parameters as mentioned above. The expression of the five coefficients $\beta_{i,k}$ in terms of the parameters $\gamma, z_i, A_{1,0}, A_{1,1}$ is too complicated to be written here. However, even if the final expression is involved, we are able to compute also in this case the partition function of bosonic string theory.

CONCLUSIONS

In this paper we have studied in which way the spin structures are realized on a algebraic curve Σ. The answer is that they live on coverings $\Delta_r(\Sigma)$ which are also algebraic curves. All the λ differentials ψdz^λ carrying spin structures and therefore the Green functions of b-c systems, are determined in the sense of ref. [37] by the behavior at the branch points a_i of Σ and at their zeros z_i of order r. The points z_i appear as degenerate branch points of $\Delta_r(\Sigma)$. In fact when $\lambda = 0$, deriving eq. (4.5) in z and putting $z = z_i$ we get a zero since z_i are the roots of $\tilde{P}_0$. The same conclusion is valid for $\lambda \neq 0$ using the function $\tilde{y}(z, y)$ of section 3 instead of $f_{\bar{u}_r}(z)$. Let us remind that $\tilde{y}$ is the part of ψ generating the multivalued behavior due to the spin structure. The situation is well exemplified by the Z_3- symmetric algebraic curve of eq. (5.8). Here we have a holomorphic half differential $\nu_*(z)dz^{\frac{1}{2}} = \frac{\sqrt{\tilde{y}}dz}{y}$ with $\tilde{y} = y + q_2(z)$. The function

$$\sqrt{\tilde{y}} = \sqrt{\sqrt[3]{-q^2(z) + q_2^3(z)} + q_2(z)}$$

has branch points when $-q^2(z)+q_2^3(z) = 0$ as the original function y and degenerate branch points when $q_2(z) = 0$. The method of λ-differentials of [37] is able to cope with equations like (4.5) which belongs to group a), but stumbles against the obstacle of degenerate branch points. Unfortunately the odd spin structures have Green functions of complicated form [51-52] and it is very difficult to grasp how the method of λ-differentials can be extended in this case. Probably the even spin structures are more simple in this sense; they will be treated elsewhere.

One of the byproducts of our work is the construction of an algorithm to solve the IJP on algebraic curves for any spin structure symplifying a previous algorithm of [20]. Moreover it solves explicitly the IJP for one odd spin structures in many examples of Riemann surfaces. This fact allowed us to derive an explicit expression for the partition functions of string theory at genus 3 and 4. Unfortunately some interesting aspects of the subject exposed in this paper have been neglected or just briefly discussed due to the lack of space: The already mentioned construction of the Green functions of free fermions with even spin structure, the generalization to the supersymmetric Beilinson Manin formula [53], the modular invariance, the Beltrami differentials and finally the variational formulas of [26]. These further developments are unavoidable in order to know how far the Knizhnik's conjecture can be pushed and in order to get a deeper understanding of conformal field theories on Riemann surfaces.

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FIGURE CAPTIONS

fig. 1 The canonical polygon of a Riemann surface of genus g. The polygon is intersected by the branch lines connecting the branch points.

fig. 2 A canonical basis of homology cycles for an hyperelliptic Riemann surface of genus g. Bold lines characterize the parts of cycles lying on the first sheet, dashed lines the parts lying on the second sheet.

fig. 3 A canonical basis of homology cycles for the Riemann surface associated to the algebraic equation:$y^3 = \prod_{i=1}^{q} \left(\frac{z-a_i}{z-a'_i}\right)$. The part of the cycles lying on the third sheet are denoted by twisted lines.

fig. 4 The branch cuts for the curves of group b). The $n-2$ branch lines on the left connect consecutive sheets pairwise. The other $g+1$ branch lines are intersected by the homology cycles as in the hyperelliptic case.

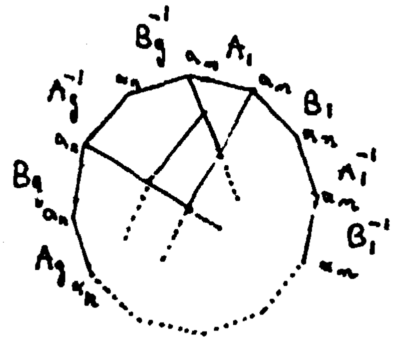

fig. 1

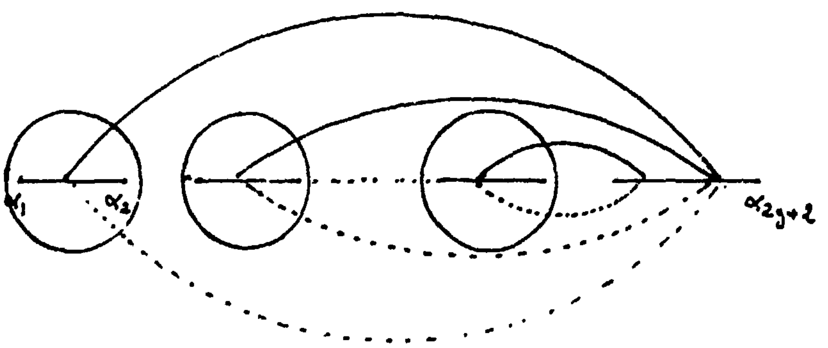

fig. 2

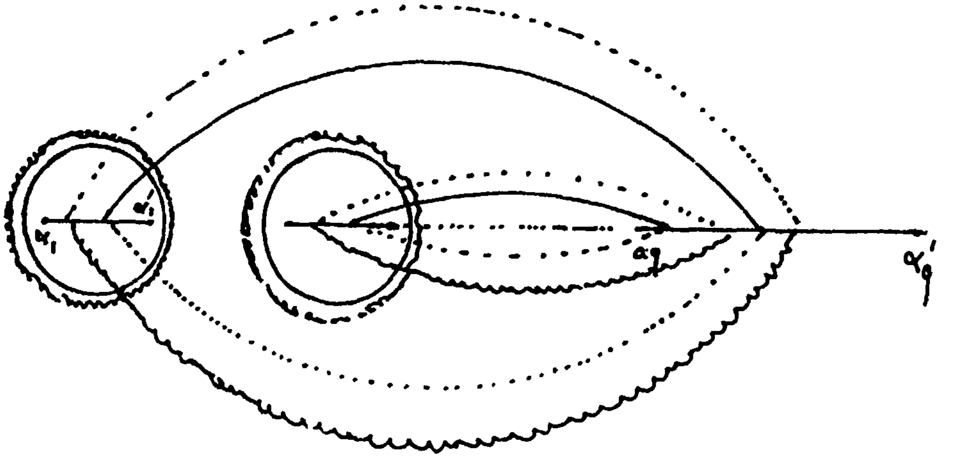
α_1
α_1'
α_g
α_g'

fig. 3

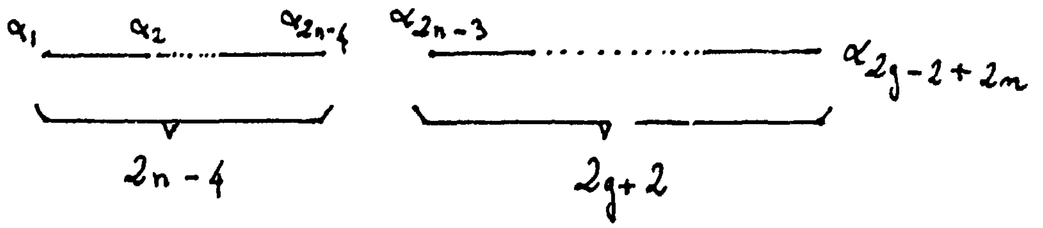
α_1
α_2
α_{2n-4}
α_{2n-3}
$\alpha_{2g-2+2n}$
$2n-4$
$2g+2$

fig. 4