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WESS–ZUMINO–WITTEN MODEL AS A THEORY OF FREE FIELDS

II. A PIECE OF GROUP THEORY

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WESS-ZUMINO-WITTEN MODEL AS A THEORY OF FREE FIELDS. II. A PIECE OF GROUP THEORY: Preprint ITEP 89-70/

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Some results from group theory are collected, which are important for free field representation of Kac-Moody algebra and WZW model on the lines of ref.1.

Fig. - 3, ref. - 5

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3. A PIECE OF GROUP THEORY

Let us collect here some basic information from group theory [1], which seems relevant for future study of the free field realization of WZWM and coset models.

3.1. ROOT SYSTEMS

Let $\mathfrak{h}$ be a vector space of finite dimension, $\dim \mathfrak{h} = r$. We shall give the definitions of root system and its Weyl group in $\mathfrak{h}$. Consider reflection τ_α with respect to the hyperplane, orthogonal to the vector $\alpha \in \mathfrak{h}$ and passing through the origin:

$$\tau_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha , \quad \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} . \qquad (3.1.1)$$

We are interested in finite groups, generated by the reflections (3.1.1). Consider a finite set of non-vanishing vectors $\Delta = \{\alpha\}$, satisfying the following conditions:

$$\begin{aligned} &1.\ \Delta \text{ generates } \mathfrak{h} \text{ as a vector space;} \\ &2.\ \tau_\alpha \Delta = \Delta \quad \text{for every } \alpha \in \Delta \\ &3.\ (\alpha^\vee, \beta) \in \mathbb{Z} \quad \text{for every } \alpha, \beta \in \Delta . \end{aligned} \qquad (3.1.2)$$

The system of vectors Δ is reffered to as a <u>root system</u>, and the group W generated by all the reflections (3.1.1) for all $\alpha \in \Delta$ – as a <u>Weyl group</u> of Δ.

Let the space $\mathfrak{h}$ be a direct sum of k subspaces, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \ldots \oplus \mathfrak{h}_k$ and let Δ_i be a root system in $\mathfrak{h}_i$.

Then the union $\Delta = \Delta_1 \cup \Delta_2 \ldots \cup \Delta_k$ will be a root system in $\mathfrak{h}$. The Weyl group is identified with the product $W_1(\Delta_1) \ldots W_k(\Delta_k)$. If no such expansion exists, the root system is called irreducible.

It may be shown, that if α is a root, then only the root $-\alpha$ is collinear to it (this is obvious) and 2α (-2α) may also be collinear. It turns out, that except for the latter case there is a one-to-one correspondense between irreducible root systems and complex simple Lie algebras.

Consider a hyperplane in the space $\mathfrak{h}$, such that it does not contain any root. The roots, lying on one side from the hyperplane are said to be positive with respect to the ordering introduced. Let us denote these by Δ_+. Evidently, $\Delta = \Delta_+ \cup (-\Delta_+ = \Delta_-)$. Among positive roots Δ_+ one can uniquely define a subset of roots Π with the property, that any $\alpha \in \Delta_+$ is a linear combination of roots $\beta \in \Pi$ with non-negative integer coefficients,

$$\alpha = \sum_{\beta \in \Pi} n_\alpha^\beta \beta . \qquad (3.1.3)$$

The roots $\beta \in \Pi$ are reffered to as simple and Π is a system of simple roots. Evidently, simple roots form a basis of the space $\mathfrak{h}$. The number $ht_\alpha = \sum_{\beta \in \Pi} n_\alpha^\beta$ is called the height of a root α.

Different ordering in the space defines another subsets Δ'_+ and Π'. Two subsets $\Delta_+ (\Pi)$, $\Delta'_+ (\Pi')$ are conjugate under the action of Weyl group $W(\Delta)$. Moreover, any $\alpha \in \Delta$ is conjugate to some simple root: $\tau\alpha \in \Pi$, $\tau \in W$.

Root systems of rank r=2 are isomorphic to one of the following three types: A_2, C_2, G_2 (see Fig. 1).

In the set of positive roots one may find a maximal one, i.e. a root $\theta \in \Delta_+$, such that $\theta + \alpha \bar{\in} \Delta$ for any $\alpha \in \Delta_+$. The number

$$h = ht_\theta + 1 \tag{3.1.4}$$

is reffered to as Coxeter number. It is independent of the ordering in $\mathfrak{h}$ and thus of the choice of a system Δ_+.

The roots $\alpha \in \Delta$ generate a root lattice $Q = \mathbb{Z}\Delta$ in $\mathfrak{h}$. The dual lattice $Q^\vee$ is generated by weights λ, related to roots by the bracket product,

$$\langle \lambda, \alpha \rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = (\lambda, \alpha^\vee). \tag{3.1.5}$$

The basis in $Q^\vee$ is formed by the set of fundamental weights, satisfying

$$\langle \lambda_j, \alpha_k \rangle = \delta_{jk}, \quad \alpha_k \in \Pi. \tag{3.1.6}$$

Let ρ be a half-sum of all positive roots,

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{3.1.7}$$

Then the reflection (3.1.1) with $\alpha_j \in \Pi$ acts on ρ as

$$r_{\alpha_i}\rho = -\frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha \in \Delta_+, \alpha \neq \alpha_i} \alpha = \rho - \alpha_i. \tag{3.1.8}$$

The first equality follows from the fact, that r_{α_i} permutes positive roots α $(\alpha \neq \alpha_i)$ Comparison of (3.1.8) and (3.1.1) leads to the following relations:

$$(\rho, \alpha_i^\vee) = 1, \quad \rho = \sum \lambda_j \tag{3.1.9}$$

$$(\rho, \alpha) = \sum \frac{(\alpha_i, \alpha_i)}{2} n_\alpha^i. \tag{3.1.10}$$

3.2. GAUSS DECOMPOSITION

Let $\mathfrak{g}$ be a simple Lie algebra. This means, that $\dim \mathfrak{g} > 1$ and the adjoint representation of $\mathfrak{g}$ is irreducible. Consider an element $x \in \mathfrak{g}$ with the minimal possible dimension of the zero-mode eigenspace of the corresponding adjoint operator, $ad_x y = [x, y]$. A maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which contains x is called Cartan subalgebra. Because of commutativity it is possible to consider common eigenspaces $\mathfrak{g}_\alpha$ for all $h \in \mathfrak{h}$. The corresponding eigenvalues are linear functionals on $\mathfrak{h}$,

$$[h, \mathfrak{g}_\alpha] = \alpha(h)\mathfrak{g}_\alpha, \quad \alpha(h) = (\alpha, h). \tag{3.2.1}$$

The eigenspaces $\mathfrak{g}_\alpha$ are called root subspaces of $\mathfrak{g}$. In accordance with (3.2.1) there is a decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{g}_\alpha. \tag{3.2.2}$$

Dimension r of $\mathfrak{h}$ is called the rank of $\mathfrak{g}$. (Complex) dimensions of all root subspaces $\mathfrak{g}_\alpha$ are equal to unity. ($\mathfrak{g}_\alpha = a\, e_\alpha$, a is a complex number, e_α is a step generator.)

Let G be a simple Lie group with the Lie algebra $\mathfrak{g}$. G acts by adjoint representation on $\mathfrak{g}$:

$$Ad_G \mathfrak{g} : \mathfrak{g} \to g\, \mathfrak{g}\, g^{-1}. \tag{3.2.3}$$

By means of (3.2.3) one can "diagonalize" $\mathfrak{g}$:

$$\mathfrak{g} = Ad_g\, \mathfrak{h}. \tag{3.2.4}$$

Thus $\mathfrak{h}$ parametrizes the set of conjugancy classes of $\mathfrak{g}$. Consider the subgroup M generated by transformations, preserving Cartan subalgebra $\mathfrak{h}$,

$$M=\{g\in G \mid g h g^{-1}=h_1 \ , \ h, h_1\in \mathfrak{h}\}. \tag{3.2.5}$$

In other words, M is normalizator of $\mathfrak{h}$. It is worthwhile to note, that M is a subgroup of maximal compact subgroup U of G. Let M' be a subgroup generated by transformations, commuting with $\mathfrak{h}$ (M' is centralizator of $\mathfrak{h}$),

$$M'=\{g\in M \mid g h g^{-1}=h \ , \ h\in \mathfrak{h}\} . \tag{3.2.6}$$

The factor group M/M' is finite and acts on the set of root subspaces $\mathfrak{g}_\alpha$ by permutations.

Relation between the previously considered theory of root systems and the theory of simple Lie algebras is based on the observation, that the set of linear functionals $\alpha(h)\in\mathfrak{h}^\vee$ is a root system in $\mathfrak{h}^\vee$ with respect to the Weyl group, which is isomorphic to the factor-group M/M'.

Choosing an ordering in $\mathfrak{h}^\vee$ one can rewrite (3.2.2) as

$$\begin{gathered}\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha=\mathfrak{h}+\mathfrak{g}^{+}+\mathfrak{g}^{-},\\ \mathfrak{g}^{\pm}=\sum_{\alpha\in\Delta^{+}}\mathfrak{g}_{\pm\alpha} .\end{gathered} \tag{3.2.7}$$

This decomposition is called Gauss decomposition of the algebra, and it may be integrated for almost all elements g of the group G:

$$\begin{gathered} g=g_L g_D g_U \quad g_L\in \exp\mathfrak{g}^{-}=G_L \quad g_D\in\exp\mathfrak{h} \\ g_U\in\exp\mathfrak{g}^{+}=G_U . \end{gathered} \tag{3.2.8}$$

Let e_α be generators, corresponding to $\mathfrak{g}_\alpha$. In terms of decomposition (3.2.2) the commutation relations take the following form:

$$\begin{aligned}
&[h, e_\alpha] = \alpha(h) e_\alpha \\
&[e_\alpha, e_{-\alpha}] = h_\alpha \qquad \alpha(h_\alpha) = 2 \\
&[e_\alpha, e_\beta] = \begin{cases} 0 & \alpha+\beta \bar{\in} \Delta \\ e_{\alpha+\beta} & \alpha+\beta \in \Delta \end{cases}
\end{aligned} \tag{3.2.9}$$

If $G = sl(n,C)$, then D is the subgroup of diagonal matrices, $G_L (G_u)$ is the subgroup of lower (upper) triangular matrices; $M' = D$, M/M' is generated by permutation matrices of the form $\mathrm{diag}(1, 1, \underset{i}{0}\, 1 \ldots 1\, \underset{j}{0}\, 1 \quad 1) + (E_{ij} - E_{ji})$

(E_{ij} is a matrix with zero entries, except for the element (i,j), which is equal to 1.) $\mathfrak{h}$ is subgroup of traceless diagonal matrices, $\mathfrak{h} = \{\mathrm{diag}(h_1, \ldots h_n)\ ,\ \Sigma h_j = 0\}$ the root generators are $e_\alpha = E_{ij}\ (i<j)$, $e_{-\alpha} = E_{ij}\ (i>j)$ the roots $h_\alpha = \mathrm{diag}(0 \ldots \underset{i}{1}\, 0 \ldots 0\, \underset{j}{-1}, 0 \ldots 0)$.
For other examples see next ss.3.3.

Note now, that by means of decomposition (3.2.7) one can construct the maximal compact subalgebra $\mathfrak{u} \subset \mathfrak{g}$. Let us remind, that $\mathfrak{h}$ is an r-dimensional complex space $\mathfrak{h} = \mathfrak{h}^R + i\mathfrak{h}^R$ and $\{e_\alpha\}$ are generators of $\mathfrak{g}$ ($\mathfrak{g}_\alpha = a e_\alpha$). Then $i\mathfrak{h}^R$ is the Cartan subalgebra of $\mathfrak{u}$, and

$$i(e_\alpha + e_{-\alpha})\ ,\ \ (e_\alpha - e_{-\alpha})$$

are generators of $\mathfrak{u}$. The particular case is compact subalgebra su(2) of $\mathfrak{g} = sl(2,C)$, with $\mathfrak{h}^R = c\,\sigma_3$, $e_{\pm\alpha} = \sigma^{\pm}$.

Let us introduce now an invariant non-degenerate scalar product on $\mathfrak{g}$. Invariance means, that

$$([z,x],y)+(x,[z,y])=0 \qquad \text{for any} \quad z\in\mathfrak{g}.$$

Such product is known under the name of Cartan-Killing form. Its existence is equivalent to semisimplicity of the algebra. Explicit expression for this invariant form is

$$(x,y)=tr\,(ad_x\cdot ad_y)\frac{1}{C_V} \quad ad_x=[x,\cdot\,]. \tag{3.2.10}$$

In terms of structure constants

$$g_{ab}=C_{ac}^{d}\,C_{bd}^{c}\,\frac{1}{C_V} \tag{3.2.11}$$

which is obviously invariant and non-degenerate for any simple Lie algebra.

From (3.2.10) and (3.2.1) we deduce, that for any

$$(h_1,h_2)=\frac{1}{C_V}\sum_{\alpha\in\Delta_+}(\alpha,h_1)(\alpha,h_2). \tag{3.2.12}$$

The universal coefficient C_V is in fact quadratic Casimir eigenvalue in the adjoint representation (it will be discussed later, in ss.3.5).

Root subspaces $\mathfrak{g}_\alpha$ are mutually orthogonal, except for $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$:

$$(\mathfrak{g}_\alpha,\mathfrak{g}_\beta)\sim\delta_{\alpha+\beta} \tag{3.2.13}$$

In fact eq.(3.2.12) and (3.2.13) describes the form (3.2.10) on $\mathfrak{g}$ in coordinates (3.2.7).

3.3. SOME EXAMPLES OF GAUSS DECOMPOSITION

In the previous ss.3.2 we have already discussed the form of Gauss decomposition for the group $sl(n)$. Here we generalize this construction to other classical groups.

The key fact, allowing consideration of arbitrary simple groups is that there is a basis in adjoint representation of the Lie algebra $\mathfrak{g}$ in which the Cartan subalgebra has a diagonal form, and the subalgebras $\mathfrak{g}^{\pm}$ may be realized in terms of the corresponding triangular matrices. Let us proceed to concrete examples.

3.3.1. $\mathfrak{g} = so(2n+1)$ (B_n):

Group elements satisfy the condition $g J g^T = J$ where J is non-degenerate symmetric matrix. For an element of algebra we have: $xJ + Jx = 0$. In the above mentioned basis matrix J has the form of

$$J = \begin{pmatrix} 0 & & j \\ & 1 & \\ j & & 0 \end{pmatrix} \qquad \text{where} \quad j = \underbrace{\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}}_{n}. \tag{3.3.1}$$

Then $x \in SO(2n+1)$ if

$$x = \begin{pmatrix} A & \tilde{\xi} & B \\ \eta & 0 & -\xi \\ C & -\tilde{\eta} & -\hat{A} \end{pmatrix} \quad \begin{array}{l} \hat{A} = j A^T j, \quad B = -\tilde{B}, \quad C = -\tilde{C} \\ \xi = (\xi_1, \ldots \xi_n), \quad \eta = (\eta_1, \ldots \eta_n) \\ \tilde{\xi} = j\xi^T \quad \tilde{\eta} = j\eta^T \end{array} \tag{3.3.2}$$

Cartan element is

$$h = diag(h_1, \ldots h_n, 0, -h_n, \ldots -h_1). \tag{3.3.3}$$

The root subspaces, corresponding to the roots $e_i - e_j$ $(1 \le i, j \le h,\ i \ne j)$ belong to the matrix A (and $\widetilde{A}$), those corresponding to $e_i + e_j$ – to B, C, $e_j \sim \xi, \eta$? Generic element of subgroup G_L is

$$\begin{pmatrix} A & 0 & 0 \\ \widetilde{\eta} & 1 & 0 \\ C & -\widetilde{A}^{-1}\eta & \widetilde{A}^{-1} \end{pmatrix} \quad \begin{aligned} &(\widetilde{CA^{-1}}) + \widetilde{A}^{-1}\eta\widetilde{\eta}A + CA^{-1} = 0 \\ &\widetilde{\eta} = \eta^T J \qquad A_{ii} = 1 \\ &\qquad\quad\ A_{ik} > 0,\ i < k \end{aligned} \tag{3.3.4}$$

and $G_U = G_L^T$.

3.3.2. $\mathcal{G}$ = sp(n) (C_n):

Defining relations are now

$$xJ + Jx = 0 \qquad J = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}. \tag{3.3.5}$$

Thus

$$x = \begin{pmatrix} A & B \\ C & -\widetilde{A} \end{pmatrix} \qquad \begin{aligned} B &= \widetilde{B} \\ C &= \widetilde{C} \end{aligned} \tag{3.3.6}$$

$$h = \mathrm{diag}\,(h_1, \ldots h_n, -h_n, \ldots -h_1)$$

The root subspaces, corresponding to the roots $e_i \pm e_j$ $(i \ne j)$ are the same that those in the case of B_n. For the roots $2e_j$ (j=1,...,n) the root subspaces are matrix elements (j,n-j) of matrices B and C. Subgroup G_L consists of the matrices

$$\begin{pmatrix} A & 0 \\ C & \widetilde{A}^{-1} \end{pmatrix} \qquad \begin{aligned} &A_{ii} = 1,\ A_{ij} = 0 \quad i < j \\ &\widetilde{A}C = \widetilde{C}A \end{aligned} \tag{3.3.7}$$

and $G_U = G_L^T$.

3.3.3 $\mathcal{G}$ = so(2n) (D_n):

The form of matrices is just the same as in the case of so(2n+1). It is necessary only to omit the central row and column.

3.4. CHARACTERS OF LIE ALGEBRAS

Irreducible finite-dimensional representations of a Lie algebra $\mathfrak{g}$ may be described in terms of the weight lattice Q^{V}. There is a correspondence between a vector $\lambda \in Q^{V}$ and an irreducible representation π_λ in a space R_λ. Representations π_λ and $\pi_{w\lambda}$ are isomorphic for any element w of Weyl group W. In other words λ, which is called the highest weight of representation π_λ is defined by λ up to conjugations from the Weyl group. Representations π_{λ_j} with λ_j given by eq.(3.1.6) are called fundamental.

Representation π_λ is completely characterized by the following formulae:

$$\pi_\lambda(h)\xi = \lambda(h)\xi \qquad \xi \in R_\lambda \qquad h \in \mathfrak{h}$$
$$\pi_\lambda(e_\alpha)\xi = 0 \qquad \alpha \in \Delta_+ \tag{3.4.1}$$

The finite-dimensional space R_λ is spanned by the vectors:

$$\eta = \pi(e_{-\alpha_1})^{n_1} \dots \pi(e_{-\alpha_r})^{n_r}\xi \qquad \alpha_j \in \Pi,\ n_j \in \mathbb{Z} \tag{3.4.2}$$

These are eigenvectors of operators $\pi_\lambda(h)$:

$$\pi_\lambda(h)\eta = \mu(h)\eta \tag{3.4.3}$$

where eigenvalues $\mu(h)$ are given by the following expression:

$$\mu(h) = \lambda(h) - \sum_{i=1}^{r} n_i \alpha_i(h) \tag{3.4.4}$$

which is in accordance with (3.4.1).

Evidently, $\mu \in Q^{V}$ and the set of weights $P_\lambda = \{\mu\}$ (the weight diagramm) is invariant under the action of Weyl group.

Let us define now the characters of representations of $\mathfrak{g}$. The character of representation π_λ is a complex function, defined by

$$ch_\lambda(x) = tr\,\pi_\lambda(x), \quad x \in \mathfrak{g}. \tag{3.4.5}$$

In view of (3.2.4) we can restrict x under the trace sign to elements of Cartan subalgebra, $h = gxg^{-1}$. Then due to (3.4.4) we can rewrite (3.4.5) as follows:

$$ch_\lambda(h) = \sum_{\mu \in P_\lambda} \dim V_\mu \, e^{\mu(h)} \tag{3.4.6}$$

where $\dim V_\mu$ stands for multiplicity of μ.

It follows from (3.4.2), that $ch_\lambda(h)$ should naively look like

$$e^\lambda \prod_{\alpha \in \Delta_+} (1 + e^{-\alpha} + e^{-2\alpha} + \dots) = \frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2})}. \tag{3.4.7}$$

This formula is not correct, since it contains the infinite seria of descendants. In fact $ch_\lambda(h)$ is invariant under the action of Weyl group. This property is a consequence of its invariance with respect to the action of Ad_g. Since the denominator in (3.4.7) is Weyl-antiinvariant,

$$\prod_{\alpha\in\Delta^+}(e^{\alpha/2} - e^{-\alpha/2}) = \sum_{s \in \Delta} \det s \, e^{s\rho}$$

it is necessary to take antiinvariant combination in the numerator of (3.4.7) as well. After this correction we immediately obtain a finite combination of exponentials $e^{\mu(h)}$. This is the celebrated Weyl formula for characters:

$$ch_\lambda(h) = \frac{\sum_{s\in W} \det s \, e^{s\lambda(h) - s\rho(h)}}{\sum_{s \in W} \det s \, e^{s\rho(h)}}. \tag{3.4.8}$$

As follows from (3.4.5), in the limit $h \to 0$ (3.4.8) turns into a formula for dimension of representation ,

$$\dim \pi_\lambda = \prod_{\alpha\in\Delta_+} \frac{(\lambda+\rho, \alpha)}{(\rho,\alpha)}. \tag{3.4.9}$$

3.5. CASIMIR OPERATORS

Consider representation π_λ of $\mathfrak{g}$. There are some polinomial combinations of generators of $\mathfrak{g}$ which commute with all the generators in representation space. They are reffered to as Casimir operators. In accordance with Schur lemma they become c-numbers within any irreducible representation. The algebra of Casimir operators is finite generated. The orders of Casimir operators in original generators of $\mathfrak{g}$ are called invariants of the Lie algebra $\mathfrak{g}$. The lowest order is equal to two, while the highest one - to the Coxeter number h. Let e^a_λ be generators of the algebra $\mathfrak{g}$ in π_λ. Then the second-order Casimir operator has the form of

$$c_2^\lambda = g_{ab}\, e^a_\lambda e^b_\lambda \qquad (3.5.1)$$

where g_{ab} is the Cartan-Killing form (3.2.10). It is easy to check, that $[c_2^\lambda, e^a_\lambda] = 0$. Omitting the sign of representation, we may rewrite (3.5.1) in terms of orthonormal basis $(h_1,\dots,h_r)$ in $\mathfrak{h}$ and step generators e_α (see (3.2.9)):

$$c_2^\lambda = \sum_{j=1}^{r} h_j^2 + \sum_{\alpha\in\Delta_+} e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha . \qquad (3.5.2)$$

Using relations (3.2.9), one gets the expression:

$$c_2^\lambda = \sum h_j^2 + \sum_{\alpha\in\Delta_+} (e_{-\alpha} e_\alpha + h_\alpha). \qquad (3.5.3)$$

Thus, according to (3.4.1) the eigenvalue of C_2^λ is equal to

$$c_2^\lambda \to (\lambda, \lambda + 2\rho) \qquad (3.5.4)$$

Note, that the highest weight of the adjoint representation coincides with the maximal root θ. Therefore for this representation

$$c_v = (\theta, \theta + 2\rho)\frac{1}{2} \tag{3.5.5}$$

Let g be a dual Coxeter number,

$$g = 1 + \sum_{j=1}^{r} m_j \tag{3.5.6}$$

where integers m_i are defined from decomposition

$$\frac{\theta}{(\theta,\theta)} = \sum_{j=1}^{r} m_j \frac{\alpha_j}{(\alpha_j, \alpha_j)} \quad : \tag{3.5.7}$$

$$m_j = \frac{2(\theta, \lambda_j)}{(\theta, \theta)} \tag{3.5.8}$$

Now from (3.5.5) we obtain the important identity:

$$c_v = \frac{(\theta,\theta)}{2}\left(1 + \frac{2(\theta,\rho)}{(\theta,\theta)}\right) = 1 + \sum_{j=1}^{r} m_j = g. \tag{3.5.9}$$

3.6. KIRILLOV-KOSTANT CONSTRUCTION [3]

3.6.1. The structure of Lie algebra $\mathfrak{g}$ gives rise to symplectic structures on some representations R of $\mathfrak{g}$. In order to have Poisson brackets on functions on R, which satisfy Jacobi identities, one needs at least an invariant antisymmetric tensor of the third rank. Such tensor obviously exists in the case of adjoint or coadjoint representation and consists of structure constants c_{k}^{ij} of the algebra $\mathfrak{g}$. Whenever Cartan-Killing form on $\mathfrak{g}$ is non-degenerate (i.e. $\mathfrak{g}$ is semisimple) adjoint and coadjoint representations are equivalent. It is no longer the case for non-semisimple Lie algebras.For example,when there U(1)-factors, the action of group in coadjoint representation is preferable,since U(1)-generators do not act in adjoint representation at all. In what follows we shall take R to be coadjoint representation and consider semisimple Lie algebras $\mathfrak{g}$.

Let us denote generators of the algebra $\mathfrak{g}$ by e^j $j = 1,\dots,D=\dim G$, and coordinates in coadjoint representation by x^k. Let us consider the set of functions on the dual space $\mathfrak{g}^*$. Then there is a Lie-Berezin bracket [4],

$$\{f(x), \varphi(x)\} = C_k^{ij} x^k \partial_i f \, \partial_j \varphi \tag{3.6.1}$$

which satisfies Jacobi identity, since it is satisfied by the tensor of structure constants C_k^{ij}. (Note,that eq. (3.6.1) may be rewritten in terms of vector fields

$V_f^j = C_k^{ji} x^k \partial_i f$:

$$[V_f, V_\varphi] = V_{\{f,\varphi\}} \quad + \text{possible cocycles} \tag{3.6.2}$$

where $[V_f, V_\varphi]$ stands for ordinary commutator of differential operators.) However, the matrix $\omega^{ij}(x) = C^{ij}_k x^k$ in (3.6.1) is degenerate at any X, since there are invariant functions $f_a(X)$ such, that

$$\{X^k, f_a(X)\} = 0. \tag{3.6.3}$$

Independent functions of this kind are labeled by the subscript a, running through r $=\mathrm{rank}\,\mathcal{G}$ integer values, which lie between 2 and Coxeter number h, and are related to Casimir operators. If a Casimir operator looks like

$C_a = g_{i_1 \ldots i_a} e^{i_1} \ldots e^{i_a}$ then corresponding

$$f_a(X) = g_{i_1 \ldots i_a} X^{i_1} \ldots X^{i_a}. \tag{3.6.4}$$

Other invariant functions, which satisfy (3.6.3) are arbitrary functions of these independent f_a.

If the values of all independent functions are fixed,

$$f_a(X) = \mu_a \tag{3.6.5}$$

the whole space of functions becomes restricted, and in this restricted space the Lie-Berezin form ω^{jk} from (3.6.1) is non-degenerate. Since functions $f_a(X)$ are invariant, conditions (3.6.5) define orbits of coadjoint representation of G. The inverse of non-degenerate restriction of Lie-Berezin form on a coadjoint orbit, ω_{jk}, is known as Kirillov-Kostant form.

Let us consider two infinitesimal variations of point X within the same orbit, $\delta_1 X^a$, $\delta_2 X^b$. Since G acts transitively on the orbit, these may be represented as

$$\delta X = ad^*_Y \cdot X. \tag{3.6.6}$$

Kirillov-Kostant form allows one to construct two invariant functions on the orbit:

$$\hat{\Omega} = \omega_{jk}(X)\,\delta_1 X^j \delta_2 X^k \tag{3.6.7}$$

and

$$\Omega = \omega_{jk}(X)\, Y^j Y^k = c_{ijk} X^i Y^j Y^k, \tag{3.6.8}$$

We shall reffer to the second one, Ω ,as Kirillov-Kostant form, since in ss.4.6 we shall demonstrate,that Ω (but not $\hat{\Omega}$) appears related to the WZW action.

3.6.2. We shall often consider generators of algebra $\mathfrak{g}$ and elements of the dual space $\mathfrak{g}^*$ as elements of matrices (see the end of ss.3.2). For example,in the case of $\mathfrak{g}$ =sl(n) *these are nxn traceless matrices. Relation between* $\mathfrak{g}$ and $\mathfrak{g}^*$ is dictated by the pairing

$$\langle x, y \rangle = tr(xy). \tag{3.6.9}$$

Adjoint action of the group G an $\mathfrak{g}$ is defined by

$$Ad_g\, y = g y g^{-1} \tag{3.6.10}$$

and it is convenient to define coadjoint action by

$$Ad^*_g x = g^{-1} x g. \tag{3.6.11}$$

Invariant Casimir operators look like

$$Tr\, x^k. \tag{3.6.12}$$

The orbit of coadjoint representation, $Tr(X)^k = \mu_k$, may be alternatively defined by pointing out one of its points, X_o. Then any other point on the orbit is

$$X = g^{-1} X_o g \tag{3.6.13}$$

for some $g \in G$. The 1-form Y in (3.6.6) is given by

$$\delta X = \delta g^{-1} X_o g + g^{-1} X_o \delta g = ad^*_{g^{-1}\delta g} X \quad , \text{ i.e.}$$

$$Y = g^{-1} \delta g \tag{3.6.14}$$

Kirillov-Kostant 2-form Ω in (3.6.8) is equal to

$$\Omega = \langle X, [Y,Y] \rangle = \langle X, [g^{-1}\delta g\, g^{-1}\delta g] \rangle = \tag{3.6.15}$$
$$= \langle X_o [\delta g g^{-1}, \delta g g^{-1}] \rangle$$

Generic orbit of coadjoint representation may be naturally parametrized with the help of Gauss decomposition

$$g = g_L g_D g_U = g_L g'_U g_D \qquad g_L = g_L(\chi),\ g_U = g_U(\varphi) \tag{3.6.16}$$

If X_o is taken to be generic diagonal matrix, then diagonal elements g_D in fact do not act on X_o, and the orbit is parametrized by χ and twisted $\tilde{\varphi}$ (its dimension is thus D-r). Kirillov-Kostant form in this parametrization is

$$\Omega = Tr(X_o\, g_L^{-1} \delta g_L(\chi)\, \delta g'_U(\tilde{\varphi}) [g'_U(\tilde{\varphi})]^{-1}). \tag{3.6.17}$$

(For particular choices of X_o this form is still degenerate, and the orbit has lower dimension and is parametrized by some subset of χ's and $\tilde{\varphi}$'s.)

Since two-form Ω is closed, it is possible to define a 1-form α , $\alpha = d^{-1}\Omega$,and α gives rise to the action

$$\mathcal{A} = \int \alpha \tag{3.6.18}$$

which is generalization of the short action

$$\int p\, dq \tag{3.6.19}$$

arising in the case of Heisenberg group. The form of Kirillov--Kostant form, arising when Gauss decomposition is used, in fact appears very close to representation (3.6.19) after appropriate choice of variables. Before we proceed to detailed discussion of this point in the next section, let us note, that Gauss decomposition is not valid at some manifolds of non-vanishing co-dimension on the orbit. Therefore appropriate boundary conditions should be specified at these points. See the first paper of ref.3 for detailed discussion of boundary conditions in the case of finite-dimensional groups G. In the case of infinite chiral algebras the problem of these boundary conditions is closely related to accurate construction of Felder's projection operators [5].

In ss.4.6 we shall briefly discuss generalization of Kirillov-Kostant construction to the case of KM algebra. For this purpose group elements g should be considered as functions of z, and all formulae should be accurately central extended.

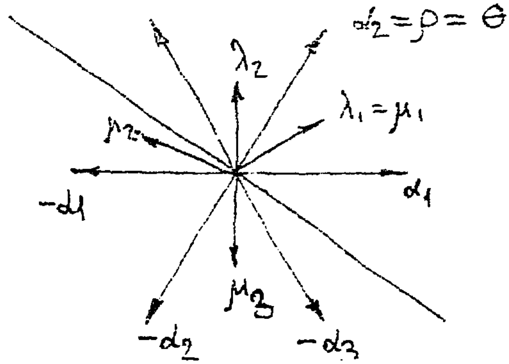


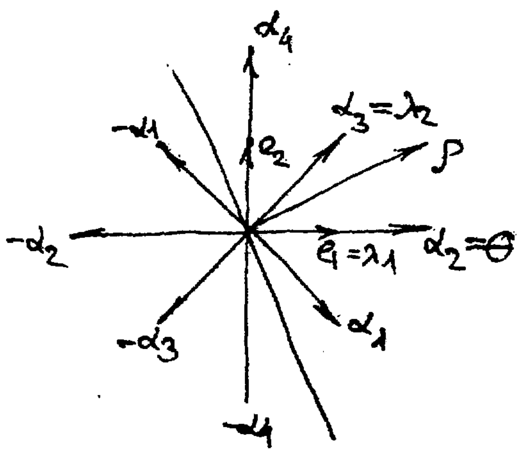
Fig.3

a) The roots of algebra $sl(3) \simeq A_2$

$\alpha_1=e_1-e_2$; $\alpha_2=e_1-e_3$; $\alpha_3=e_2-e_3$; $\alpha_1, \alpha_3 \in \Pi$; $\rho=\alpha_2=e_1-e_3$;

$\lambda_1=e_1$, $\lambda_2=-e_3$; $ht_{\alpha_1}=ht_{\alpha_3}=1$, $ht_{\alpha_3}=2$

Correspondense between the fields $\tilde{W}_j$ and positive root subspaces : $\tilde{W}_j \Rightarrow g_{\alpha_j}$

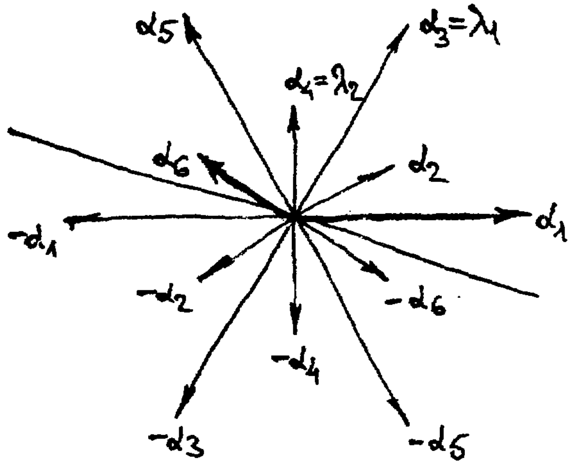


b) The roots of algebra $sp(2) \simeq C_2$

$\alpha_1=e_1-e_2$; $\alpha_2=2e_1$, $\alpha_3=e_1+e_2$, $\alpha_4=2e_2$; $\alpha_1, \alpha_4 \in \Pi$;

$\rho=2e_1+e_2$; $\lambda_1=e_1$; $\lambda_2=e_1+e_2$; $ht_{\alpha_1}=ht_{\alpha_4}=1$, $ht_{\alpha_3}=2$, $ht_{\alpha_2}=3$

$\tilde{W}_j \rightarrow g_{\alpha_j}$



c) The roots of algebra G_2.

$\alpha_1 = -2e_3 + e_1 + e_2$; $\alpha_2 = e_1 - e_3$; $\alpha_3 = 2e_1 - e_2 - e_3$; $\alpha_4 = e_1 - e_2$

$\alpha_5 = -2e_2 + e_1 + e_3$; $\alpha_6 = e_3 - e_2$; $\alpha_1, \alpha_6 \in \Pi$; $\rho = 5\alpha_6 + 3\alpha_1$

$\lambda_1 = \alpha_3$, $\lambda_2 = \alpha_4$

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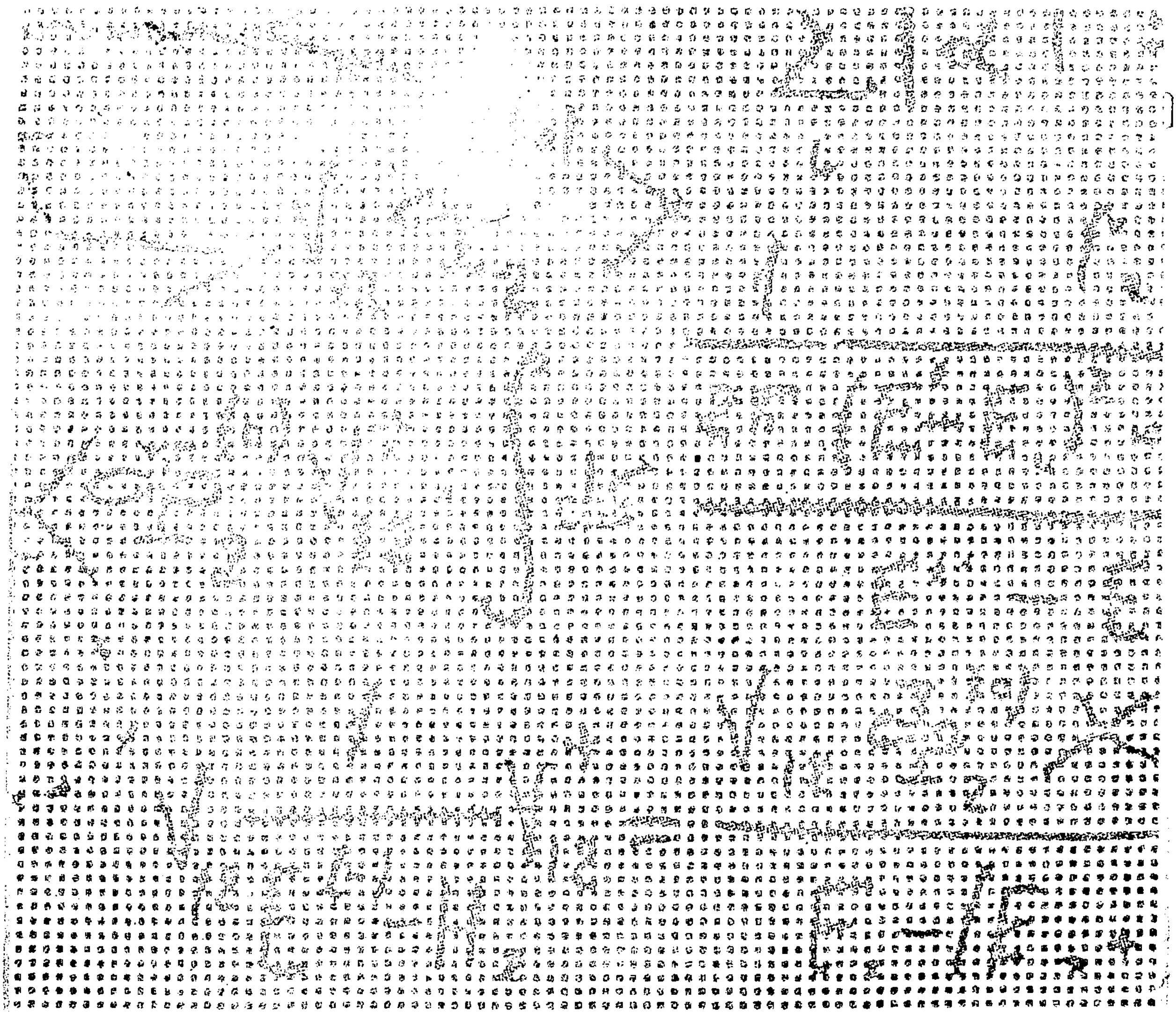

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