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Galilei-Invariant Quantum Field Theories with Pair Interaction

Review[1]

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Abstract

We exhibit a class of quantum field theories where particles interact with pair potentials and for which the time evolution exists in the Heisenberg representation. The essential condition for existence is the stability in the thermodynamic sense and this is achieved by having the interaction fall off with the relative momenta of the particles. This can be done in a Galilei-invariant manner. We show that these systems have some mixing behaviour which one postulates in ergodic theory but difficult to prove for classical systems.

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1 Introduction

The quantum dynamics of N particles interacting via 2-body potentials is by now well understood. Even if the potentials have $1/r$ singularities (or slightly worse) the time evolution determined by

$$H_N = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m_i} + \sum_{i>j} v(\vec{x}_i - \vec{x}_j) \tag{1.1}$$

exists as a 1-parameter group of automorphisms of the observable algebra. This is in contradistinction to the classical problem where for $v \sim 1/r$ (and $N \geq 4$) there are orbits where particles go to infinity in a finite time [1]. That quantum mechanics got rid of these pathologies [2] is not all what people studying bulk matter want. There one does not count the number of particles and one needs results which hold uniformly for large N. In these theories N should be considered as a dynamical variable. Frequently even phase relations between states with different local particle numbers become important. This situation is conveniently described by second quantization where (1.1) is expressed by creation and annihilation operators $a^*(x)$, $a(x)$ as

$$H = \frac{1}{2m} \int dx \vec{\nabla} a^*(x) \vec{\nabla} a(x) + \int dx dx' a^*(x) a^*(x') v(x - x') a(x') a(x) \tag{1.2}$$

and we have to make sense out of that. Now $N = \int dx a^*(x) a(x)$ is not fixed any more. What one usually does is to put the system in a volume V to go to the Fock representation, to quantize the k-modes, and then in perturbation theory to get rid again of V by replacing $\frac{1}{V}\sum_k$ by $\int \frac{d^3k}{(2\pi)^3}$. Since in reality V is fictitious and no system is completely isolated but has some interactions with the outside it seems more reasonable to take (1.2) as it stands in infinite space $\mathbf{R}^3$. Thus one faces the problem whether it determines a time evolution in the Heisenberg representation by

$$a(x,t) = e^{iHt} a(x) e^{-iHt}. \tag{1.3}$$

It is usually assumed that this works, at least if v is a sufficiently well-behaved function. In this review we shall investigate this question and find that even for very smooth v the formal expression will not determine a time evolution unless the potential v is stable, i.e., there is a constant A such that

$$H_N > -AN \quad \forall N. \tag{1.4}$$

The importance of stability for time evolution was first emphasized in [3]. Lest the reader might think that this is a question of ε-tics we hasten to say that it is just a question of realizing the far reaching physical implications of (1.3). If it determines a time evolution then the latter would be independent of the state of the system. Though from a philosophical viewpoint one wants a time evolution of the observables irrespective of the state of the system the equation (1.3) actually requires a lot. Since N is determined by the state and will be ∞ for a finite density of particles we need some limit for the velocities of the particles as $N \to \infty$. This is guaranteed by (1.4) and what happens for potentials where (1.4) fails is that a hot cluster develops and the mean kinetic energy per particle goes with N to infinity. This has been shown for classical mechanics by recent computer solutions of the equations of motion for $v(x) = -e^{x^2}$ [4,5]. To the extent that (1.3) also has to contain the classical limit, these investigations show that such a phenomenon will also occur in the quantum case and prevents time evolution to exist on the algebraic level.

The success of an optimistic treatment of (1.3) is due to the fact that the Coulomb interaction which is the relevant potential for condensed matter physics is stable for fermionic electrons [6,7]. Generally the question whether a potential is stable is rather delicate. Clearly, if v is all attractive ($v < 0$), then $\sum_{i>j} \sim -N^2$ and H_N is not stable. But even Coulomb-type potential energies

$$\sum_{i>j} e_i e_j v(x_i - x_j), \qquad e_i = \pm e, \qquad \sum e_i = 0,$$

may or may not be stable. One finds:

if $v(\vec{x}) = 1/|x|$, H_N is stable for fermions but not for bosons [6,7],

if $v(\vec{x}) = \dfrac{1 - e^{-\mu|x|}}{|x|}$, H_N is stable for fermions and bosons [8],

if $v(\vec{x}) = |\vec{x}|e^{-|x|}$, H_N is not stable, neither for fermions nor for bosons [8].

There is no local potential $v \neq 0$ known such that both v and $-v$ are stable. This shows that any expansion of (1.3) in a power series in v is doomed to divergence. If this series were to converge for any sign of v it would determine a time evolution for v and $-v$ and this is not possible. However, the preceding intuitive argument points to the correct cure for the instability. Taking a potential which becomes ineffective for high (relative) velocities of the particles the velocities should remain bounded for $N \to \infty$ and there is a chance that (1.3) gives a time development on the level of the algebra of observables. In this paper we shall exhibit conditions when it actually does.

The reader might wonder why the Hamiltonian (1.2) with a smooth v, which looks so nice and selfadjoint can fail to determine a time evolution. (1.2) contains two dangers. It is unbounded since $\rho(x) = a^*(x)a(x)$ is an unbounded operator and because $\int d^3x d^3x'$ goes over $\mathbf{R}^3 \times \mathbf{R}^3$. The latter, though preventing H from becoming an element of the observable algebra, does not make any difficulty in a time derivative for a local operator $\dot{a}(x) = i[H, a(x)]$. As $\rho(x')$ and $a(x)$ commute for $x \neq x'$ for a v with finite range the integrals go effectively only over a finite region. The real trouble maker is $\rho(x)$ which in some representation like the Fock representation is well defined but in others, like the one based on the tracial state, is truly infinite. Though there is no doubt that for a smooth v H from (1.2) determines a time evolution in the Fock representation, an automorphism of the algebra of observables, which would be valid in any state, cannot be expected in general. For our potentials with momentum cut-off H will still not be a bounded operator but in any representation with a time invariant state it can be renormalized in such a way that it becomes selfadjoint. Then the unitaries e^{iHt} exist but they do not belong to the observable algebra but depend on the state. Nevertheless, the time evolution $a \to e^{iHt} a e^{-iHt}$ will be state independent.

2 The Classical Many-Body System

The problem of existence of the time evolution in quantum field theory is not primarily a matter of mathematical sophistication but of understanding the physics of the many body problem. For this purpose we shall first consider the motion of N classical particles as determined by (1.1). We are not interested in difficulties arising from a singularity of the potential or of its long range and hence we shall take v to be smooth and short range. The question is whether some typical feature arise in the limit $N \to \infty$. Recent computer studies [4,5] have shown that there is a dramatic difference between potentials which are stable or unstable in the sense of (1.4). Typically $N = 400$ particles were studied on a torus T^2 with a density such that mean particle distance = 10× range of the potential. For stable potentials, $\sum e_i e_j e^{-|x_i - x_j|}$, $\sum e_i = 0$ [9], what happened was exactly what one learns in statistical physics: spatial inhomogeneities diffused away and the momentum distribution tended to a Maxwell-Boltzmann distribution. For unstable potentials and low total energy E the opposite happened, a hot cluster developed out of a homogeneous distribution. The lower the energy the greater the fraction of all particles which the cluster contained and the bigger the temperature difference which was building up. This means that the system has a negative (microcanonical) specific heat and thermodynamics works differently than usual. Nevertheless, the standard ergodic arguments to justify thermodynamics apply since these systems are presumably K-systems [10] and thus certainly ergodic. This means that one particular macroscopic feature dominates the energy shell and almost any orbit will end up in these regions of phase space and the typical feature will appear. Though an analytic solution of the Hamilton equation is beyond the reach of present day mathematics the dominant features in phase space can be easily singied out and will be done first.

Classically the microcanonical state is given by the probability measure $\delta(H_N - E)e^{-S(E)}$ in phase space. It becomes more suggestive when projected down to configuration space V^N. If we denote a point in configuration space $(x_1, x_2, \ldots, x_N)$ by X and the potential energy $\sum_{i>j} v(\vec{x}_i - \vec{x}_j)$ by $\Phi(X)$ then (in two dimensions, and $m = 1$)

$$\rho(X) = \int_{-\infty}^{\infty} d^2p_1 \ldots d^2p_N \delta(H_N - E)e^{-S(E)} = \frac{2^N \pi^N}{(N-1)!} e^{-S(E)}(E - \Phi(X))^{N-1}. \tag{2.1}$$

Those X have the highest probability where $|\Phi(X)| \gg |E|$ and, if the system is unstable and $E \sim N$, there are always X where this holds. For instance, if $v(x) = -e^{-x^2}$ then $\inf_X \Phi(X) = -N(N-1)/2$ and those X where all particles sit on top of each other have the highest weight. Of course, these X fill only a small volume and we shall now estimate the probability of one big cluster compared with, say, two somewhat smaller clusters. If $V_0 \subset V$ is a volume of the size of the range of v then to have N_c particles in one cluster small enough that all particles interact with each other costs volumewise $(V_0/V)^{N_c-1}$. On the other hand we have seen that there the probability density $\rho(x)$ has for $|E| \ll N_c^2$ a value $(N_c^2)^{N-1}$ so that to have one cluster with N_c particles has a probability

$$P_1 \sim \left(\frac{V_0}{V}\right)^{N_c-1} (N_c^2)^{N-1} \tag{2.2}$$

where N_c^{2N-2} is the value of $\rho(X)$ and $(V_0/V)^{N_c-1}$ arises since the center of mass of the cluster may be anywhere. P_1 attains its maximum for $N_c \sim \dfrac{2N}{\ln(V/V_0)}$ and thus the fraction $2/\ln(V/V_0)$ of all particles likes to be in one cluster. To have 2 clusters with $N_c/2$ particles each has by the same argument a probability

$$P_2 \sim \binom{N_c}{N_c/2} \left(\frac{V_0}{V}\right)^{N_c-2} \left(\left(\frac{N_c}{2}\right)^2\right)^{N-1} \tag{2.3}$$

and for large N we see $P_2 = P_1 \cdot \dfrac{V}{V_0} 2^{-N}$ and thus it becomes much more likely to have one big cluster. These rough arguments can be substantiated by exact calculation by discretized versions of (1.1) [1,11]. In the computer simulation of the dynamics V/V_0 is about 10^4 and thus one expects about 1/3 of all particles in the cluster. In fact, one found in the computer solutions that N_c fluctuates around 140 ± 10.

Next we turn to the question of the local temperature and define a map $T : V^N \to \mathbf{R}^+$ by ($m = 1$)

$$T(X) = \int_{-\infty}^{\infty} d^2p_1 \ldots d^2p_N \frac{\vec{p}_1^2}{2} \delta(E - H_N) e^{-S(E)} = \langle \frac{\vec{p}_i^2}{2} \rangle_X \tag{2.4}$$

such that the local temperature is just the expectation value of the kinetic energy of a single particle at a given point X in configuration space.

Remarks (2.5)

1. At a given point $X \in V^N$ the distribution is the same for all $\vec{p}_i$ (we could have integrated $1/N \sum_i \vec{p}_i^2$ instead of $\vec{p}_1^2$). This means that in equilibrium the particles outside the cluster are just as hot as the ones inside.

2. The microcanonical state differs from the canonical state by the X-dependence of T. If we replace $\delta(H_N - E)e^{-S(E)}$ by $e^{-\beta(H_N - F(\beta))}$ then $T = 1/\beta$ irrespective of X. F is the free energy. Of course, since the cluster can sit anywhere T depends only on the correlations between the particles and $V^{N-1} \int d^2x_2 \ldots d^2x_N T(X)$ becomes independent of X and in the limit $N \to \infty$ equals the microcanonical temperature $(\partial S(E)/\partial E)^{-1}$.

3. It has to be remembered that $T(X)$ is the temperature for the configuration X in the equilibrium state. In the dynamical studies the following transient features appear. If a cluster with N_c particles is formed then locally the gain both in potential and in kinetic energy is $\sim N_c^2$. Thus in the cluster the system heats up to a temperature $\sim N_c$. However, the cluster is not thermally insulated, and eventually its temperature is distributed over all particles to give a common equilibrium temperature $\sim N_c^2/N$.

With the Hamiltonian (1.1) we calculate ($m = 1$)

$$T(X) = \frac{1}{N}(E - \Phi(X)). \tag{2.6}$$

Remarks (2.7)

1. Since $-\Phi \sim N_c^2$ we get $T \sim N_c^2/N$ as soon as $E \ll N_c^2$, for instance, if we start with $E \sim N$ and $N_c \sim N/3$.

2. The definition (2.4) gives an X-dependent temperature even for thermodynamically stable systems. However, if $\Phi(X) > -cN$ and $E \sim N$ there will be only fluctuations around $T = \frac{1}{N}(E - V^{-N} \int d^2x_1 \ldots d^2x_N \Phi(X))$.

Since we have seen $N_c/N \sim 2/\ln(V/V_0)$ we see $T \sim N_c^2/N \sim N/(\ln V/V_0)^2$ and thus for $N \to \infty$ T increases indefinitely, irrespective of whether V or V/N remains constant in this limit. Thus if we start with the same particle density and mean velocity but increase N we end up with a motion which gets faster and faster. Actually, the derivation of the Vlasov equation [12] can be carried over to unstable situations. There the limit $N \to \infty$ of the time evolution from

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2} + \frac{1}{N} \sum_{i>j} v(\vec{x}_i - \vec{x}_j)$$

was considered. We do not have $1/N$ in front of Φ but we can produce N in front of the kinetic energy by rescaling the time $t \to \tau := t/\sqrt{N}$. Also for quantum systems with attractive regularized $1/r$-potentials one can show that the limit $N \to \infty$ of the time evolution exists only after some rescaling of t [13]. In these cases the result is that the time evolution is governed by a mean field theory provided we rescale the time to resolve the speeded up motion. This obviously excludes the possibility of a limiting dynamics without scaling of the time and thus a time evolution in quantum field theory without reference to the state which determines the number of particles involved. Such a dynamics is possible only for Hamiltonians satisfying the stability condition and in section 4 we shall demonstrate that it is actually true for a reasonably large class of potentials.

3 Quantum Observables and Their Time Development

Taking the thermodynamic limit $N, V \to \infty$ means that we concentrate our interest on properties that are independent of the exact number of particles, i.e. we concentrate on quantities like density, the speed of particles and correlations between finite parts of the system. These properties which we can measure we call the observables. We can think of an observable A independently of a particular physical situation. Such a physical realization corresponds to a state ω, where $\omega(A) \in \mathbf{R}$ is the average of the results of measurements of A in the state ω. As mathematical quantities we can add and multiply observables, so they form an algebra. Apart from the algebraic structure we can consider a topological structure: First we assume that results of measurements are bounded, so the observables are bounded. Next we call observables close to one another, if measurements give essentially the same results independently of the physical realization. So we can define a norm as $\|A - B\| = \sup_\omega |\omega(A) - \omega(B)|$. In fact we could deal with a Jordan algebra, where the product is symmetric but not associative. But it is mathematically by far more convenient to pass to a C^*-algebra $\mathcal{A}$ with associative product that is not commutative. This requires the introduction of another mathematical operation, the adjunction $A \to A^*$. In standard quantum mechanics this corresponds to going to the hermitian conjugate and the following rule relates this process to the norm

$$\|A\| = \|A^*\| = \|A^*A\|^{1/2} = \|AA^*\|^{1/2} \tag{3.1}$$

For a deeper insight into this algebraic approach to quantum statistics we refer to [14,15,16]. According to the previous arguments it seems clear that an observable has to remain an observable in the course of time. Thus it appears as a minimal requirement that the time evolution corresponds to an automorphism group of $\mathcal{A}$. For bulk matter this time evolution should be the limit of the time evolution of an increasing number of particles. For finitely many particles we have a Hamiltonian time evolution (Heisenberg representation)

$$\tau_t^N(A) = e^{iH_N t} A e^{-iH_N t}$$

so the question is whether

$$\lim_{N \to \infty} \tau_t^N(A) = \tau_t A \tag{3.2}$$

exists and is independent of a particular limiting procedure chosen.

Of course, such a time automorphism group $\tau_t : \mathcal{A} \to \mathcal{A}$ of the observable algebra also defines a time evolution of the set of states E (Schrödinger representation):

$$\omega_t(A) = \omega(\tau_t A) =: (\tau_t^* \omega)(A). \tag{3.3}$$

For $N \to \infty$ we can also consider

$$\lim_{N\to\infty} \omega \circ \tau_t^N = \tau_t^* \circ \omega,$$

but the existence of such a τ_t^* on a convenient subset of E does not yet guarantee that there is an automorphism τ_t with $\tau_t^*\omega = \omega \circ \tau_t$. To see the various problems more clearly we shall first look at some elementary examples. Some difficulties occur even before the thermodynamic limit.

Examples (3.4)

1. The Observable Algebra of One Particle.
   Usually one considers the Weyl algebra as the relevant algebra, i.e., the algebra built by $W(u,v) = \exp[i(pu + vx)]$, $u, v \in \mathbf{R}^d$. For this algebra the free time evolution is given by the automorphisms group

   $$\tau_t W(u,v) = W(u + vt, v) = e^{iH_0t} W(u,v) e^{-iH_0t}.$$

   Also for the harmonic oscillator the algebra is stable,

   $$\tau_t W(u,v) = W(u\cos t + v\sin t, v\cos t - u\sin t),$$

   but for completely harmless potentials V with $H = H_0 + V$ the time translate $\tau_t W(z) = e^{iHt}W(z)e^{-iHt}$ does not belong to the norm closure of $W_f = \sum_{z\in I} f(z)W(z)$, and I finite [17]. Taking instead as C^*-algebra the algebra of compact operators every e^{iHt} would implement an automorphism. Since density matrices are compact operators all states of W_f defined by density matrices have a well defined time evolution. Of course enlarging W_f to its strong closure in the unique continuous representation ensures the existence of the time evolution as an automorphism group in this case.

2. For bosons one considers the CCR algebra [18] as relevant C^*-algebra, namely the algebra created by

   $$e^{i(a(f)+a^\dagger(f))} = W(f),$$

   $$W(f)W(g) = W(f+g)e^{\sigma(f,g)} \qquad \text{with } \sigma(f,g) = \langle f|g\rangle - \langle \bar{f}|\bar{g}\rangle,$$

   $$\|W(f)\| = 1 \qquad \text{and} \qquad \|W(f) - W(g)\| = 2 \text{ if } f \neq g,$$

   where $a(f)$, $a^*(f)$ are annihilation and creation operators for bosons with wave function f ($a(f) = \int dx \bar{f}(x)a(x)$).

   Here we have two choices:

   (a) $f \in L^2(\mathbf{R}^3)$ and has compact support. Then the observables are called strictly local, which has structural advantages. But free time evolution immediately takes one out of this algebra.

   (b) $f \in L^2(\mathbf{R}^3)$ without restriction. Free time evolution is now an automorphism. But the thermodynamic limit of states becomes more problematic, because strictly local observables are no more dense in the algebra.

3. For fermions we take the CAR-algebra [18]. Here the creation and annihilation operators satisfy $[a(f), a^*(g)]_+ = \langle f|g\rangle$ and are bounded

$$\|a(f)\| = \|f\|_2.$$

Now the algebra built by functions with compact support is dense in the algebra of L^2-functions. The CAR-algebra is separable, since $L^2(\mathbf{R}^3)$ is separable. Free time evolution $\tau_t a(f) = a(e^{ih_0 t} f)$ with $h_0 = p^2/2$ is an automorphism.

4. A favorite test algebra for structural considerations is the CAR-algebra on a fermi lattice. Here $f \in L^2(\mathbf{R}^3)$ is replaced by $f \in L^2(Z^\nu)$. Quasifree time automorphisms are defined by $\tau_t a(f) = a(e^{iht} f)$. Furthermore reasonable interactions lead to a well-defined automorphism as long as the interactions are sufficiently decreasing for long distances [18,19].

5. As last example consider a system of particles with hard-cores and with short range potentials. From a physical point of view the time evolution does not seem to rise any problem, the system is stable and only finitely many particles can influence another particle at a given time. But we are not allowed to take the algebra of free or interacting particles (i.e. the algebra built by creation and annihilation operators) any more, because the possible states differ: For hard-cores every state reduced to a bounded region Λ corresponds to a Fock state representation $\mathcal{H}_{\Lambda,a}$ where the wave functions $\psi(x_1, \ldots, x_n)$ vanish if $|x_i - x_j| < a$ [20], where $n \leq N_{\max}(\Lambda)$. Instead we can consider the C^*-algebra as the inductive limit of $\mathcal{B}(\mathcal{H}_{\Lambda,a})$. But this algebra is not separable any more nor can we express its observables in such an explicit way that we could handle the questions of how to extend the time evolution of $\mathcal{H}_{\Lambda,a}$ to the limiting algebra. In which sense time evolution exists for hard-core particles in the thermodynamic limit is an open problem.

In the following we shall always think about electrons and thus take as observable algebra the CAR-algebra $\mathcal{A}$ over $\mathbf{R}^3$. Therefore we shall use the terminology that $A \in \mathcal{A}$ is a local observable. Global observables will arise by the weak closure $\pi(\mathcal{A})''$ in a particular representation π and will be state dependent. For the convenience of the reader who is not so familiar with this terminology we shall summarize it in the Appendix.

Once we will have shown the existence of a time automorphism of an infinite system this will enable us to find relations between the thermodynamic behaviour and properties of the time evolution. A first requirement for an equilibrium state ω is its time invariance $\omega \circ \tau_t = \omega$. Whereas there is no general result about the approach to equilibrium the conditions for return to equilibrium can be spelt out easily in quantum mechanics. Let

$$\omega_A(B) := \omega(A^*BA)/\omega(A^*A), \qquad A, B \in \mathcal{A}, \qquad \omega(A^*A) \neq 0,$$

be a local perturbation of the state ω. We say that ω is stable under local perturbations if

$$\lim_{t \to \pm\infty} \omega_A(\tau_t B) = \omega(B) \quad \forall\, B \in \mathcal{A}.$$

This property is related to notions of mixing [21]
(ω is mixing $\Longleftrightarrow$ $\lim_{t\to\pm\infty} \omega(A\tau_t(B)C) = \omega(AC)\omega(B)) \forall A, B, C \in \mathcal{A}$) and asymptotic abelianness [18,22,23,24] (ω is weakly asymptotic abelian if $\lim_{t\to\pm\infty} \omega(A[\tau_t(B), C]D) = 0 \,\forall A, B, C, D \in \mathcal{A}$). In fact, in the purely quantal context where the center Z_ω is trivial these notions coincide. (The center are those elements of the weak closure of $\mathcal{A}$ in the representation given by ω which commute with all elements of $\mathcal{A}$.)

Lemma (3.5): If ω is an invariant state with $Z_\omega = \{z1 | z \in \mathbf{C}\}$ then the following are equivalent:

(i) ω is stable under local perturbations,

(ii) ω is mixing,

(iii) $\tau_t(B) \rightharpoonup \omega(B)$,

(iv) $(\mathcal{A}, \tau, \omega)$ is weakly asymptotically abelian.

Proof:

(i) $\Rightarrow$ (ii) by polarization,

(ii) $\Rightarrow$ (i) is trivial,

(ii) $\Leftrightarrow$ (iii) is trivial,

(ii) $\Rightarrow$ (iv) $\omega(A[\tau_t(B), C]D) \to \omega(ACD)\omega(B) - \omega(ACD)\omega(B) = 0$,

(iv) $\Rightarrow$ (ii) Any weak limit point of $\tau_t(B)|t \in \mathbf{R}$ has to be in the center and therefore a multiple of unity, $c\mathbf{1}$. Since ω is τ-invariant $\tau_{t_n}(B) \rightharpoonup c\mathbf{1}$ implies $c = \omega(B)$ and thus all weak limit points are c and $\tau_t(B) \rightharpoonup \omega(B)$. This implies (ii) resp. (iii).

Remarks (3.6)

1. The operator topology in which the limits are attained is essential for us. In (iii) and (iv) only weak convergence is possible. (iii) means that for any vector $| \perp \rangle$ in $\mathcal{H}_\omega$ perpendicular to $|\Omega\rangle$ the limit of $e^{iHt} | \perp\rangle$ is zero. As strong limit this would contradict the unitarity of e^{iHt}. Similarly, $\tau_t(a) \to \omega(a) \forall a \in \mathcal{A}$ would imply $\tau_t(a^* a) \to \omega(a^* a)$ and $\tau_t(a^* a) \to \omega(a^*)\omega(a)$ since strong limits respect products. This would mean
$$0 = \omega(a^* a) - \omega(a^*)\omega(a) = \omega((a^* - \omega(a^*))(a - \omega(a)))$$
or that ω is dispersionfree for all $a \in \mathcal{A}$. For non-commutative systems this is impossible.

2. Bounded regions are weak* compact in von Neumann algebras but not norm compact in C^*-algebras. Thus any infinite bounded sequence in $\pi(\mathcal{A})$ has weak accumulation points but they may not belong to $\pi(\mathcal{A})$ but only to $\pi(\mathcal{A})''$. Only norm limits are surely in $\pi(\mathcal{A})$ but norm convergence is in many cases impossible.

3. The conditions (3.5) imply that ω is extremal invariant, i.e. not of the form $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ with ω_i invariant, $\omega_1 \neq \omega_2$ and $0 < \lambda < 1$. If this were the case then $\pi_\omega = \pi_{\omega_1} \oplus \pi_{\omega_2}$ and (iii) required $\pi_{\omega_1}(\tau_t a) \rightharpoonup \omega(a)$. Taking the expectation value with ω_1 gives $(1 - \lambda)(\omega_1(a) - \omega_2(a)) = 0$ implying $\omega_1 = \omega_2$.

4. The essential message of (3.5) is the following. The opposite requirements of the system to be completely quantal ($\mathbf{Z}_\omega = c\mathbf{1}$) and to be classical for large times (asymptotic abelianness) constrains it to the extent that all observables have to approach their equilibrium value (or equivalently all states tend to equilibrium). This is a special bonus of quantum theory, classically the construction of mixing systems is nontrivial.

Deeper are the results concerning the existence and the properties of those states which are the generalization of the canonical Gibbs states.

Definition (3.7): A state is called an equilibrium or KMS state [25,26,27,28] if it has the KMS property, i.e. $\omega(A\tau_t B) = g_{AB}(t)$ is a continuous function in the strip $\{t = t_0 + i\gamma; t_0 \in \mathbf{R}, 0 \le \gamma \le \beta\}$, analytic in the interior and it satisfies $g_{AB}(t + i\beta) = \omega(\tau_t B \cdot A)\ \forall\ A, B \in \mathcal{A}$.

For finite systems with Hamiltonian H the KMS states are given by

$$\omega(A) = \mathrm{Tr}\ e^{-\beta H} A / \mathrm{Tr}\ e^{-\beta H}$$

and the question arises how such states can be constructed for the infinite system. For a quasifree time evolution $\tau_t a(f) = a(e^{iht} f)$ the KMS state for given temperature β and chemical potential μ over the CAR is given by

$$\Phi_\beta(a^*(f)a(g)) = \langle g | \frac{1}{1 + e^{\beta(h-\mu)}} | f \rangle$$

and vanishing truncated $n \neq 2$ point functions. A special state is the tracial state

$$\Phi_0(a^*(f)a(g)) = \frac{1}{2}\langle g | f \rangle.$$

The following lemma shows that once we have solved the problem for the time development it is easy to find a limiting KMS state.

Lemma (3.8): Assume ω_n is a KMS state for an automorphism group τ_t^n. Assume that $\forall A \in \mathcal{A}$

$$\lim_{n\to\infty} \|\tau_t^n A - \tau_t A\| = 0.$$

Let ω be a weak* limit point of ω_n. Then ω is τ-KMS.

Proof:

$$\lim_{n\to\infty} \omega_n(A\tau_t^n B) = \lim_{n\to\infty} \omega_n(A\tau_t B) = \omega(A\tau_t B)$$

for $t \in \mathbf{R}$ resp. $t \in R + i\beta$, so ω_n converges on a set of uniqueness. Furthermore these functions are equicontinuous in compact regions in the strip. Therefore they converge in the strip and the limit is again an analytic function [29] with the KMS-property.

The lemma can be used in two ways: either we consider an increasing set of finite subsystems $\mathcal{A}_n$, $n \in N$, such that $\mathcal{A} = \mathcal{A}_n \otimes \mathcal{A}_n^c\ \forall n \in N$ and $\lim_{n\nearrow\infty} \mathcal{A}_n = \mathcal{A}$ and a time evolution $\tau^n : \mathcal{A}_n \to \mathcal{A}_n$ with corresponding KMS states ω_n, i.e.

$$\tau^n A = e^{iH_n t} A e^{-iH_n t}, \qquad A, H_n \in \mathcal{A}_n \qquad \text{and} \qquad \omega_n A = \mathrm{Tr}\ \rho_n A \otimes \omega_{|\mathcal{A}_n^c},$$

where $\rho_n = e^{-\beta H_n}/\mathrm{Tr}\ e^{-\beta H_n}$ is trace class. $\omega_{|\mathcal{A}_n^c}$ can be chosen at will and disappears in the weak* limit. Because of the weak* compactness of the state space $\mathrm{Tr}\ \rho_n \otimes \omega_{|\mathcal{A}_n^c}$ always has some weak* cluster points $\bar{\omega}$. According to the lemma $\bar{\omega}$ are τ-KMS if $\tau^n \to \tau$. Uniqueness does not follow because the limit can depend on the chosen subsequence.

An alternative to find KMS states is to choose τ^0 to be an automorphism group of $\mathcal{A}$ with KMS state ω_0. Let τ_t^n be a local perturbation of the dynamics, i.e.

$$\frac{d}{dt}\tau_t^n A = i[V_n, \tau_t^n A] + \delta_0 \tau_t^n A$$

where $V_n \in \mathcal{A}$ and $\delta_0 A = \frac{d}{dt}\tau_t^0|_{t=0}$ so that

$$\tau_t^n A = \tau_t^0 A + \sum_{k=1}^{\infty}(i)^n \int_0^t dt_k \int_0^{t_k} dt_{k-1} \dots \int_0^{t_1} dt_0 [V_n, \dots \tau_{t_2}^0[V_n, \tau_{t_1}^0[V_n, \tau_{t_0}^0 A]]].$$

Then a corresponding τ_n-KMS-state [8,18] is defined by

$$\omega_n(A) = \frac{\omega_0(\exp\frac{H_0}{2}\exp-\frac{H_0+V_n}{2}A\exp-\frac{H_0+V_n}{2}\exp\frac{H_0}{2})}{\omega_0(\exp\frac{H_0}{2}\exp-(H_0+V_n)\exp\frac{H_0}{2})} \tag{3.9}$$

where $\exp iH_0 t$ implements τ_t^0 and

$$\exp\frac{H_0}{2}\exp-\frac{H_0+V_n}{2} = T\int_0^{\beta/2} e^{-\tau_{i\gamma}^0 V_n} d\gamma$$

belongs to $\pi_0(\mathcal{A})''$.

Again weak* limit points of ω_n must exist and are τ-KMS-states if $\lim \tau_t^n A = \tau_t A$. So the existence of a time automorphism constructed as limit of τ^n with known KMS states already guarantees the existence of equilibrium states. The converse is not true. $\omega_n \to \omega$ does not yet guarantee that limit points of τ_n represent an automorphism group of $\mathcal{A}$. A counterexample is constructed in [30]. Alternatives, where there exists a time evolution corresponding to the limit state, though not on the C^* level but only for the weak closure, are discussed in [31,32,33] and occur in the framework of mean field theory [34].

If in addition we were to know that the time automorphism is asymptotically abelian, we could also motivate why we consider exactly the extremal KMS states as equilibrium states:

Lemma (3.10): Let $(\mathcal{A},\tau)$ be a C^*-dynamic system $\forall A,B,C,P \in \mathcal{A}$. Assume that $\int|\omega([P,\tau_t A])|dt \; \exists$ and further that

$$\lim_{|t_1|\to\infty,|t_2|\to\infty,|t_1-t_2|\to\infty} \omega(A\tau_{t_1}(B)\tau_{t_2}(C)) = \omega(A)\omega(B)\omega(C).$$

Then ω is an extremal KMS state for some $\beta \in R\cup\{\pm\infty\}$ iff

$$\int \omega([P,\tau_t A])dt = 0 \qquad \forall P, A \in \mathcal{A}. \tag{3.11}$$

The proof can be found in [8,18,35]. There are two equivalent physical interpretations of this condition. One is dynamical stability: Let P correspond to a local perturbation of the dynamics. Then we demand that in the neighbourhood of ω (more precisely in its local folium) there exists exactly one τ^P invariant state. This state can be obtained as $\lim_{t\to\pm\infty}\omega(\tau_t^P A) = \omega^P(A)$. First order perturbation theory in P gives as condition for the equality of the two limits the above equation (3.11) [35]. The other interpretation is the time honoured condition of adiabatic invariance in the setting of infinite quantum systems [36]. Let ω be an τ_t invariant state. Let the time dependent perturbation $P(t)$ be adiabatically switched on and off. Then the time dependent state returns to ω iff (3.11) holds.

We have seen that asymptotic abelianness is the key feature which makes quantum ergodic theory work. This property does not occur for finite systems where the Hamiltonian has a point spectrum and quasiperiodic observables. The latter may also appear in infinite systems where some finite parts are completely isolated. Therefore asymptotic abelianness has always been one of the questionable unproven assumptions. Intuitively it is plausible that Galilei or Poincaré invariance will exclude that some finite parts are somewhere completely locked in and might open the way for asymptotic abelianness. Lattice systems do not have this invariance and actually there we do not have any proof of asymptotic abelianness apart from quasifree systems [37]. In section 5 we will show that Galilei invariance guarantees some asymptotic abelianness provided the time evolution exists at all on the algebraic level. In section 6 we will discuss which further mixing properties follow. Though from the fundamental point of view the Poincaré group [38,39] would be even better we will not use it since for these systems the existence of the time evolution on the algebraic level has been demonstrated only for free fields.

4 Field Theory of Particles with Pair Potentials

Usually in many body theory with particles interacting via pair potentials one uses the formal Hamiltonian

$$H = \int d^3x \vec{\nabla} a^*(x) \vec{\nabla} a(x) + \int d^3x d^3x' a^*(x) a^*(x') v(x-x') a(x') a(x). \tag{4.1}$$

As we have explained in section 3 one cannot hope in general that this Hamiltonian satisfies the stability condition (1.4) and there is no chance that it defines a time evolution of $\mathcal{A}$. To stabilize (4.1) for fermions we will cut off the high momenta in v. This will reduce the potential energy of a cluster of fermions since by their concentration in configuration space they have to get high momenta. We realize the cut off by letting v also depend on the momentum. To exploit this we shall use instead of creation and annihilation operators in configuration space, $a^*(x)$ and $a(x)$, such operators in phase space. In this way we shall be able to retain Galilei invariance [40] which is lost in other cut-off procedures [41,42,43].

The operators localized in phase space are defined by

$$a_{q,p} = \pi^{-3/4} \int d^3x e^{-(q-x)^2/2+ipx} a(x). \tag{4.2}$$

They have the virtue of being bounded operators, in fact we have normalized them such that $\|a_{q,p}\| = 1$. Since the coherent states are total in Hilbert space, any $a(f)$ can be written as limits of linear combinations of $a_{q,p}$ and so the $a_{q,p}$ generate the whole algebra. Accordingly we replace the potential energy by

$$V = (2\pi)^{-6} \int d^3q d^3p d^3q' d^3p' a^*_{q,p} a^*_{q',p'} v(q-q', p-p') a_{q',p'} a_{q,p}. \tag{4.3}$$

Remarks (4.4)

1. If v is momentum independent, $v(x,p) = v(x)$, then V equals the second term in (4.1).

2. The kinetic energy H_0 and particle number N become expressed in phase space

$$(2\pi)^{-3} \int d^3q d^3p a^*_{q,p} a_{q,p} (p^2 - 3/2) \text{ resp. } (2\pi)^{-3} \int d^3q d^3p a^*_{q,p} a_{q,p}.$$

3. Though $a_{q,p}$ is bounded all these operators are unbounded since we integrate over unbounded regions. Thus we will have to inspect whether the time derivative $dA/dt = i[H, A]$ also belongs to $\mathcal{A}$.

Proposition (4.5)

$$V \geq -N\|v\|_1, \qquad \|v\|_1 = \int \frac{d^3x d^3p}{(2\pi)^3} |v(x,p)|.$$

Proof: To ease the notation we shall henceforth denote a point in phase space (q,p) by $z \in \mathbf{R}^6$ and $dz = d^3q d^3p/(2\pi)^3$. Because of the general inequality

$$\|a_z^* a_{z'}^* a_{z'} a_z\| \leq a_z^* a_z \|a_{z'}^* a_{z'}\| = a_z^* a_z$$

we have

$$V \geq -\int dz dz' a_z^* a_{z'}^* a_{z'} a_z |v(z-z')| \geq -\int dz dz' a_z^* a_z |v(z-z')| = -N\|v\|_1. \qquad \square$$

Remarks (4.6)

1. Since $H_0 \geq 0$ we have $H \geq -N\|v\|_1$ and thus we have proved stability for fermions if $\|v\|_1 \leq \infty$. In the local limit $v(x,p) = v(x)$ we have $\|v\|_1 = \infty$ and in this case stability requires special features of v [6,7,8].

2. For bosons $a_{q,p}$ is unbounded and our argument does not work. In fact, in this case even for nice v, f.i., $v(x,p) = -e^{-x^2-p^2}$, there is no stability. This can be seen by taking as trial function
$$|N\rangle = \frac{1}{\sqrt{N!}} a_{0,0}^N |0\rangle,$$
$|0\rangle$ the Fock vacuum. In this case $\langle N|H_0|N\rangle \sim N$ but $\langle N|V|N\rangle \sim -N^2$.

To construct the time evolution τ_t generated by $H = H_0 + V$ we use the standard perturbation expansion

$$\tau_t(a) = a_t + \sum_{n\geq 1} i^n \int_{0\leq t_n \leq \ldots \leq t_1 \leq t} [V_{t_n}, [V_{t_{n-1}}, [\ldots V_{t_1}, a_t] \ldots]] \tag{4.7}$$

with $a_t = e^{iH_0 t} a e^{-iH_0 t}$. A priori this is derived for V bounded. Our V is not bounded but can be considered as the limit of $V_\Lambda = \int_\Lambda dz_1 \int_\Lambda dz_2 a_{z_1}^* a_{z_2}^* v(z_1 - z_2) a_{z_2} a_{z_1}$ where Λ is a region of finite volume in $\mathbf{R}^6$. In the following we see that in (4.7) $\Lambda \uparrow \mathbf{R}^6$ exists as norm limit and thus we shall work with V without dwelling on this point. We shall now show for any a_z that

(i) each term in $\sum_n$ is a bounded operator,

(ii) $\sum_{n=1}^\infty$ converges in norm for $|t| < t_0$ where $t_0(1+t_0^2)^{3/4} = 1/32\|v\|_1$.

Since the series is norm convergent the usual formal rearrangements to show $\tau_t(ab) = \tau_t(a)\tau_t(b)$ are legitimate and τ_t can for $t < t_0$ be extended to an automorphism of $\mathcal{A}$. Furthermore (4.7) guarantees $\tau_{t_1+t_2} = \tau_{t_1} \circ \tau_{t_2}$ if $|t_1|$, $|t_2|$, $|t_1 + t_2|$ are $< t_0$. Then we can for $|t| < 2t_0$ define consistently $\tau_t = \tau_{t_1} \circ \tau_{t_2}$ for any $|t_1|, |t_2| < t_0$, $t_1 + t_2 = t$. By iteration we get a one parameter group τ_t, $t \in \mathbf{R}$ of automorphisms.

Proof of (i): Since

$$V_t = \int dz dz' a^*_{z,t} a^*_{z',t} v(z-z') a_{z',t} a_{z,t}$$

the n-th order term in (4.7) can be written $i^n \int_{0 \le t_1 \le \ldots \le t_n \le t} dt_1 dt_2 \ldots dt_n I_n$.

$$I_n = \int dz_1 dz_2 \ldots dz_{2n} v(z_1 - z_2) \ldots v(z_{2n-1} - z_{2n}) [a^*_{z_{2n},t_n} a^*_{z_{2n-1},t_n} a_{z_{2n-1},t_n} a_{z_{2n},t_n}, [a^*_{z_{2n-2},t_{n-1}} \cdots$$

$$\ldots [a^*_{z_2,t_1} a^*_{z_1,t_1} a_{z_1,t_1} a_{z_2,t_2}, a_{z,t}] \ldots]].$$

From the free time evolution of Gaussian wave packets we know

$$[a_{z,t}, a^*_{z',t'}]_+ \equiv S(z,t;z',t')$$

with

$$|S(z,t;z',t')| = (1+(t-t')^2)^{-3/4} \exp[-\frac{1}{4}(p-p')^2 - \frac{1}{4}\frac{(q-q'+(p+p')(t-t'))^2}{1+(t-t')^2}].$$

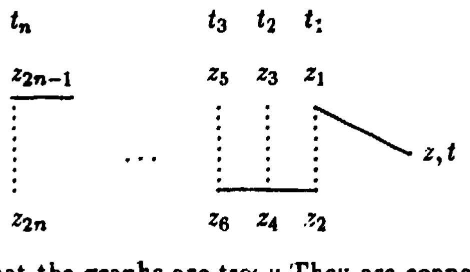
Thus the innermost commutator is

$$(a^*_{z_1,t_1} S(z,t;z_2,t_1) - a^*_{z_2,t_1} S(z,t;z_1,t_1)) a_{z_1,t_1} a_{z_2,t_2}.$$

In the next commutator each term produces a sum of $2(2+1)$ similar terms and finally we end up with $2 \cdot 2 \cdot 3 \cdot 2 \cdot 5 \ldots 2 \cdot (2n-1) = 2^n \frac{(2n)!}{2^n n!}$ terms. Since $\|a_{z,t}\| = 1$ we can majorize $\|I_n\|$ by replacing v and S by $|v|$ and $|S|$ resp. and dropping the n factors a^* and $n+1$ factors a. Thus $\|I_n\|$ is bounded by $(2n)!/n!$ terms of the form

$$\int dz_1 \ldots dz_{2n} \prod_{j=1}^{n} |v(z_{2j} - z_{2j-1})| \prod_{j=2}^{n} |S(z_{\alpha_j}, t_j; z_{\beta_j}, t_{r_j})| \cdot |S(z_1, t_1; z, t)|.$$

To each term corresponds a distribution of $\alpha_j, \beta_j \in (1,2,\ldots,2n)$, $r_j \in (1,2,\ldots,j-1)$ which is best illustrated in a graph where the v's are dashed lines and the S's solid lines. Typically



By construction one sees that the graphs are trees: They are connected because z_{2k} and z_{2k-1} are connected by v and one of them is connected by an S with $z_i, i < 2k-1$. They do not contain closed loops since each commutator brings in an S with new z's. To carry out the z-integrations we note that

$$\int dz_\alpha |S(z_\alpha, t_j; z_\beta, t_r)| = 2^3(1+(t_r - t_j)^2)^{3/4} \le 2^3(1+t^2)^{3/4}$$

independent of z_β. As a consequence we may keep integrating the end points of the graph until it is reduced to $\|v\|_1^n 8^n (1+t^2)^{3n/4}$. Since in I_n we have a finite number of graphs and the t_i-integrals go over finite regions we have established the existence of the n-th time derivative of a_z.

Proof of (ii): To prove real analyticity we recall that I_n has $(2n)!/n!$ graphs contributing. Thus with $\int_{0\le t_1\le \ldots \le t_n \le t} dt_1 dt_2 \ldots dt_n = t^n/n!$ the sum in (4.7) is bounded by

$$\|\sum_n \| \le \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{(2n)!}{n!} [8\|v\|_1 (1+t^2)^{3/4}]^n.$$

Since $(2n)!/(n!)^2 \sim 4^n$ we have established convergence for $|t| < t_0$ with

$$t_0(1+t_0^2)^{3/4} = \frac{1}{32\|v\|_1}.$$

Remarks (4.8)

1. The time unit which enters in the $1+t^2$-factor has the physical significance of (spread in x-space)/(spread in velocity space) in $a_{q,p}$. However this is chosen, the resulting t_0 goes to zero in the limit of the local interaction (4.3). For them $v(x,p)$ becomes independent of p in which case $\|v\|_1 = \infty$.

2. In previous proofs of the divergence of the perturbation expansion in bosonic field theories [44,45,46] the same combinatoric situation arose. There one calculated a matrix element of the remaining operators $\prod_{\alpha=1}^n a_{x_\alpha}^* \prod_{\beta=1}^n a_{x_\beta}$ and an additional factor $n!$ from the number of possible combinations of the a's with the a^*'s appear. For bosons this gives a permanent where each contribution has the same sign. Then the extra $n!$ reduces t_0 to zero. For fermions we get a determinant and the alternating signs give a lot of cancellations. In fact, our norm estimate proves that in absolute value it has to be less than one.

3. Computer studies of $N = 132$ to 1600 particles interacting with $v = -e^{-x^2-p^2}$ potentials show [9] that though clusters are formed by increasing N they do not heat up beyond a temperature determined by v. Thus it is possible that τ_t also exists for bosons but our method of proof is not adequate for this case.

4. That we need a velocity cut-off in v may seem artificial for local interactions which are stable. However, since we are *free* to cut-off at arbitrarily high velocities we can argue that for stable interactions where these high velocities do not arise the cut-off should not change the dynamics essentially.

5. Since the $a_{x,t}$ are complex analytic operators in t we could even prove complex analyticity but this will not be needed for our purpose.

Summarizing our results we can state

Theorem (4.9): If $\|v\|_1 < \infty$, then $H_0 + V$ defines the time development as a one parameter group τ_t of automorphisms of $\mathcal{A}$. It is continuous in the sense that

$$\lim_{t_1 \to t_2} \|\tau_{t_1} A - \tau_{t_2} A\| = 0 \quad \forall A \in \mathcal{A}.$$

Corollary (4.10): For all $0 < \beta < \infty$ there are KMS states for τ_t.

Proof: Consider

$$V_n = \int_{|q_{1,2}|<n, |p_{1,2}|<n} dz_1 dz_2 a^*_{z_1} a^*_{z_2} v(z_1 - z_2) a_{z_2} a_{z_1}.$$

This is a local perturbation and $H_0 + V_n$ generates an automorphism τ^n_t. It has the KMS states ω_n given by (3.9). Since we have seen that in the construction of τ_t all phase space integrals converge in norm we infer

$$\lim_{n\to\infty} \|\tau^n_t A - \tau_t A\| = 0.$$

Thus by (3.8) the weak* limit points of $\{\omega_n\}$ are τ-KMS states.

Remark (4.11): There is no norm convergence $\lim_{n\to\infty} \sup_{\|A\|=1} \|\tau^n_t(A) - \tau_t(a)\| = 0$. Consequently also the representations given by the KMS states ω and ω_0 corresponding to τ and τ^0 will not be quasiequivalent, which means that there will be no weakly continuous isomorphism between the corresponding von Neumann algebras. This is not possible by general theorems. ω_0 is space translation invariant and the extremal translation invariant components of ω will not admit another normal translation invariant state.

5 Consequences of Galilei Invariance

Having shown that our mollified pair potentials do define a time evolution τ_t for the infinite system we want to inspect to what extent it possesses the properties we need for the foundations of statistical mechanics. The hope is that they are not a special feature of some interactions but hold if the time evolution exists at all. We will show in this section that this hope is born out to some extent and discuss in the next section how these results fit into the general ergodic hierarchy.

We have seen in (3.8) that the existence of the time automorphism as limit of inner perturbations already guarantees the existence of equilibrium states. In this sense our continuous systems are neither better nor worse than the known examples of quantum lattice systems [18]. But for quantum lattice systems a proof under which circumstances time evolution is asymptotically abelian is missing. The existence of the time evolution – as in section 4 – is shown by expansion in t, and this makes us lose the control over large time behaviour. But in our situation the boost relates space translation and time translation. For the shift the large distance behaviour is well under control [47] and, as will be shown, the time translation inherits some of its properties.

Let us first collect general results:

Lemma (5.1): Let $\mathcal{A}$ be the C^*-algebra built by fermionic creation and annihilation operators $a(f)$, $a^*(f)$ with $f \in L^2(\mathbf{R}^d)(L^2(\mathbf{Z}^d))$. Let ν_α be the gauge groups $\nu_\alpha a(f) = e^{i\alpha} a(f)$. Consider an automorphism group α_n that satisfies $\alpha_n \circ \nu_\alpha = \nu_\alpha \circ \alpha_n$ and

$$\lim_{n\to\infty} \|[a(f), \alpha_n a^*(g)]_+\| = 0.$$

Then

$$\lim_{n\to\infty} \|[A, \alpha_n B]_-\| = 0 \qquad A \text{ or } B \text{ even} \in \pi(\mathcal{A}). \tag{5.2}$$

Let ω be a gauge invariant α-invariant faithful state. Then

$$\text{w-}\lim [A, \alpha_n B] = 0 \qquad \forall A, B \in \pi(\mathcal{A})''. \tag{5.3}$$

Proof: (5.2) follows simply by algebraic calculation, writing then even B as product of creation and annihilation operators. Now take ω to be an invariant state. Since it is faithful there exists a modular automorphism group m such that (see Appendix (A.4,2))

$$\omega(AB) = \omega(m_i B \cdot A).$$

More precisely, vectors of the form $D|\Omega\rangle$, D analytic with respect to m_t are dense in $\mathcal{H}_\pi$ so that (5.3) is equivalent to $\lim_{n\to\infty} \omega(C[\alpha_n A, B]) = 0 \; \forall \, A, B \in \pi(\mathcal{A})''$, C analytic. Furthermore, Kaplansky's density theorem tells us that for every $A \in \pi(\mathcal{A})''$ with $\|A\| = 1$ there exists a sequence $\{A_k, A_k \in \pi(\mathcal{A})\}$ such that $\|A_k\| = 1$ and st-$\lim_{k\to\infty}(A - A_k)\Omega = 0$ (Ω the GNS vector of ω).

Therefore, for A or B even elements $\in \pi(\mathcal{A})''$

$$\begin{aligned} &\lim_{n\to\infty} \omega(C[\alpha_n A, B]) = \\ &\lim_{k\to\infty} \lim_{n\to\infty} [\omega(C \cdot \alpha_n A \cdot B_k) - \omega(C \cdot B \cdot \alpha_n A_k)] = \\ &\lim_{k\to\infty} \lim_{n\to\infty} [\omega(\alpha_n A_k B_k \cdot m_{-i} C) - \omega(B_k \alpha_n A_k \cdot m_{-i} C) = \\ &\lim_{k\to\infty} \lim_{n\to\infty} \omega(C[\alpha_n A_k, B_k]) = 0. \end{aligned} \tag{5.4}$$

Now take A and B odd. Then again it suffices to study

$$\lim_{n\to\infty} \omega(C[\alpha_n A, B]), \qquad A, B \in \pi(\mathcal{A})'', \quad C \text{ analytic}.$$

It converges to zero if $\lim \omega(C\alpha_n A) = 0 \, \forall \, A$ odd, C arbitrary. If C is even, $C\alpha_n A$ is odd and $\omega(C\alpha_n A) = 0 \, \forall \, n$. So we can concentrate on C odd. We know from the CAR that

$$\lim_{n\to\infty} \omega(\alpha_n AB + B\alpha_n A) = \lim \omega((B + m_i B)\alpha_n A) = 0 \quad \forall \, A \in \pi(\mathcal{A}),\, B \in \pi(\mathcal{A})'' \text{ and analytic.}$$

Now $(B + m_i B)|\Omega\rangle = (1 + e^{-s})B|\Omega\rangle$ is dense in the odd subspace, i.e., there exists to every $C \in \pi(\mathcal{A})''$ some B with $\|(C - (1 + e^{-s})B)|\Omega\rangle\| \leq \varepsilon$. Therefore

$$\lim \omega(C\alpha_n A) = 0. \qquad \square$$

Corollary (5.5): Let α_n be a quasifree automorphism, i.e., $\alpha_n a(f) = a(e^{inh} f)$ with h some operator acting in the one particle space. Then α_n satisfies the condition of (5.1) if h has purely absolutely continuous spectrum. Special examples are the boost and space translation

$$\sigma_x a(f) = a(f_x), \qquad f_x(y) = f(x + y),$$

$$\gamma_b a(f) = a(e^{ibx} f),$$

which means for $a_{p,q}$ of (4.2)

$$\sigma_x a_{p,q} = e^{-ipx} a_{p,q+x},$$

$$\gamma_b a_{p,q} = a_{p+b,q}.$$

Proof:

$$[a^\dagger(f), a(e^{inh}g)]_+ = \langle g|e^{-inh}|f\rangle.$$

For h an operator with purely absolutely continuous spectrum the right hand side goes to zero according to Riemann-Lebesgue and so does the norm of the anticommutator. □

Lemma (5.6): Though in this article we deal mainly with fermions, we refer to the corresponding result for bosons: For h as above with purely absolutely continuous spectrum define $\alpha_n W(f) = W(e^{inh}f)$. Then

$$\lim \|[W(f), \alpha_n W(g)]\| = 0.$$

Proof: With σ of (3.4,2) (the difference of two scalar products)

$$[W(f), \alpha_n W(g)] = W(f + \alpha_n g) i \sin \sigma(f, e^{inh}g).$$ □

Corollary (5.7): The free time evolution is weakly asymptotically abelian.

Proof: $h = p^2/2m$ has absolutely continuous spectrum. □

It seems plausible that our interactions should not make mixing properties worse. Clusters of particles that might be formed by the interaction should keep some kinetic energy which makes their effect on the commutator for local operators disappear in the course of time. This feeling can be substantiated by using Galilei invariance.

Lemma (5.8): Consider the shift σ_x, the boost γ_b (defined in (5.5) and the gauge transformation ν_α (defined in 5.1). They are continuous in the sense that

$$\lim_{x\to 0} \|\sigma_x A - A\| = \lim_{\alpha\to 0} \|\nu_\alpha A - A\| = \lim_{b\to 0} \|\gamma_b A - A\| = 0.$$

They satisfy

$$\sigma_x \circ \nu_\alpha = \nu_\alpha \circ \sigma_x, \qquad \gamma_b \circ \nu_\alpha = \nu_\alpha \circ \gamma_b, \qquad \gamma_b \circ \sigma_x = \sigma_x \gamma_b \nu_{-bx}.$$

Proof: The commutation relations between the automorphisms are determined on the level of an annihilation operator and can be seen by direct calculation, and the continuity properties follow from $\|a(f) - a(g)\| = \|f - g\|$. □

Definition (5.9): A time evolution is called gauge and Galilei invariant, if

$$\tau_t \circ \nu_\alpha = \nu_\alpha \circ \tau_t, \qquad \tau_t \circ \sigma_x = \sigma_x \circ \tau_t, \qquad \tau_t \circ \gamma_b = \gamma_b \circ \tau_t \circ \sigma_{-bt} \circ \nu_{-b^2t/2}.$$

Remark: If we turn to finite systems (with finitely many particles in an infinite space) we consider potentials of the form $\sum_{1\le i<j\le N} V(x_i - x_j)$, and these potentials are invariant under σ, γ and ν (which acts trivially when we fix the particle numbers). Added to the kinetic energy they generate a time evolution satisfying (5.9). We refer to [48] for the study of the structure of the Galilei group in this context (ν acts trivially) and its representations. (The correct commutation relations in the second quantized theory can be checked by the action of the free time evolution on one annihilation operator.)

Lemma (5.10): The time evolution of (4.9) is Galilei-invariant.

Proof: The free time evolution τ^0 is Galilei-invariant. Therefore (with the usual region for the time integration)

$$\begin{aligned}\sigma_x \tau_t a &= \sigma_x \tau_t^0 a + \sum i^n \sigma_x \int dt_1 \ldots dt_n [\tau_{t_n}^0 V, \ldots [\tau_{t_1}^0 V, \tau_t^0 a] \ldots] \\ &= \tau_t^0 a + \sum i^n \int dt_1 \ldots dt_n [\tau_{t_n}^0 V, \ldots [\tau_{t_1}^0 V, \tau_t^0 \sigma_x a] \ldots] = \tau_t \sigma_x a,\end{aligned}$$

where we used $\sigma_x V = V$ or more carefully in the sense of section 4

$$\lim_{n\to\infty} [\sigma_x V_n, A] = \lim_{n\to\infty} [V_n, A] \text{ for } A \text{ local.}$$

Similarly

$$\gamma_b \tau_t a = \gamma_b \tau_t^0 a + \sum i \int dt_1 \ldots dt_n [\tau_{t_n}^0 V, \ldots [\tau_{t_1}^0 V, \gamma_b \tau_t^0 a] \ldots] = \tau_t \gamma_b \sigma_{bt} \nu_{b^2 t/2}$$

since again

$$\lim_{n\to\infty} [\gamma_b V_n, A] = \lim_{n\to\infty} [V_n, A]$$

and thus $\gamma_b \tau_{t_i}^0 V = \tau_{t_i}^0 \gamma_b \sigma_{bt_i} \nu_{b^2 t_i/2} V = \tau_{t_i}^0 V$. To study the consequences of Galilei invariance we have to distinguish between two cases: first we concentrate on states which are invariant under all automorphisms σ, γ, ν and τ. Among the states ϕ_β from (3.7) this holds only for the Fock state ϕ_∞ and the tracial state ϕ_0, the latter corresponds to $\beta = 0$ and satisfies $\phi_0(AB) = \phi_0(BA)$. For the CAR algebra this state is uniquely determined by this commutativity behaviour. In ϕ_∞ and ϕ_0 all automorphisms are unitarily implemented and we could use the representation theory of the Galilei group [47,48] to study the mixing behaviour. We prefer to extract the mixing behaviour directly from (5.9) because this method enables us to make some statement also for other space- and time-invariant states.

Lemma (5.11): Let $A \in \mathcal{A}$. Then to every $\varepsilon > 0$ there exists some $\alpha > 0$ such that

$$\|A - \frac{\pi}{\alpha} \int da db e^{-(a^2+b^2)/\alpha} \sigma_a \gamma_b A\| < \varepsilon.$$

Proof: This follows from the continuity of σ_a and γ_b (5.8) and the fact that $f_\alpha(a) = \sqrt{\pi/\alpha}\, e^{-a^2/\alpha}$ tends to $\delta(a)$ for $\alpha \to 0$.

Theorem (5.12): Take B with $\phi_0(B) = 0$. Then

$$\lim_{t\to\infty} \phi_0(A \tau_t B) = 0.$$

Proof: According to (5.11) and the fact that τ_t is an automorphism and therefore norm preserving we may replace

$$\begin{aligned}
\lim_{t\to\infty} |\phi_0(A\tau_t B)| &= \lim_{\alpha\to 0}\lim_{t\to\infty} \left| \int da d\bar{a} db d\bar{b} f_\alpha(a) f_\alpha(b) f_\alpha(\bar{a}) f_\alpha(\bar{b}) \phi_0(\sigma_a \gamma_b A \tau_t \sigma_{\bar{a}} \gamma_{\bar{b}} B) \right| \\
&= \lim_{\alpha\to 0}\lim_{t\to\infty} \left| \int da db d\bar{a} d\bar{b} f_\alpha(a) f_\alpha(b) f_\alpha(\bar{a}) f_\alpha(\bar{b}) \phi_0(\tau_{-t} A \gamma_{-b} \sigma_{\bar{a}-a-bt} \nu_{-b^2 t/2} \gamma_{\bar{b}} B) \right| \\
&\leq \lim_{\alpha\to 0}\lim_{t\to\infty} \|A\| \Big[\int da db \dots d\bar{b} dc \dots d\bar{d} f_\alpha(a) \dots f_\alpha(c) \dots f_\alpha(\bar{d}) \\
&\quad \phi_0(\gamma_{-b} \sigma_{\bar{a}-a-bt} \nu_{-b^2 t/2} \gamma_{\bar{b}} B^* \gamma_{-d} \sigma_{c-\bar{c}-dt} \nu_{-d^2 t/2} \gamma_{\bar{d}} B) \Big]^{1/2} = 0.
\end{aligned}$$

Now we have to see what happens for $t \to \infty$. For B a monomial in creation and annihilation operator $\nu_{t(b^2-d^2)/2}$ gives an oscillating factor ($= 1$ for gauge invariant elements) and $\sigma_{c-\bar{c}+a-\bar{a}+(b-d)t}$ makes the expectation value go to zero for $b \neq d$ according to (5.1) and (5.5). Thus we have a bounded integrand which goes for fixed $b - d$ for $t \to \infty$ to zero almost everywhere. By Lebesgue's lemma the integral then goes to zero.

We can immediately generalize this result to the

Corollary (5.13):

$$\lim_{t\to\infty} \phi_0(A\tau_t B) = \lim_{t\to\infty} [\phi_0(A\tau_t(B - \phi_0(B)) + \phi_0(A)\phi_0(B)] = \phi_0(A)\phi_0(B).$$

Again by using the same estimates as in (5.4) this holds for all $A, B \in \pi_0(\mathcal{A})''$.

The Fock representation π_∞ is not faithful. Here the automorphisms are inner for $\pi_\infty(\mathcal{A})''$ and there exist noncommuting time-invariant elements in $\pi_\infty(\mathcal{A})''$. Thus $(\pi_\infty(\mathcal{A})'', \tau_t)$ is not an asymptotically abelian system. Nevertheless, we have some kinds of clustering. We first concentrate on the properties of space translation.

Lemma (5.14):

$$\text{st-}\lim_{x\to\infty} \pi_\infty(\sigma_x a(f)) = 0, \qquad \text{w-}\lim_{x\to\infty} \pi_\infty(\sigma_x \prod_i a^*(f_i)) = 0.$$

By Wick-ordering a general element of $\mathcal{A}$ can be written as $A = \phi_\infty(A) + \sum a^*(f_1) \dots a^*(f_n) a(g_1) \dots a(g_m)$. Then

$$\text{st-}\lim_{x\to\infty} \pi_\infty(\sigma_x A) = \phi_\infty(A) \quad \text{for } m \neq 0,$$

$$\text{w-}\lim_{x\to\infty} \pi_\infty(\sigma_x A) = \phi_\infty(A) \quad \text{for } m = 0.$$

Proof: This can be seen by direct calculation using the relations (5.5). (See [47].) □

The analogous results for τ_t warrant a more detailed investigation.

Theorem (5.15):

a) $\text{st-}\lim_{t\to\infty} \pi_\infty(\tau_t a(f)) = 0$,

b) $\text{w-}\lim_{t\to\infty} \pi_\infty(\tau_t a^*(f)) = 0$.

c) $\text{st-lim}_{t\to\infty} \pi_\infty([a^*(g), \tau_t a^*(f)]) = 0$.

For the above general A of (5.14)

d) $\text{st-lim}_{t\to\infty} \pi_\infty(\tau_t A) = \phi_\infty(A)$ for $m \neq \infty$,

e) $\text{w-lim}_{t\to\infty} \pi_\infty(\tau_t A) = \phi_\infty(A)$ for $m = 0$,

f) $\text{st-lim}_{t\to\infty}([A, \tau_t B]_\pm) = 0$ where $\pm$ depends on the grading of the elements.

Proof: The proof needs two distinct facts which we collect in separate lemmas.

Lemma (5.16): $\tau_t(\pi_\infty(a(f)))$ converges strongly to zero for $t \to \pm\infty$.

Proof: In $\mathcal{H}_\infty$ the $N = 0$ sector consists only of the vacuum and since $a(f)|0\rangle = 0$ we can turn to the $N > 0$ sector. We may separate the motion of the N particles in a center of mass and an internal motion such that

$$\mathcal{H}^{(N)} = \mathcal{H}_c \otimes \mathcal{H}_I, \qquad U_t^{(N)} = e^{-itP^2/2N} \otimes e^{-iH_I t}.$$

Since the total momentum P has an absolutely continuous spectrum $e^{-itP^2/2N} \rightharpoonup 0$. We shall now show that $a(f)$ acts like a compact operator and converts this weak convergence into strong convergence. To see this we note that $a^*(f)a(f)$ acts in the momentum representation of $\mathcal{H}^N$ as the integral kernel

$$a^*(f)a(f) \leftrightarrow K(p_1 \dots p_N; p_1' \dots p_N') = \sum_{j=1}^{N} \delta(p_1 - p_1') \dots \tilde{f}(p_j)\tilde{f}^*(p_j') \dots \delta(p_n - p_n').$$

The partial trace over $\mathcal{H}_c$ corresponds to the integral kernel

$$\text{tr}_{\mathcal{H}_c} a^*(f)a(f) \leftrightarrow \int dq\, K(p_1 + q, \dots p_n + q; p_1' + q, \dots p_N' + q) =$$

$$= N \prod_{j=1}^{N-1} \delta(p_j - p_j') \int dq |\tilde{f}(q)|^2 \leftrightarrow 1 \cdot N \|f\|_2^2$$

in the space $\mathcal{H}_I$ with $\sum_i p_i = \sum p_i' = 0$.

Since the $a(f)$ are bounded operators it suffices to demonstrate strong convergence on a total set of vectors. For these we take vectors of the form $\psi = v_c \otimes v_I$, $v_c \in \mathcal{H}_c$, $v_I \in \mathcal{H}_I$ and we know

$$U_t^{(N)}\psi = e^{-itP^2/2N} v_c \otimes e^{-itH_I} v_I \quad \text{and} \quad v_c(t) = e^{-itP^2/2N} v_c \rightharpoonup 0.$$

In finite dimensions weak and strong convergence coincide and thus for any projection $q \in \mathcal{B}(\mathcal{H}_c)$ with $\text{tr}_{\mathcal{H}_c} q < \infty$ and any $\varepsilon > 0 \; \exists T$ such that $\|qv_c(t)\| < \varepsilon \; \forall t > T$. Furthermore, $\forall A \in \mathcal{B}(\mathcal{H}^{(N)})$ with $\|\text{tr}_{\mathcal{H}_c} A^*A\| < \infty$ exists a projection $q \in \mathcal{B}(\mathcal{H}_c)$ with $\text{tr}_{\mathcal{H}_c} q < 0$ and $\|(1-Q)A^*A(1-Q)\| < \varepsilon^2 \Leftrightarrow \|A(1-Q)\|\varepsilon$ for $Q = q \otimes 1$. For such an A we have $AU_t^{(N)}\psi \to 0$ since

$$\|AU_t^{(N)} v_c \otimes v_I\| = \|(A(1-Q+Q)v_c(t) \otimes v_I(t)\| \leq \|A(1-Q)\| + \|A\|\|qv_c(t)\|\|v_I(t)\| \leq (1 + \|A\|)\varepsilon$$

assuming $\|v_c\| = \|v_I\| = 1$. Now $a(f)$ qualifies in $\mathcal{H}^{(N)}$ for A if $\|f\| < \infty$ and thus Lemma (5.16) is proved.

Corollary (5.17): A of the form (5.14) converges weakly to $\phi_\infty(A)$ if $m = 0$, otherwise strongly.

Proof: $A_t \to 0$ implies $A_t^* \to 0$ and $B_t A_t \to 0$ if $\|B_t\| < c\ \forall t$. Furthermore $A_{t,z}^{(1,2)} \to 0$ implies $A_t^{(1)} \cdot A_z^{(2)} \to 0$.

Lemma (5.18): $[a^*(g), \tau_t a^*(f)]_+$ converges strongly to zero for $t \to \pm\infty$.

Proof: A general vector in the N-particle sector can be written $A_N^*|0\rangle$ with $A_N^* = \int dx_1 \ldots dx_N \psi(x_1 \ldots x_N) a^*(x_1) \ldots a^*(x_N)$. From the identity

$$(a^*(g)\tau_t(f) + (\tau_t a^*(f))a^*(g))A_N^* =$$
$$(\tau_t a^*(f))a^*(g)A_N^* + (-)^N a^*(g)A_N^* \tau_t A^*(f) + a^*(g)[(\tau_t a^*(f))A_N^* + (-)^{N-1} A_N^* \tau_t A^*(f)]$$

we infer that (5.18) is satisfied if $((\tau_t a^*(f))A_N^* + (-)^N A_N^* \tau_t a^*(f))|0\rangle$ goes strongly to zero. The vector

$$(\tau_t a^*(f) A_n^* + (-)^n A_n^* \tau_t a^*(f)]|0\rangle$$

corresponds to the wave function

$$e^{-iH_{n+1}t} \sum_{j=1}^{n+1} f(x_j)(-)^j e^{iH_n t} \psi_n(x_1 \ldots \hat{x}_j, \ldots x_{n+1}) - \sum_{j=1}^{n+1} (-)^j f_t(x_j) \psi_n(x_1, \ldots \hat{x}_j, \ldots x_{n+1}),$$

$\hat{x}_j$ means that this argument is deleted. This vector converges strongly iff

$$e^{iH_1 t} e^{-iH_{n+1}t} e^{iH_n t} \to 1.$$

We call this property

Definition (5.19): Asymptotic triviality of the Möller operators holds if

$$\text{st-}\lim e^{iH_1 t} e^{-iH_n t} e^{iH_{n-1}t} \Rightarrow 1$$

or, equivalently,

$$\text{st-}\lim e^{iH_1 t} e^{-iH_n t} e^{-iH_1 t} e^{i(H_{n-1}+H_1)t} = \text{st-}\lim e^{-i(H_{n-1}+H_1+V_1(t))t} e^{i(H_{n-1}+H_1)t} = 0$$

with $V_1(t) = e^{iH_1 t} V e^{-iH_1 t}$.

Lemma (5.20): For our Hamiltonian (4.1,4.3) in three (or more) dimensions the Möller operators are asymptotically trivial.

Remark: The Lemma (5.20) is proven in [49]. It is plausible because st-lim $V_1(t) = 0$, since by H_1 only particle 1 is carried away to infinity and so it will not interact with the other particles anymore. On the other hand, it should be noted that this argument is not sufficient. In one dimension one particle can overrun the other so that nevertheless interaction occurs and in fact in the proof of [49] the spreading of the wave functions in three and more dimensions becomes an important ingredient in the estimates.

The results (5.16) - (5.20) prove Theorem (5.15). Its intuitive content is the following. For $t \to \pm\infty$ $\tau_t(a^*(f))$ converges weakly to zero since it creates a particle far away and such vectors become orthogonal to the other vectors in $\mathcal{H}_\infty$ which live mainly in finite regions. But norms of vectors where particles are far away do not go to zero and thus $\tau_t(a^*(f))$ does not converge strongly to zero. On the other hand, $\tau_t(a(f))$ wants to annihilate particles which are far away. Since far away there are no particles it converges strongly to zero. Norm convergence is impossible for automorphisms. Finally $[a^*(g), \tau_t a^*(f)]_+$ should go to zero since creations of particles at vastly different points should not influence each other. Here it is possible that our results are not optimal and $[\,]_+$ converges even in norm.

Though the asymptotic abelian properties deduced so far will allow us to draw quite strong conclusions in § 6 we have been falling short of norm asymptotic abelianness. Therefore we have no proof of weak asymptotic abelianness for the KMS states and the mixing properties guaranteed by (3.5). This is the case for space translations and (3.5) assures as that all extremal space translation invariant states are spatially clustering.

Passing from space clustering to time clustering is asking for too much, because clustering can only occur in an extremal invariant state and we must keep in mind the possibility of a crystal where the space translation invariant state is the average over a crystal cell. Therefore it is not extremal time invariant and cannot cluster in time. Examining the proof of Lemma (5.12) we notice that the states are clustering in the average in the following strong sense:

Lemma (5.21): Suppose ω is an gauge and time invariant and extremal space invariant state. Then

$$\lim_{\alpha\to 0}\lim_{t\to\infty}\int \omega\circ\gamma_b(A\tau_t B)f_\alpha(b)db = \omega(A)\omega(B).$$

Proof:

$$\begin{aligned}\lim_{\alpha\to 0}\lim_{t\to\infty}\int\langle\Omega|\gamma_b A\gamma_b\tau_t B|\Omega\rangle f_\alpha(b)db &= \lim_{\alpha\to 0}\lim_{t\to\infty}\int\langle\Omega|\gamma_b A\gamma_b\tau_t\sigma_{\bar a}B|\Omega\rangle f_\alpha(b)f_\alpha(a)dadb\\ &= \lim_{\alpha\to 0}\lim_{t\to\infty}\int\langle\Omega|A\tau_t\sigma_{a+bt}\nu_{-bt^2/2}B|\Omega\rangle f_\alpha(b)f_\alpha(a)dadb\end{aligned}$$

where we used the fact that $\lim_{b\to 0}\|\gamma_b A - A\| = 0$ and therefore we returned to the previous estimates (5.12).

Remarks (5.22)

1. With ω also $\omega\circ\gamma_b$ is gauge and time invariant and extremal space translation invariant. The convex combination $\omega_\alpha = \int db f_\alpha(b)\omega\circ\gamma_b$ tends for $\alpha \to 0$ to ω and the lemma shows $\lim_{\alpha\to 0}\lim_{t\to\infty}\omega_\alpha(A\tau_t B)$ factors whereas clustering of ω means the same for $\lim_{t\to\infty}\lim_{\alpha\to 0}\omega_\alpha(A\tau_t B)$.

2. Whereas the property $\omega(A\tau_t B) \to \omega(A)\omega(B)$ is lost by taking convex combinations of states the property $\omega(C[A, \tau_t B]D) \to 0$ is preserved. Thus our method to deduce the latter from the former is not optimal and it may be that in the example of a crystal quoted above asymptotic abelianness holds whereas clustering fails.

6 The Ergodic Hierarchy

Boltzmann's ideas about the behaviour of large dynamical systems have developed into two mathematical disciplines, topological dynamics and measuretheoretic ergodic theory [10,50]. In our present setting they can be described as the study of the asymptotic behaviour of a one parameter automorphism group τ_t of a C^*-algebra $\mathcal{A}$ and the properties of invariant states ω and their associated W^*-algebras $\pi_\omega(\mathcal{A})''$. The latter also contain extrapolations depending on the observer described by ω whereas $(\mathcal{A}, \tau)$ should be looked at as the objective reality. Thus we will be mainly concerned with the topological dynamical system $(\mathcal{A}, \tau)$ but the most useful information will come from the existence of faithful extremal invariant states. We start with discussing a list of ergodic properties for a dynamical C^*-system $(\mathcal{A}, \tau)$.

Proposition (6.1) [21]: Between the conditions

(i) The only invariant elements of $\mathcal{A}$ are multiples of 1.

(ii) The only quasiperiodic elements of $\mathcal{A}$ are multiples of 1. (Quasiperiodic elements A are those for which there exists $\forall \varepsilon > 0, t$ a $t' > t$ with $\|\tau_{t'}(A) - A\| < \varepsilon$.)

(iii) $\mathcal{A}$ does not contain proper invariant subalgebras.

there are the implications (iii) $\Rightarrow$ (i) $\Leftarrow$ (ii).

Proof:

(ii) $\Rightarrow$ (i): invariant elements are quasiperiodic.

(iii) $\Rightarrow$ (i). $\{a \in \mathcal{A} | \tau a = a\}$ is an invariant subalgebra which cannot be all of $\mathcal{A}$ since we have the standing assumption $\tau \neq id$.

Remarks (6.2)

ad (i) Traditionally ergodicity is defined by (i). Unfortunately the strength of this condition depends very much on how large $\mathcal{A}$ has been chosen. Consider for instance the motion on the $E = 0$ energy shell on $T^*(R)$ which is determined by the double well potential $-x^2 + x^4$. There are three orbits whose projections on R are $(-1,0)$, $\{0\}$, $(0,1)$. If $\mathcal{A}$ are the continuous functions there is no other invariant function then constants since they have to be constant on the orbits. There are two linearly independent L^∞ functions $\theta(x)$ and $\theta(-x)$. If one considers all bounded functions then the cardinality of linearly independent invariant functions equals the cardinality of the orbits. Taking the latter algebra (i) could be never satisfied if there is more than one orbit which classically is always the case if the energy shell has more than one dimension. In any case (i) is strong enough to exclude inner automorphisms and thus the usual finite quantum systems. If $\tau(A)$ were $U^{-1}AU$, $U \in \mathcal{A}$, then U would be a constant $\neq c1$. (For a finite quantum system where τ is not inner, see [51].)

ad (ii) By a famous theorem of Poincaré almost all classical orbits on a compact energy shall have quasiperiodic properties. Nevertheless, for such classical systems the continuous functions may satisfy (ii) and never come close to their original forms. This does not contradict Poincaré's theorem since neighbouring orbits will return close to their starting point at quite different times. What (ii) excludes is a time evolution $\tau(A) = UAU^{-1}$ where U has a pure point spectrum. In particular finite quantum systems and systems with finite dimensional invariant subalgebras do not qualify for (ii). (We did not use the absence of invariant finite dimensional subalgebras as ergodic criterion because the usual algebras of continuous functions do not have finite dimensional subalgebras to start with.)

ad (iii) The trouble with (iii) is that on the one hand it is too weak to imply (ii) as is shown by the trivial example:

$$\mathcal{A} = \{z_1 + Az_2, z_i \in C, A^2 = 1\}, \qquad \tau A = -A.$$

$\mathcal{A}$ does not have proper subalgebras but A is quasiperiodic. On the other hand, for our purposes (iii) is far too strong and will never be satisfied for the CAR algebra if τ respects the particle number. There the subalgebras of elements of the form

$$z + \sum_{n \geq n_o} a^*(f_1)a^*(f_2)\dots a^*(f_n)a(g_1)a(g_2)\dots a(g_n)$$

are for $n_0 > 1$ proper and invariant. (iii) would exclude the splitability, i.e. $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\tau = \tau_1 \otimes \tau_2$, but systems with the best mixing properties, namely K-systems can be splitable (as Bernoulli shift) and in any case have sub-K-systems. In fact the tensor product of two K-systems is in an obvious way again a K-system.

In (3.5) we introduced mixing of state and one might wonder how this property can be expressed in topological dynamics. There the convergence of observables to their equilibrium values is impossible since the structure of $(\mathcal{A}, \tau)$ only contains norm convergence. Classically the weak convergence of any function $f(x)$ to its equilibrium value $1 \cdot \int f$ means that every part of f becomes dispersed over all of phase space such that $\int g f_t \to \int g \int f$ for all functions g. Thus also $\sup_x |g(x) f_t(x)|$ should approach $\sup_x |g(x)| \sup_x |f_t(x)|$ which motivates the

Definition (6.3): $(\mathcal{A}, \tau)$ is called mixing if

$$\lim_{t \to \infty} \|A \tau_t B\| = \|A\| \|B\| \quad \forall A, B \in \mathcal{A}.$$

[47,52,53,54]

Remarks (6.4)

1. If $\mathcal{A}$ is abelian then it is the algebra of continuous functions of a compact Hausdorff space X and τ is induced by a homeomorphism $\tau^* : X \to X$ such that $(\tau f)(x) = f(\tau^* x)$ $\forall f \in \mathcal{A}$. In this case (6.3) agrees with the classical notion of topologically mixing. It is equivalent to the existence of a dense orbit or that every open invariant set of X is empty or dense in X.

2. If $||A|| = ||B|| = 1$ then (6.3) means for operators in a Hilbert space $\mathcal{H}$ that $\forall\varepsilon > 0$ $\exists T \in \mathbf{R}$ and $\psi \in \mathcal{H}$ such that $|(A\psi|B_t\psi)| > 1-\varepsilon \; \forall t > T$. If A and B are propositions ($A^2 = A = A^*$, $B^2 = B = B^*$), and we call propositions compatible if they have some common eigenvectors then (6.3) means that eventually all propositions become compatible within ε. Thus the system is chaotic in the sense that no definite predictions about the future can be made.

(6.3) clearly excludes the possibility of quasiperiodic observables.

Proposition (6.5): (6.3) $\Rightarrow$ (6.1,(ii)).

Proof: Suppose $A \in \mathcal{A}$ is quasiperiodic. Then also $A + A^*$ and $A/||A||$ are quasiperiodic so that we might assume $A = A^*$ and $||A|| = 1$. If $A \neq 1$ then it must in addition to 1 have another spectral value α with $|\alpha| < 1$. Now pick a continuous function $f \in C(\mathbf{R})$ such that $f(\alpha) = 1$ and $\sup_{x\in\mathbf{R}} |xf(x)| = |\alpha|$. Then $f(A) \in \mathcal{A}$, $||f(A)|| = 1$ and $||Af(A)|| = |\alpha| < 1$. Now consider $||f(A)\tau_t A|| \; \forall\varepsilon > 0, T \,\exists\, t'$ with $||f(A)\tau_{t'}(A)|| \leq ||f(A)A|| + ||f(A)|| ||A - \tau_{t'}(A)|| \leq |\alpha| + \varepsilon$. Since we may choose $\varepsilon < 1 - |\alpha|$ this contradicts (6.3).

Next we will show that our Galilei-invariant quantum field theories are indeed mixing.

Theorem (6.6): Let $\mathcal{A}$ be the C^*-algebra generated by the creation and annihilation operators $a^*(f)$, $a(f)$, $f \in L^2(\mathbf{R}^3)$ and τ the time evolution constructed in § 4. Then $(\mathcal{A}, \tau)$ is mixing.

Proof: Since $\mathcal{A}$ is simple $\pi_\infty\mathcal{A})$ is faithful. Furthermore we know that the Fock vacuum is mixing for $(\mathcal{A}, \tau)$ and $(\pi_\infty(\mathcal{A}), \tau)$ is asymptotically abelian (or antiabelian) [49] which means st-$\lim_{t\to\pm\infty}[A, B_t]_\pm = 0$ with $B_t = \tau_t(B)$ and $\pm$ depending on the grading of the elements. Now use the general operator inequality

$$CD_tAB_tB_t^*A^*D_t^*C^* \leq ||AB_t||^2 CD_tD_t^*C^*, \qquad A, B, C, D \in \mathcal{A}.$$

Taking the expectation value with the Fock state ϕ_∞ we deduce

$$\frac{\phi_\infty(CD_tAB_tB_t^*A^*D_t^*C^*)}{\phi_\infty(CD_tD_t^*C^*)} \leq ||AB_t||^2.$$

Now let t go to ∞. Irrespective of the grading of the elements the left hand side converges to

$$\frac{\phi_\infty(CAA^*C^*)\phi_\infty(DBB^*D^*)}{\phi_\infty(CC^*)\phi_\infty(DD^*)}.$$

Now use faithfulness of π_∞ which implies

$$\sup_{C\in\mathcal{A}} \frac{\phi_\infty(CAA^*C^*)}{\phi_\infty(CC^*)} = ||A||^2$$

and similarly take $\sup_D$ in the second factor. Then we arrive at

$$||A||^2||B||^2 \leq \lim_{t\to\infty} ||AB_t||^2.$$

But generally $||AB_t|| \leq ||A|| ||B_t|| = ||A|| ||B||$ and thus we have verified the condition of (6.3).

Our results so far have shown Galilei-invariant QFT ⇒ topologically mixing ⇒ absence of quasiperiodic observables $\neq z\mathbf{1}$ ⇒ absence of constant observables $\neq z\mathbf{1}$. Thus we have climbed up the ergodic ladder a few steps but we have not yet reached the top. The most satisfactory foundation of quantum statistical mechanics is provided by the so-called K(olmogorov) systems [55,56,57,58].

K-systems have the maximal possible uniformity in A and B in the convergence $\omega(A\tau_t B) \to \omega(A)\omega(B)$ and this entails [56,57] the desired behaviour of relative and dynamical entropies. It is to be hoped that all the systems for which we have shown the existence of τ_t are actually K-systems but so far there is no proof for this conjecture.

Appendix

For details we refer to [15,18,59].

A.1 The GNS Construction

A C^*-algebra is an algebra with a norm which satisfies (3.1) and is complete in this norm. Positive operators are those which can be written A^*A, $A \in \mathcal{A}$, and states are linear functionals ω with $\omega(A^*A) \geq 0$, $\omega(1) = 1$. If $A \neq 0 \Rightarrow \omega(A^*A) > 0$ we call ω ***faithful***. Since $\mathcal{A}$ is a Banach space any $A \in \mathcal{A}$ is represented as a bounded linear operator on this Banach space by $B \to AB$, $B \in \mathcal{A}$. If ω is faithful it endowes $\mathcal{A}$ with the structure of a pre-Hilbert space by $\langle A|B\rangle = \omega(A^*B)$ and thus any A is represented by a linear operator $\pi_\omega(A)$ in a Hilbert space $\mathcal{H}_\omega$, the completion of $\mathcal{A}$ in the $\langle\cdot\rangle$ norm. (If ω is not faithful one has first to divide by the null space $\{A : \omega(A^*A) = 0\}$.) In $\mathcal{H}_\omega$ the vector $|\Omega\rangle$ corresponding to the unit element is *cyclic* which means that $\mathcal{A}|\Omega\rangle$ spans all of $\mathcal{H}_\omega$. If ω is faithful $|\Omega\rangle$ is also separating which means $\pi_\omega(A)|\Omega\rangle = 0 \Leftrightarrow A = 0$. In this case $\pi_\omega(\mathcal{A})$ is also faithful. ($\pi_\omega(A) \neq 0$ if $A \neq 0$.)

A.2 C^* and W^* Dynamical Systems

A ***C^*-dynamical system*** consists of a C^*-algebra $\mathcal{A}$ and an automorphism $\tau \neq id$ of $\mathcal{A}$. A state ω is called invariant if $\omega \circ \tau = \omega$. The τ can be represented unitarily in $\mathcal{H}_\omega$, $\pi_\omega(\tau(A)) = U\pi_\omega(A)U^{-1}$, by defining $U\pi_\omega(A)|\Omega\rangle = \pi_\omega\tau(A)|\Omega\rangle$. If τ in addition can be imbedded in a one parameter group τ_t with $\lim_{t\to 0}\|\tau_t(A) - A\| = 0 \,\forall A \in \mathcal{A}$ then by Stone's theorem U_t can be written e^{-iHt} with the Hamiltonian H a selfadjoint operator in $\mathcal{H}_\omega$ though neither U_t nor H may belong to $\mathcal{A}$. In this case τ_t can be extended to an automorphism of a larger algebra, the weak closure of $\pi_\omega(\mathcal{A})$. We shall denote this by $\pi_\omega(\mathcal{A})''$ since by von Neumann's famous theorem it coincides with the double commutant of $\pi_\omega(\mathcal{A})$ (the commutant is defined by $\pi_\omega(\mathcal{A})' = \{B \in \mathcal{B}(\mathcal{H}_\omega), [B, A] = 0 \,\forall A \in \pi_\omega(\mathcal{A})\}$). $\pi_\omega(\mathcal{A})''$ is not only a C^*-algebra but a W^* (or von Neumann) algebra which enjoys the following additional properties:

(a) For every increasing bounded sequence in $\pi(\mathcal{A})''$ the supremum is also contained in $\pi(\mathcal{A})''$.

(b) As Banach space $\pi(\mathcal{A})''$ is the dual space of another space, the predual. Thus by a general theorem the closed bounded sets in $\pi(\mathcal{A})''$ are compact in the weak* topology given by the predual. The positive normalized elements of the predual are called the *normal* states. Any state over $\mathcal{A}$ can be extended to a state over $\pi(\mathcal{A})''$ by the theorem of Hahn-Banach but not all extensions will be normal.

We shall call $(\pi(\mathcal{A})'', \tau_t)$ a W^*-dynamical system and it has the virtue that it contains many limiting elements not contained in $\mathcal{A}$. Typically time limits or time averages exist in $\pi(\mathcal{A})''$ but not in $\mathcal{A}$. On the other hand, the additional elements in $\pi(\mathcal{A})''$ may destroy some of the nice features of $\mathcal{A}$ and we always have to worry what extends to $\pi(\mathcal{A})''$ and what not.

A.3 Examples

1. The CAR algebra is simple and therefore has trivial center. However, $\mathbf{Z}_\omega = \pi_\omega(\mathcal{A})' \cap \pi_\omega(\mathcal{A})''$ may be nontrivial as is the case if ω is a convex combination of two states leading to inequivalent representations.

2. The CAR algebra is weakly asymptotically abelian for the free time evolution in the Fock representation π_∞. However, $\pi_\infty(\mathcal{A})''$ is not asymptotically abelian since $\pi_\infty(\mathcal{A})''$ contains the generators of rotations and translations which are constant but do not all commute.

3. If $\omega \circ \tau \neq \omega$ then in general τ cannot be extended to $\pi_\omega(\mathcal{A})''$. Furthermore, if $\omega \circ \tau = \omega$ then τ can be extended to $\pi(\mathcal{A})''$ but if $\lim_{t\to 0} \|\tau_t(A) - A\| = 0 \, \forall A \in \mathcal{A}$, $\forall A \in \pi(\mathcal{A})''$ we can only conclude that the strong limit of $\tau_t(A)$ goes to A.

If the state is faithful, even when extended to $\pi(\mathcal{A})''$, then the tie between the C^*-algebra and its weak closure becomes stronger due to the existence of a modular automorphism group.

A.4 The modular automorphism

1. If $A \in \pi(\mathcal{A})''$ and $A|\Omega\rangle = 0$ implies $A = 0$ then Ω is not only cyclic but also separating for $\pi(\mathcal{A})''$ and therefore also separating and cyclic for $\mathcal{A}'$.

2. There exists a modular automorphism group m_t that can be extended to the strip $0 \leq \text{Im}\, t \leq 1$ such that
$$\omega(AB) = \omega(m_i B \cdot A).$$
m_i itself is only defined on a dense set of analytic operators. Analytic operators are easily constructed, for instance,
$$\tilde{A}_\alpha(t) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} dt' e^{-(t-t')^2/\alpha} m_{t'} A$$
is analytic in t. Because m_t is strongly continuous on $\pi(\mathcal{A})''$ we see that for $\alpha \to 0$ $\tilde{A}_\alpha(t)$ converges strongly to $m_t A$. Thus analytic operators are strongly dense in $\pi(\mathcal{A})''$. For the expectation value the analytic extension is possible for all $A, B \in \pi(\mathcal{A})''$.

3. If the representation has a nontrivial center the modular automorphism leaves the center pointwise invariant.

4. If ω is invariant under the automorphism τ then $\tau \circ m = m \circ \tau$.

5. If τ is weakly asymptotically abelian on $\mathcal{A}$ then it is weakly asymptotically abelian on $\pi(\mathcal{A})''$ (see 5.4).

A.5 Decomposition of states: In section 5 we referred to the decomposition theory of states. The basic facts are the following:

1. Consider two states ω_1 and ω_2 such that $\omega_1(A) > \lambda\omega_2(A)$. Then there exists some $T \in \pi_1(\mathcal{A})'$ with $0 < T < \lambda$ such that

   $$\omega_2(A) = \langle\Omega_1|T\pi_1(A)|\Omega_1\rangle.$$

   Thus if ω can be written as linear sum of other states we can find corresponding positive operators in the commutant that add up to 1.

2. If ω is invariant under an automorphism group and is decomposed into other invariant states then the corresponding operators $\in \pi_\omega(\mathcal{A})'$ are also invariant.

3. If we know that the invariant operators from $\pi_\omega(\mathcal{A})'$ belong to the center – as is the case if the automorphism group is asymptotically abelian – they form an abelian algebra and we can use the isomorphism of an abelian algebra to $L^\infty(d\mu)$ for some measure $d\mu$ to find a decomposition into extremal invariant states: the decomposition of 1 in $L^\infty(d\mu)$ into a finite sum of projectors defines a corresponding decomposition of the state. Refinement of this decomposition has a unique upper limit [22] and this gives the (unique) decomposition into extremal invariant states. Since the invariant operators are contained in the center, the decomposition into extremal invariant states is coarser than the one determined by the center, which is the factor decomposition (and for KMS states corresponds to the decomposition into extremal KMS states).

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