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GEOMETRIC APPLICATIONS OF EVOLUTION

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GEOMETRIC APPLICATIONS OF EVOLUTION *

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1. Curve shortening

Let $\varphi : S^1 \to \mathbf{R}^2$ be a smooth embedded closed curve in the plane. For $x \in S^1$ let $\nu(\varphi(x))$ = inward unit normal vector. If φ is parametrized by arc length, then its <u>curvature at</u> $\varphi(x)$ is

$$k(\varphi(x)) = |\varphi''(x)| .$$

Consider a 1–parameter family $(\varphi_t)_{t \geq 0}$ of such embedded curves, such that

$$\left.\begin{aligned} \frac{\partial \varphi_t}{\partial t} &= k(\varphi_t)\,\nu(\varphi_t) \\ \varphi_t\big|_0 &= \varphi . \end{aligned}\right\} \qquad (1)$$

Theorem There exists a unique solution of the evolution equation (1). As $t \to \infty$ the curves become convex. The limit is a round point. (We leave to the reader the pleasant task of defining "round point", in this context.)

That theorem was proved by M. Grayson [J. Diff. Geo. 26 (1987)], following earlier contributions by M. Gage and R. Hamilton [J. Diff. Geo. 23 (1986)], who established a special case, assuming that the initial curve $\varphi(S^1)$ is itself convex.

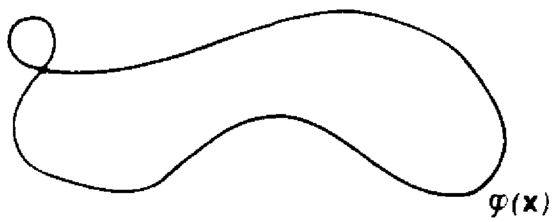
Now let us replace the plane by a surface N with a Riemannian metric h. (We can think of N as a surface in Euclidean space $\mathbf{R}^3$, with induced h – which permits us to measure angles between vectors on N, as well as lengths of smooth curves.)

Say that N <u>is convex at</u> ∞ if the convex hull of every compact subset is compact.

Grayson [Ann. Math. 129 (1989), 71–111] has generalized the preceding result as follows:

Theorem Let N be a surface which is convex at ∞. For any smooth embedded curve $\varphi : S^1 \to N$ equation (1) has a unique solution for $0 \leq t < t_\infty$. If $t_\infty < \infty$ then $\varphi_t(S^1)$ converges to a point. If $t_\infty = \infty$, then its curvature converges to 0 in norm. [It could happen that $\varphi_t(S^1)$ converges to a closed geodesic as N.]

Remark If $\varphi : S^1 \to N$ is immersed; i.e., self intersections are permitted:

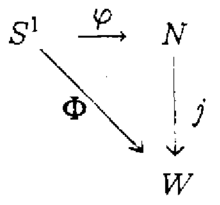


the singularities can develop, in the form of loops which *probably* pinch off to cusps.

In that theorem we need to describe the curvature (= geodesic curvature) k of a curve on N. We do so now:

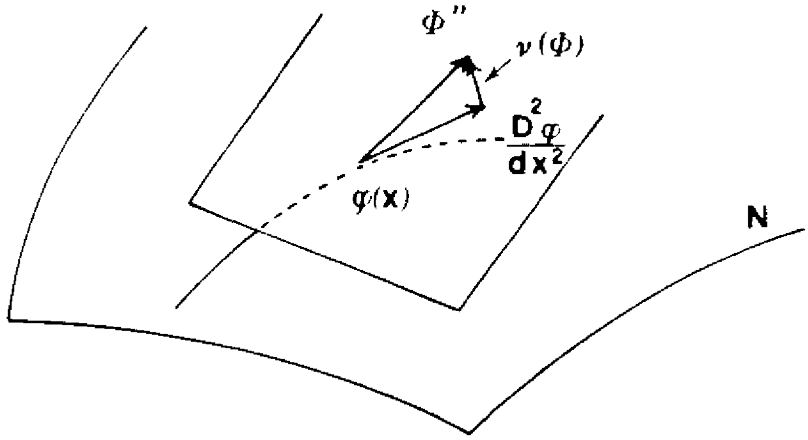
2. Closed geodesics

Let W be a Euclidean space of some finite dimension; and $j : N \hookrightarrow W$ a closed submanifold. Consider a map



Let $\Phi = j \cdot \varphi$

We let $\varphi'(x) = \frac{d\varphi(x)}{dx}$ denote the tangent vector of φ at the point x (if x is thought of as time then $\varphi'(x)$ is the velocity of φ).



Of course, we have the acceleration vector $\Phi''(x) \in W$. Define the acceleration vector $\frac{D^2\varphi(x)}{dx^2}$ φ at x as the orthogonal projection of $\Phi''(x)$ on the tangent plane of N at $\varphi(x)$:

$$\Phi'' = \frac{D^2\varphi}{dx^2} + \nu(\Phi) .$$

If $\varphi : S^1 \to N$ is parametrized by arc length, we say that φ is a (closed) geodesic of N iff $\Phi'' \perp N$ at every point, i.e., iff

$$\frac{D^2\varphi}{dx^2} \equiv 0 \quad \text{on} \quad S^1 .$$

The vector $\frac{D^2\varphi(x)}{dx^2}$ is the geodesic curvature of φ at x; and in the notation of Section 1 has the form $k(\varphi(x))\nu(x)$, with respect to the orientation of $\varphi(S^1)$.

Theorem If N is compact, then there is a unique solution to the evolution equation

$$\left.\begin{aligned} \frac{\partial\varphi_t}{\partial t} &= \frac{D^2\varphi_t}{dx^2} \\ \varphi_t\big|_{t=0} &= \varphi\,. \end{aligned}\right\} \tag{2}$$

That solution subconverges uniformly to a closed geodesic (possibly a point) of N. That theorem was proved by Eells–Sampson [Am. J. Math. 84 (1964)]; the curvature hypothesis in their proof is unnecessary for one–dimensional domains. See also K. Ottarsson [J. Geo. Phys. 2 (1985)].

Remark Without curvature restrictions on N I do not know whether the solution of (2) actually converges to a closed geodesic. In the theorem "subconverges" signifies that $\exists(t_j)$ such that $t_j \to \infty$ and φ_{t_j} converges uniformly.

3. Higher dimensional domains

Suppose we replace the domain S^1 by a compact Riemannian manifold M. (You can think of $M = S^1 \times S^1$ with its product metric; or $M = S^m$, the Euclidean sphere in $\mathbb{R}^{m+1}$.) Then associated with M is its Laplacian on functions

$$\Delta = g^{ij}\left(\frac{\partial^2}{\partial_x{}^i\partial_x{}^j} - \Gamma_{ij}^h\frac{\partial}{\partial_x{}^k}\right).$$

We use that in place of the second derivative of Sections 1 and 2.

Proceeding as in Section 2, given a map φ:

$$\begin{array}{ccc} S^1 & \xrightarrow{\varphi} & N \\ & \searrow^{\Phi} & \downarrow j \\ & & W \end{array}$$

Let $\Phi = j \cdot \varphi$

we have $\Delta\Phi : M \to W$. Define the tension field $\tau(\varphi)(x)$ of φ at x as the orthogonal projection of $\Delta\Phi(x)$ onto the tangent space of φ at x:

$$\Delta\Phi = \tau(\varphi) + \nu(\Phi)\,.$$

We say that a map $\varphi : M \to N$ is harmonic iff $\Delta\Phi \perp N$ at every point. I.e., iff

$$\tau(\varphi) \equiv 0 \quad \text{on} \quad M .$$

$\tau(\varphi)$ plays the role of $\dfrac{D^2\varphi}{dx^2}$ in Section 2.

Once again, we can form the evolution equation for maps $\varphi : M \to N$, as initial conditions:

$$\left.\begin{aligned} \frac{\partial\varphi_t}{\partial t} &= \tau(\varphi_t) \\ \varphi_t\big|_0 &= \varphi . \end{aligned}\right\} \tag{3}$$

This is a semi–linear parabolic system. (3) always has a unique smooth solution for a positive time interval $0 \leq t < t_1$.

Basic problem If dim $M = 2$ and both M and N are compact, are the solutions of (3) defined for all $0 \leq t < \infty$?

In case dim $M \geq 3$, there is a wide variety of situations in which that basic problem has a negative answer (Y.M. Chen and W–Y. Ding [Inv. Math. 99 (1990)]).

By way of contrast, for all compact M, there are various geometric restrictions on N which insure that (3) does have full solutions, and which converge (or subconverge) to a harmonic map $M \to N$. For that extensive story we refer to Eells–Lemaire [Bull. London Math. Soc. 10 (1978) and 20 (1988), Sec.3].