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INTEGRABLE LATTICE MODELS AND QUANTUM GROUPS

SALEUR H.- ZUBER J.B.
CEA Centre d'Etudes Nucleaires de Saclay, 91 - Gif-sur-Yvette (FR).
Service de Physique Theorique

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Integrable Lattice Models and Quantum Groups

H. Saleur * and J.-B. Zuber

Service de Physique Théorique de Saclay [1]
F-91191 Gif-sur-Yvette cedex, France

These lectures aim at introducing some basic algebraic concepts on lattice integrable models, in particular quantum groups, and to discuss some connections with knot theory and conformal field theories. The list of contents is:

Lectures given at the 1990 Trieste Spring School on String Theory and Quantum Gravity.

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* (Address after May 15, 1990: Institute for Theoretical Physics, Santa Barbara, CA 93106)

[1] Laboratoire de l' Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique

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0. INTRODUCTION

Integrable lattice models have undergone a very fast and spectacular development over the last few years. Previously known models have been generalized in several directions. In particular, the celebrated six-vertex model and the related XXZ spin $\frac{1}{2}$ quantum chain have been recognized to be the first of several hierarchies, involving either higher spin representations of $SU(2)$ or higher rank algebras or both. This progress has been made possible by the algebraic formulation of the Yang-Baxter integrability condition in a form which involves a deformation of Lie groups. The concept of "quantum group" has thus become a major theme of study.

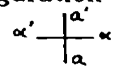
In parallel, connections between integrable lattice models and other fields have been discovered: conformal field theories, topological field theories and knot theory. Some of these connections are only imperfectly understood as yet but are certainly a remarkable contribution to the unification of concepts in two-dimensional physics.

The purpose of these lectures is to present some of these topics in a hopefully pedagogical way. We focus throughout on the algebraic aspects –though at an elementary level– and do not discuss the relevance of these lattice models in statistical mechanics. Moreover as we are interested in making contact with conformal theories, we concentrate on the so-called trigonometric solutions of the Yang-Baxter (YB) equation. We first present the various forms of this equation on the $\check{R}$ and R matrices and illustrate them on the six-vertex model (Lecture 1). The concept of quantum group, or more correctly, of quantum algebra $\mathcal{U}_q\, sl(2)$, is then introduced in Lecture 2 following the method of Sklyanin, and shown to lead to new solutions of the YB equation. In Lecture 3, we develop a different standpoint in which the quantum algebra appears as a symmetry of the model; the properties of the representations of $\mathcal{U}_q\, sl(2)$ are sketched. In Lecture 4, we show how vertex models may be reinterpreted as face models. For q a root of unity, this allows a restriction of the face model as discussed in Lecture 5. Contact with conformal field theory is shortly made at this point. The last lecture is a brief introduction to yet another topic: knot theory. It is shown how solutions to the Yang-Baxter equation of the type discussed previously enable one to construct polynomials that yield a topologically invariant description of knots.

This is by no means an exhaustive overview of the subject. Topological field theory and several aspects of the connections with conformal field theories are not touched upon. Likewise the references we give have no pretense to completeness. Among the many excellent review articles on the subject, we have deliberately selected a few references [1]-[5] that we found useful and to which we borrowed in the preparation of these notes.

1. VERTEX MODELS.

1.1. The $\check{R}$ matrix.

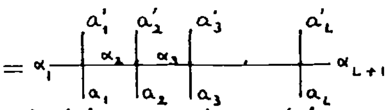
The first class of integrable models that we are going to consider are "vertex models". Degrees of freedom are attached to links of a square lattice, and interact at the vertices of the lattice. Each configuration of four links incident on a vertex is assigned a Boltzmann weight $w(a', a; \alpha', \alpha) =$ 

and the Boltzmann weight of the whole configuration on the lattice is the product of these individual contributions.

The degrees of freedom denoted by α and a may be of a rather general nature and take their values in a discrete set. As the notation a', a, α', α suggests, one may even suppose that degrees of freedom on horizontal (α', α) and vertical (a', a) links are of a different nature. For the time being, we do not specify what the horizontal variables are, and take the generic case where the vertical ones may take N independent values.

It is convenient to use a Hamiltonian picture, in which "time" flows upward on the lattice, and the various configurations of vertical links are considered as the independent possible states of the system at a given time. For a lattice of horizontal width L (with, say, periodic b.c.), the space spanned by these states is $V^{\otimes L}$, where V is the one-body vector space: $V = \mathbf{C}^N$ for the N-state model just mentionned. Time evolution is carried out by the row-to-row transfer matrix $\mathcal{T}^{(L)}$. The latter is the trace of the monodromy matrix $\tau^{(L)}_{\alpha_1 \alpha_{L+1}}$

$$\mathcal{T}^{(L)} = \sum_\alpha \tau^{(L)}_{\alpha\,\alpha} \tag{1.1}$$

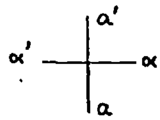
$$< a'_1 a'_2 \cdots a'_L | \tau^{(L)}_{\alpha_1 \alpha_{L+1}} | a_1 a_2 \cdots a_L > =$$ 

$$= \sum_{\alpha_2, \cdots \alpha_L} w(a'_1 a_1; \alpha_1 \alpha_2) w(a'_2 a_2; \alpha_2 \alpha_3) \cdots w(a'_L a_L; \alpha_L \alpha_{L+1}) \tag{1.2}$$

In a more compact notation:

$$\tau^{(L)}_{\alpha_1 \alpha_{L+1}} = \sum_{\alpha_2, \cdots \alpha_L} t_{\alpha_1 \alpha_2} \otimes t_{\alpha_2 \alpha_3} \otimes \cdots \otimes t_{\alpha_L \alpha_{L+1}} \tag{1.3}$$

where we have defined the n^2 $n \times n$ matrices $t_{\alpha' \alpha}$ by:

$$(t_{\alpha' \alpha})_{a' a} = w(a' a; \alpha' \alpha) = \tag{1.4}$$ 

In terms of $\mathcal{T}^{(L)}$, the partition function on a $L\times T$ doubly periodic lattice is

$$Z=\mathrm{tr}(\mathcal{T}^{(L)})^T \tag{1.5}$$

To ensure the complete integrability of the model, we actually seek a one-parameter family of Boltzmann weights $w(a'a;\alpha'\alpha|u)$, hence of monodromy and transfer matrices $t_{\alpha'\alpha}(u)$, $\tau(u)$ and $\mathcal{T}^{(L)}(u)$, such that

$$[\mathcal{T}^{(L)}(u),\mathcal{T}^{(L)}(v)]=0 \tag{1.6}$$

The interpretation of the additional "spectral parameter" u is not quite evident at this stage. In actual models, like the 6-vertex model (see below), it introduces an anisotropy in the weights, but does not affect the critical universal properties (in a certain range). In fact, its major role is to make $\mathcal{T}^{(L)}(u)$ a generating function of conserved quantities (see subsec 1.5).

Commutativity of the transfer matrices follows from the assumed existence of a non-singular $n^2\times n^2$ matrix $\check{R}(u,v)=(\check{R}_{\alpha\beta,\gamma\delta}(u,v))$ that satisfies

$$\check{R}(u,v)(t(u)\otimes t(v))=(t(v)\otimes t(u))\check{R}(u,v) \tag{1.7}$$

Here (in contrast with (1.3)), the tensor product of $t(u)$ and $t(v)$ refers to "horizontal variables" α',α. Each $t_{\alpha'\alpha}(u)$ is still a matrix acting on the vertical variables a',a and the notation $t(u)\otimes t(v)$ implies a matrix multiplication w.r.t. these variables, whereas the $\check{R}_{..,..}$ are c-numbers. In components this reads

$$\sum_{a',\gamma,\delta}\check{R}_{\alpha'\beta',\gamma\delta}(u,v)(t_{\gamma\alpha}(u))_{a''a'}(t_{\delta\beta}(v))_{a'a}=\sum_{a',\gamma,\delta}(t_{\alpha'\gamma}(v))_{a''a'}(t_{\beta'\delta}(u))_{a'a}\check{R}_{\gamma\delta,\alpha\beta}(u,v) \tag{1.8}$$

Eq. (1.7) implies a similar relation for $\tau^{(L)}$

$$\check{R}(u,v)(\tau^{(L)}(u)\otimes\tau^{(L)}(v))=(\tau^{(L)}(v)\otimes\tau^{(L)}(u))\check{R}(u,v) \tag{1.9}$$

hence

$$(\tau^{(L)}(u)\otimes\tau^{(L)}(v))=\check{R}^{-1}(u,v)(\tau^{(L)}(v)\otimes\tau^{(L)}(u))\check{R}(u,v) \tag{1.10}$$

and upon taking the trace, eq. (1.6) follows.

1.2. Yang-Baxter equation

We note that in eq. (1.7), the matrix $\check{R}$ interchanges the two operators $t(u)$ and $t(v)$ and their action on the two copies of the horizontal vector space. This suggests to introduce the permutation operator P

$$P_{\alpha'\beta',\alpha\beta} = \delta_{\alpha\beta'}\delta_{\alpha'\beta} \tag{1.11}$$

and the braiding matrix

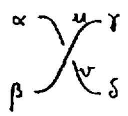
$$R = P\check{R} \qquad \text{i.e.} \qquad R_{\alpha'\beta',\alpha\beta} = \check{R}_{\beta'\alpha',\alpha\beta}. \tag{1.12}$$

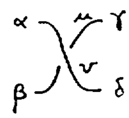
Eq. (1.7) is rephrased as

$$R(u,v).(t(u)\otimes\mathbf{1}).(\mathbf{1}\otimes t(v)) = (\mathbf{1}\otimes t(v)).(t(u)\otimes\mathbf{1}).R(u,v) \tag{1.13}$$

which allows a graphical representation

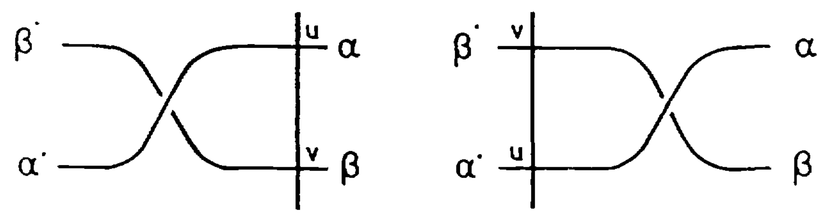
$$R_{\alpha\beta,\gamma\delta}(u,v) = \qquad\qquad R^{-1}_{\alpha\beta,\gamma\delta}(u,v) = \tag{1.14}$$





See fig. 1. Notice that each thread ($\alpha'\alpha$ or $\beta'\beta$) pertains to a definite copy of the horizontal space and that it carries its own spectral parameter.

Fig. 1.



The matrix R satisfying (1.13) is subject to a consistency condition, which originates in the two inequivalent ways to braid three threads 123 into 321. Take three copies of the horizontal space H_1, H_2 and H_3, attach spectral parameters u_1, u_2 and u_3 to them, and append a pair of indices to R to indicate on which pair of these three spaces it acts non trivially. Thus, one considers R_{12}, R_{23} and R_{13}, and for example

$$(R_{23})_{\alpha'_1\alpha'_2\alpha'_3,\alpha_1\alpha_2\alpha_3} = \delta_{\alpha'_1\alpha_1} R_{\alpha'_2\alpha'_3,\alpha_2\alpha_3}(u_2,u_3)$$

Finally, by abuse of notation, denote

$$t_1 = t(u_1)\otimes\mathbf{1}\otimes\mathbf{1}, \qquad \text{etc...}$$

Then from (1.13) (see the graphical representation in fig. 2)

$$
\begin{aligned}
t_1 t_2 t_3 &= R_{12}^{-1} t_2 t_1 t_3 R_{12} \\
&= R_{12}^{-1} R_{13}^{-1} t_2 t_3 t_1 R_{13} R_{12} \\
&= R_{12}^{-1} R_{13}^{-1} R_{23}^{-1} t_3 t_2 t_1 R_{23} R_{13} R_{12}
\end{aligned}
\tag{1.15a}
$$

but also

$$
\begin{aligned}
&= R_{23}^{-1} t_1 t_3 t_2 R_{23} \\
&= R_{23}^{-1} R_{13}^{-1} t_3 t_1 t_2 R_{13} R_{23} \\
&= R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} t_3 t_2 t_1 R_{12} R_{13} R_{23}
\end{aligned}
\tag{1.15b}
$$

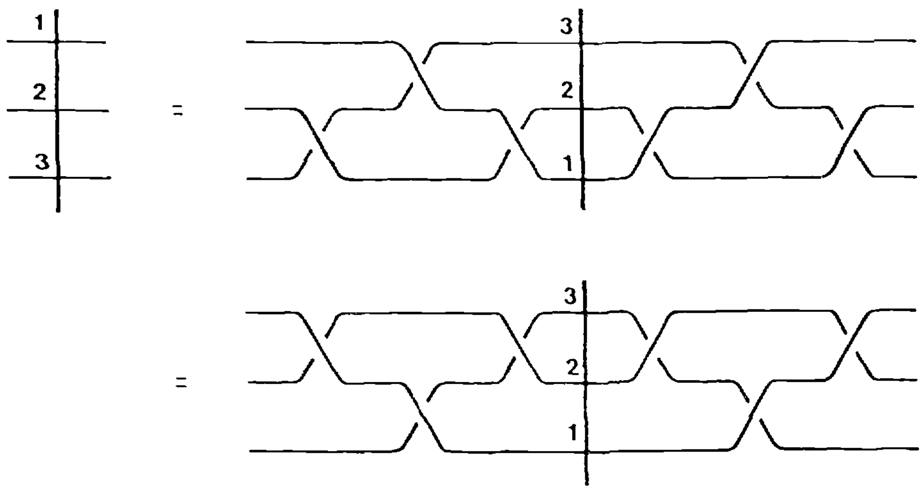
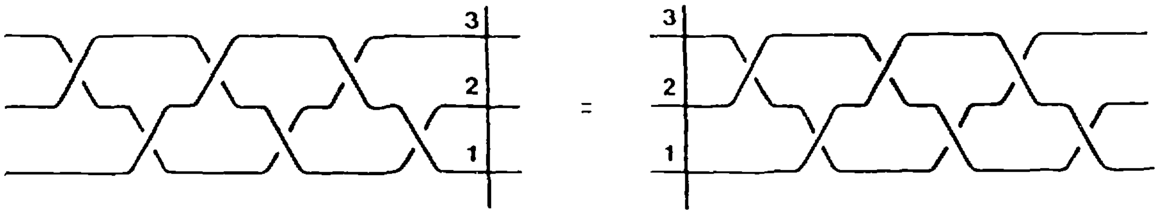


Fig. 2.

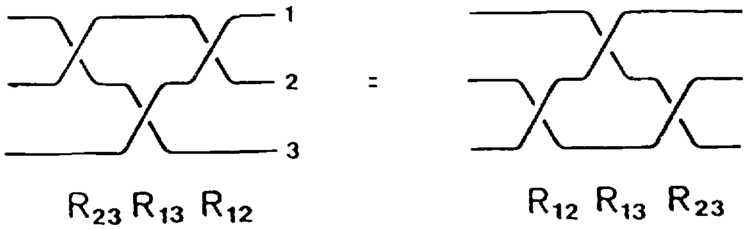
Comparing (1.15a) and (1.15b) one sees that $R_{23}R_{13}R_{12}R_{23}^{-1}R_{13}^{-1}R_{12}^{-1}$ commutes with $t_3t_2t_1$.



The simplest possibility is

$$
R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2) = R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) \tag{1.16}
$$

which is the Yang-Baxter equation.



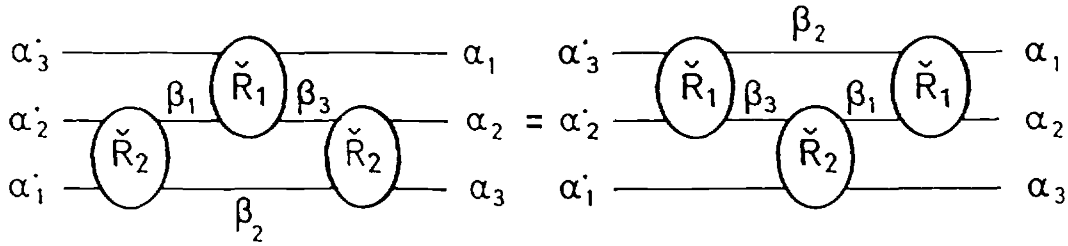
It may of course be made more explicit (and cumbersome) in terms of the components of R:

$$
\begin{aligned}
&\sum_{\beta_1,\beta_2,\beta_3} R_{\alpha'_2\alpha'_3,\beta_2\beta_3}(u_2,u_3)R_{\alpha'_1\beta_3,\beta_1\alpha_3}(u_1,u_3)R_{\beta_1\beta_2,\alpha_1\alpha_2}(u_1,u_2)\\
&\quad = \sum_{\beta_1,\beta_2,\beta_3} R_{\alpha'_1\alpha'_2,\beta_1\beta_2}(u_1,u_2)R_{\beta_1\alpha_3,\alpha_1\beta_3}(u_1,u_3)R_{\beta_2\beta_3,\alpha_2\alpha_3}(u_2,u_3)
\end{aligned}
\tag{1.17}
$$

These relations may also be rewritten in terms of $\check{R}$. Define $\check{R}_1(u,v) = PR_{12}(u,v)\otimes\mathbf{1}$, $\check{R}_2(u,v) = \mathbf{1}\otimes PR_{23}(u,v)$ acting on $H_1\otimes H_2\otimes H_3$. (More generally on $H^{\otimes T}$, $\check{R}_i(u,v) = \mathbf{1}\otimes\cdots\otimes PR_{i\,i+1}(u,v)\otimes\cdots\otimes\mathbf{1}$.) Then (1.17)reads

$$
\check{R}_2(u_1,u_2)\check{R}_1(u_1,u_3)\check{R}_2(u_2,u_3) = \check{R}_1(u_2,u_3)\check{R}_2(u_1,u_3)\check{R}_1(u_1,u_2) \tag{1.18}
$$

(see fig. 3). Of course this alternative expression carries the same nformation as (1.16), it is, however, particularly suited for the connection with the braid group (see below).

Fig. 3.


1.3. Remarks

i) Eq. (1.17) constitutes a vastly overdetermined system with n^6 equations for $3n^4$ unknowns. There are however several classes of solutions, in which symmetries reduce this overdeterminacy.

ii) In most cases known until recently, the dependence on the spectral parameters of $R(u,v)$ was only through the difference $u-v$ and was either rational, or trigonometric (or hyperbolic), or elliptic. Recently new classes of solutions involving higher genus parametrizations have been discovered [6]. In the following, we shall only consider solutions for R that are trigonometric functions of $u-v$.

iii) If we choose $u=v$ in eq. (1.7), a simple solution is provided by

$$\check{R}(u,u) = \check{R}(0) = \mathbf{1} \tag{1.19}$$

hence

$$R(u,u) = R(0) = \mathcal{P} \tag{1.20}$$

All the solutions of Y.B. that we are going to consider satisfy (1.19)-(1.20) ("regular" solutions). Incidentally, we note that the permutation operator $\mathcal{P}$ satisfies both (1.16) and (1.18).

$$\mathcal{P}_{12}\mathcal{P}_{13}\mathcal{P}_{23} = \mathcal{P}_{23}\mathcal{P}_{13}\mathcal{P}_{12} \tag{1.21a}$$

$$\mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{12} = \mathcal{P}_{23}\mathcal{P}_{12}\mathcal{P}_{23} \tag{1.21b}$$

Moreover, multiplying (1.7) by $\check{R}(v-u)$, we find

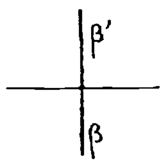
$$\check{R}(v-u)\check{R}(u-v)\big(t(u)\otimes t(v)\big) = \big(t(u)\otimes t(v)\big)\check{R}(v-u)\check{R}(u-v) \tag{1.22}$$

and it is again natural to impose as a constraint that

$$\check{R}(v-u)\check{R}(u-v) = \rho(u-v)\,\mathbf{1} \tag{1.23}$$

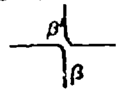
iv) Any solution of the YB equation yields a solution of the $Rtt = ttR$ equation in which one chooses

$$\big(t_{\alpha',\alpha}(u)\big)_{\beta'\beta} = R_{\alpha'\beta',\alpha\beta}(u) = \alpha' \quad \text{[diagram]} \quad \alpha \tag{1.24}$$



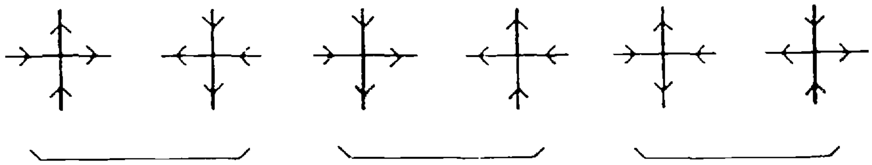
Eq. (1.17) may indeed be rewritten as:

$$\sum_{\beta_1,\beta_2,\beta_3} R_{\alpha'_1\alpha'_2,\beta_1\beta_2}(u_1-u_2)\big(t_{\beta_1,\alpha_1}(u_1-u_3)\big)_{\alpha'_3\beta_3}\big(t_{\beta_2,\alpha_2}(u_2-u_3)\big)_{\beta_3\alpha_3}$$
$$= \sum_{\beta_1,\beta_2,\beta_3} \big(t_{\alpha'_2,\beta_2}(u_2-u_3)\big)_{\alpha'_3\beta_3}\big(t_{\alpha'_1,\beta_1}(u_1-u_3)\big)_{\beta_3\alpha_3} R_{\beta_1\beta_2,\alpha_1\alpha_2}(u_1-u_2) \tag{1.25}$$

equivalent to (1.7). In this solution, the vertical and horizontal spaces are identical. If R is regular (1.19), then $t_{\alpha',\alpha}(0))_{\beta'\beta} = \delta_{\alpha'\beta}\delta_{\alpha\beta'}$ which may be represented as α'  α.

1.4. The six-vertex model

The six-vertex model of Lieb is a 2-state model in which the degrees of freedom are represented by arrows on the links of the lattice. The possible configurations are restricted by demanding that at each vertex the number of incoming arrows equals the number of outgoing ones:



The R-matrix is a $2^2 \times 2^2$ matrix and reads

$$R = \frac{1}{4}\sum_{i=0}^{3} w_i \sigma^i \otimes \sigma^i = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

where σ^i, $i = 1, 2, 3$ are the Pauli matrices, $\sigma^0 = 1$ and $w_1 = w_2$. This matrix satisfies the YB equation if it has the following dependence in the spectral parameter u

$$\begin{aligned} a &= \frac{w_0 + w_3}{4} = \rho \sin(\gamma + u) \\ b &= \frac{w_0 - w_3}{4} = \rho \sin u \\ c &= \frac{w_1}{2} = \rho \sin \gamma. \end{aligned} \tag{1.27}$$

There ρ is an overall irrelevant factor and γ is a coupling constant. By letting $\gamma = \pi - \lambda$, one finds another common parametrization $a = \rho \sin(\lambda - u)$, $b = \rho \sin u$ and $c = \rho \sin \lambda$, on which the interpretation of u as an anisotropy of the model (between weights a and b) is manifest. (The parameter γ is also sometimes referred to as anisotropy because of such an interpretation in the quantum spin chain, see below subsect. 1.5.) Both u and γ may be real or complex and it is suitable to discuss the phase structure of the model in terms of

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = \cos \gamma \tag{1.28}$$

Then one may show that for $|\Delta| < 1$ the model is in a critical massless phase.

Note that for $u = 0$, $R(0) = \rho \sin\gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \rho \sin\gamma P$ and that for u and γ small, $R \approx u\mathbf{1} + \gamma P$.

For periodic boundary conditions in the horizontal direction, there must exist an equal number of each type of c-vertex along each row. This suggests that one might modify the R_{1221} and R_{2112} entries of the matrix, keeping their product fixed while preserving the YB equation. This is indeed the case, and

$$R(u) = \begin{pmatrix} a & & & \\ & b & ce^{xu} & \\ & ce^{-xu} & b & \\ & & & a \end{pmatrix} \tag{1.29}$$

also satisfies (1.16). The merit of this form is that it leads a non trivial limit as $u \to \pm i\infty$, when $x = -i$:

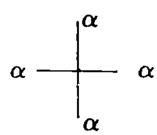
$$R(-i\infty) \sim \begin{pmatrix} 1 & & & \\ & q^{-1} & 0 & \\ & 1 - q^{-2} & q^{-1} & \\ & & & 1 \end{pmatrix} \qquad R(i\infty) \sim \begin{pmatrix} 1 & & & \\ & q & 1 - q^2 & \\ & 0 & q & \\ & & & 1 \end{pmatrix} \tag{1.30}$$

where

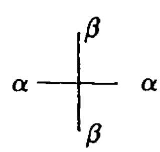
$$q = e^{i\gamma} \tag{1.31}$$

There are other more general types of such "gauge transformations" of the R-matrix.

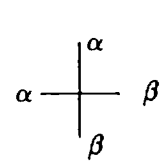
In the latter form (1.29), the six-vertex model generalizes nicely to an $N-$state model [7]. With α, β being Z_N variables, ($\alpha \neq \beta$), the Boltzmann weights read



$= \sin(u + \gamma)$ (1.32a)



$= \sin u$ (1.32b)



$= \sin\gamma\, e^{iu\, \mathrm{sign}(\alpha-\beta)}$ (1.32c)

1.5. Conserved quantities

The spectral parameter dependent transfer matrix $\mathcal{T}^{(L)}(u)$ may be regarded as the generating function of conserved quantities. The space $V^{\otimes L}$ in which $\mathcal{T}^{(L)}(u)$ acts is viewed as the Hilbert space of a quantum one-dimensional system. If one introduces the operators

$$\mathcal{H}_n = \frac{\partial^n}{\partial u^n} \log \mathcal{T}^{(L)}(u)|_{u=0} \tag{1.33}$$

the commutation of $\mathcal{T}^{(L)}(u)$ and $\mathcal{T}^{(L)}(v)$ implies the commutation of the infinite set of $\mathcal{H}_n$:

$$[\mathcal{H}_n, \mathcal{H}_m] = 0. \tag{1.34}$$

In particular, if $\mathcal{H}_1$ is regarded as the Hamiltonian of the quantum system, there is an infinite number of conserved quantities, commuting with $\mathcal{H}_1$.

It is instructive to compute $\mathcal{H}_1$ for the 6-vertex model

$$\frac{d}{du}\mathcal{T}^{(L)}(u=0) = \sum_{k=1}^{L} t_{\alpha_1\alpha_2}(0) \otimes \cdots \otimes \dot{t}_{\alpha_k\alpha_{k+1}}(0) \otimes \cdots \otimes t_{\alpha_L\alpha_1}. \tag{1.35}$$

Thus

$$\mathcal{H}_1 = \mathcal{T}^{(L)-1}(0)\dot{\mathcal{T}}^{(L)}(0) = \sum_{k=1}^{L} \mathbf{1} \otimes \cdots \otimes h_{k\,k+1} \otimes \cdots \mathbf{1} \tag{1.36}$$

where $h_{k\,k+1}$ acts on the k and $k+1$-th vertical variables, i.e. in $V_k \otimes V_{k+1}$

$$\begin{aligned} h_{k\,k+1} &= \frac{1}{\rho \sin\gamma} P\dot{R}(0) = \frac{1}{\sin\gamma}\begin{pmatrix} \cos\gamma & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \cos\gamma \end{pmatrix} \\ &= \frac{1}{2\sin\gamma}\big((\mathbf{1}\otimes\mathbf{1} + \sigma^3 \otimes \sigma^3)\cos\gamma + \sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2\big). \end{aligned} \tag{1.37}$$

Thus, up to an irrelevant constant:

$$\mathcal{H}_1 = \frac{1}{2\sin\gamma}\sum_{k=1}^{L}\left(\sigma_k^1 \otimes \sigma_{k+1}^1 + \sigma_k^2 \otimes \sigma_{k+1}^2 + \cos\gamma\, \sigma_k^3 \otimes \sigma_{k+1}^3\right) \tag{1.38}$$

This is the Hamiltonian of a (periodic) chain of $\frac{1}{2}$-spins: $S_k = \frac{1}{2}\sigma_k$ interacting with an anisotropic "XXZ"-interaction.

Thus one sees that $\mathcal{T}^{(L)}(u)$, the transfer matrix of the 6-vertex model, commutes with the XXZ spin $\frac{1}{2}$ hamiltonian. Alternatively, one may say that the latter is generated from the former in a "very anisotropic (continuum) limit" where the lattice spacing in the vertical direction is let to zero as well as the spectral parameter:

$$\mathcal{T}^{(L)}(u) \approx \mathbf{1} + u\mathcal{H}_1. \tag{1.39}$$

One may study in a similar way the other conserved quantities and prove a locality property: $\mathcal{H}_n$ couples at most $n+1$ spins in the chain.

2. QUANTUM $sl(2)$ AND THE YANG-BAXTER EQUATION.

The relations between the Yang-Baxter (YB) equation and "quantum groups" appear in several related forms in the literature [8]-[10]. We present here some of the simplest features and refer to the references for a more thorough mathematical discussion.

2.1. Classical limit of YB.

We have seen before that the R matrix of the 6-vertex model satisfies $R(u,\gamma=0)=\mathbf{1}$ (for $\rho\sin u=1$). Whenever we have an $R(u,\gamma)$ matrix depending on a parameter γ with this property, we may expand

$$R(u,\gamma)=\mathbf{1}+\gamma r(u)+O(\gamma^2) \tag{2.1}$$

The matrix $r(u)$ is called the classical limit of $R(u,\gamma)$ (γ is regarded as the $\hbar$ parameter of quantum mechanics). In the 6-vertex model, (see (1.27) for $\rho=1/\sin u$), this is simply:

$$r_{12}(u)=\frac{1}{\sin u}\begin{pmatrix}\cos u & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \cos u\end{pmatrix} \tag{2.2}$$

which for u small, reduces to

$$r_{12}(u)=\frac{1}{u}\left(\tfrac{1}{2}\mathbf{1}+2\mathbf{s}_1.\mathbf{s}_2\right) \tag{2.3}$$

where $\mathbf{s}=\frac{1}{2}\sigma$ is the spin $\frac{1}{2}$ operator. In general, this matrix $r(u)$ satisfies a limiting form of the YB equation called the classical Yang-Baxter equation:

$$[r_{12}(u),r_{13}(u+v)]+[r_{12}(u),r_{23}(v)]+[r_{13}(u+v),r_{23}(v)]=0 \tag{2.4}$$

which has its own significance in the context of classical integrable models. The remarkable property of (2.4) is that it only involves commutation relations. If g denotes a Lie algebra and X_μ a basis of generators, and if we have found a solution of (2.4) in $g\otimes g$,

$$r(u)=\sum_{\mu\nu}r^{\mu\nu}(u)X_\mu\otimes X_\nu \tag{2.5}$$

then any representation of the X_μ in a space H gives a solution of (2.5) in $H\otimes H$. For example, any spin representation $\mathbf{s}$ in (2.3) gives a solution. In that sense the forms (2.3),(2.5) are universal solutions of the classical Yang-Baxter equation.

It seems legitimate to ask if something similar can be done with the $R(u,\gamma)$ matrix.

2.2. Looking for other solutions of YB.

Writing $R(u) = \sum_{i=0}^{3} w_i(u) s^i \otimes s^i$ we can wonder to which extent the YB equation satisfied by R and the $sl(2)$ algebra

$$\begin{aligned} [s^+, s^-] &= 2s^z \\ [s^z, s^\pm] &= \pm s^\pm \end{aligned} \tag{2.6}$$

satisfied by $s = \frac{1}{2}\sigma$ are related. For instance a (too) naïve question is what happens if instead of $sl(2)$ generators acting on $V = H = C^2 = \rho_{\frac{1}{2}}$, we take $V = \rho_j$, $H = \rho_{j'}$ where ρ_j is a $sl(2)$ spin j representation? The answer is that YB is no longer satisfied in general. To proceed we consider the case where the horizontal degrees of freedom are still in the $\rho_{\frac{1}{2}}$ representation with the operator $s = \frac{1}{2}\sigma$ and $s^0 = \frac{1}{2}\mathbf{1}$ but vertical degrees of freedom may take values in some other space ρ onto which some operators S^i act. Under which conditions on these S is the relation (1.13) satisfied

$$R(u-u')(t(u)\otimes\mathbf{1})(\mathbf{1}\otimes t(u')) = (\mathbf{1}\otimes t(u'))(t(u)\otimes\mathbf{1})R(u-u') \tag{2.7}$$

where

$$\begin{aligned} t(u) &= \sum_{i=0}^{3} w^i s^i \otimes S^i \\ t(u') &= \sum_{i=0}^{3} w'^i s^i \otimes S^i \\ R(u-u') &= \sum_{i=0}^{3} w''^i s^i \otimes s^i. \end{aligned} \tag{2.8}$$

For the sake of brievity, we denote $w = w(u)$, $w' = w(u')$ and $w'' = w(u-u')$. Both sides of (2.7) are products of 4×4 matrices acting in $H^{\otimes 2} = \rho_{\frac{1}{2}}^{\otimes 2}$. The matrix elements of $(t(u)\otimes\mathbf{1})$ and $(\mathbf{1}\otimes t(u'))$ are operators in V which are linear combinations of S's. Explicitly

$$t(u)\otimes\mathbf{1} = \frac{1}{2}\begin{pmatrix} w_0S^0 + w_3S^3 & 0 & w_1S^- & 0 \\ 0 & w_0S^0 + w_3S^3 & 0 & w_1S^- \\ w_1S^+ & 0 & w_0S^0 - w_3S^3 & 0 \\ 0 & w_1S^+ & 0 & w_0S^0 - w_3S^3 \end{pmatrix}$$

and

$$\mathbf{1}\otimes t(u') = \frac{1}{2}\begin{pmatrix} w'_0S^0 + w'_3S^3 & w'_1S^- & 0 & 0 \\ w'_1S^- & w'_0S^0 - w'_3S^3 & 0 & 0 \\ 0 & 0 & w'_0S^0 + w'_3S^3 & w'_1S^- \\ 0 & 0 & w'_1S^- & w'_0S^0 - w'_3S^3 \end{pmatrix}$$

In the products, S^i must be treated as operators which do not *a priori* commute. A straightforward calculation gives then the set of relations that the S^i must satisfy for (2.7) to hold. Let us look more closely at say the first line of the matrices obtained on the two sides of (2.7). Equality implies then

$$(w_3 w_0' - w_3' w_0)[S^3, S^0] = 0 \tag{2.9}$$

and

$$\begin{aligned} a''(w_0 S^0 + w_3 S^3) w_1 S^- &= b'' w_1' S^- (w_0 S^0 + w_3 S^3) + c''(w_0' S^0 + w_3' S^3) w_1 S^- \\ a'' w_1 S^- (w_0' S^0 + w_3' S^3) &= c'' w_1' S^- (w_0 S^0 + w_3 S^3) + b''(w_0' S^0 + w_3' S^3) w_1 S^- \end{aligned} \tag{2.10}$$

i.e.

$$(a'' w_1' w_0 - c'' w_0' w_1) S^0 S^- - b'' w_1' w_0 S^- S^0 = b'' w_1' w_3 S^- S^3 + (c'' w_3' w_1 - a'' w_1' w_3) S^3 S^- \tag{2.11a}$$

$$(a'' w_0' w_1 - c'' w_1' w_0) S^0 S^- - b'' w_0' w_1 S^- S^0 = b'' w_3' w_1 S^- S^3 + (c'' w_1' w_3 - a'' w_3' w_1) S^3 S^- \tag{2.11b}$$

Using eq. (1.26), (2.11*a*) gives, for $u \neq u'$ and $\gamma \neq 0$

$$\begin{aligned} &\cos\frac{\gamma}{2}\left(\sin(u+\gamma)\cos\frac{\gamma}{2}[S^0, S^-] + \cos(u+\gamma)\sin\frac{\gamma}{2}[S^0, S^-]_+ \right) \\ &\quad = \sin\frac{\gamma}{2}\left(\cos(u+\gamma)\cos\frac{\gamma}{2}[S^3, S^-] - \sin(u+\gamma)\sin\frac{\gamma}{2}[S^3, S^-]_+ \right) \end{aligned} \tag{2.12}$$

while (2.11*b*) gives just the same with u replaced by u'. For (2.7) to hold for all u, we thus need

$$[S^0, S^-] = \tan^2\frac{\gamma}{2}[S^3, S^-]_+ \tag{2.13a}$$

$$[S^3, S^-] = -[S^0, S^-]_+ \tag{2.13b}$$

A similar calculation of other matrix elements provides the additional relations

$$[S^0, S^+] = -\tan^2\frac{\gamma}{2}[S^3, S^+]_+ \tag{2.13c}$$

$$[S^3, S^+] = [S^0, S^+]_+ \tag{2.13d}$$

$$[S^+, S^-] = 2[S^0, S^3]_+ \tag{2.13e}$$

$$[S^0, S^3] = 0 \tag{2.13f}$$

and (2.7) holds if and only if $(2.13a - f)$ are satisfied. The algebra generated by the S^i contains two central elements

$$C = (S^0)^2 + (S^1)^2 + (S^2)^2 + (S^3)^2 - \frac{1}{4\cos^2\frac{\gamma}{2}} \tag{2.14a}$$

$$D = \left(\cos\frac{\gamma}{2}S^0\right)^2 + \left(\sin\frac{\gamma}{2}S^3\right)^2 \tag{2.14b}$$

The subtractive contribution to C is introduced for later convenience. Since D is central and $[S^0, S^3] = 0$, we introduce S^z such that

$$\begin{aligned} S^0 &= \frac{q^{S^z} + q^{-S^z}}{2(q^{\frac{1}{2}} + q^{-\frac{1}{2}})} \\ S^3 &= \frac{q^{S^z} - q^{-S^z}}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \end{aligned} \tag{2.15}$$

where

$$q = e^{i\gamma}. \tag{2.16}$$

There is no loss of generality in (2.15) since by an appropriate rescaling $\mathbf{S} \to \mathbf{S}/\sqrt{D}$ which preserves (2.13), we can always set $D = \frac{1}{4}\mathbf{1}$.

2.3. The $\mathcal{U}_q\, sl(2)$ algebra

With the parametrization (2.15), relations (2.13) are equivalent to

$$\begin{aligned} [S^+, S^-] &= \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \\ [S^z, S^\pm] &= \pm S^\pm \end{aligned} \tag{2.17}$$

Notice that (2.17) is satisfied by spin $\frac{1}{2}$ generators, which is expected since the particular choice $S^i = \frac{1}{2}\sigma^i$ in (2.8) satisfies YB. Also (2.17) reproduces the $sl(2)$ relations (2.6) in the limit $q \to 1$ i.e. $\gamma \to 0$. Hence in this limit (2.7) holds for S^i a $sl(2)$ generator in an arbitrary spin representation. The most convenient framework to deal with (2.17) is to consider the associative algebra (with unit) generated by $S^\pm, S^z$ satisfying (2.17): this is a deformation of the universal enveloping algebra of $sl(2)$ denoted $\mathcal{U}_q\, sl(2)$.

For q not a root of unity (hence in particular in the non critical case $|\Delta| > 1$), finite dimensional irreducible representations of $\mathcal{U}_q\, sl(2)$ are in one-to-one correspondence with those of $sl(2)$. They are labelled by an integer or half-integer j and denoted ρ_j in what follows. The representation ρ_j has a basis $|jm>$ that satisfies

$$\begin{aligned} S^{\pm}|jm> &= \sqrt{(j \mp m)_q (j \pm m + 1)_q}|jm \pm 1> \\ S^z|jm> &= m|jm> \end{aligned} \tag{2.18}$$

In (2.18) we have introduced the q-analogue symbol

$$(x)_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{2.19}$$

such that $(x)_q \to x$ as $q \to 1$. Matrix elements of $S^{\pm}$ in (2.18) look like those in $sl(2)$ with numbers replaced by their q-analogues. In particular $S^+|jj>= S^-|j-j>= 0$. The spectrum of S^z is $-j, -j+1, \cdots, j-1, j$. The representation ρ_j has dimension $2j+1$. It is useful to also introduce the "q-dimension" defined by

$$d_j = \mathrm{tr}_{\rho_j} q^{2S^z} = (2j+1)_q \tag{2.20}$$

In ρ_j, the Casimir operator (2.14a)

$$C = S^- S^+ + \left(S^z + \tfrac{1}{2}\right)_q^2 - \left(\tfrac{1}{2}\right)_q^2 \tag{2.21}$$

is scalar and takes the value

$$C_j = \left(j + \tfrac{1}{2}\right)_q^2 - \left(\tfrac{1}{2}\right)_q^2. \tag{2.22}$$

The norm of the states $|jm>$ is defined by $<jj|jj>= 1$ and by $(S^{\pm})^{\dagger} = S^{\mp}$.

If we substitute in (2.8) the operators S^i of (2.18) acting in ρ_j we obtain an integrable vertex model with horizontal degrees of freedom in $\rho_{\frac{1}{2}}$ and vertical ones in ρ_j. Rather than considering this very anisotropic situation, we shall pursue the study of $\mathcal{U}_q\, sl(2)$ which leads to more interesting results.

An important property of $\mathcal{U}_q\, sl(2)$ is that a tensor product can be defined. This is not obvious since the commutation relations are non linear in the generators. One checks that $\Delta: \ \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$

$$\begin{aligned} \Delta(S^z) &= 1 \otimes S^z + S^z \otimes 1 \\ \Delta(S^{\pm}) &= q^{S^z} \otimes S^{\pm} + S^{\pm} \otimes q^{-S^z} \end{aligned} \tag{2.23}$$

is an algebra homomorphism. $\mathcal{U}_q\, sl(2)$ equipped with the "coproduct" Δ and some other operations is a Hopf algebra [10]. It is of course not commutative but also non cocommutative, i.e. the two components of $\mathcal{U}_q \otimes \mathcal{U}_q$ are treated in an asymmetric way by Δ. Only in the $q \to 1$ limit does $\mathcal{U}_q$ become cocommutative.

Using Δ the composition of representations can be defined. One has

$$\rho_{j_1} \otimes \rho_{j_2} = \oplus_J \rho_J \tag{2.24}$$

with the usual rule that J runs from $|j_1 - j_2|$ to $j_1 + j_2$ and that $2J = 2j_1 + 2j_2 \bmod 2$. The q-dimensions being characters factorize accordingly

$$d_{j_1} d_{j_2} = \sum_J d_J. \tag{2.25}$$

Clebsch-Gordan coefficients can also be defined as in the $sl(2)$ case

$$|(j_1 j_2) JM >= \sum_{m_1+m_2=M} < j_1 m_1 j_2 m_2 | JM > |j_1 m_1 j_2 m_2 > \tag{2.26}$$

For $j = \frac{1}{2}$ the generators of $\mathcal{U}_q\, sl(2)$ coincide with their $q = 1$ limit, i.e. the $s = \frac{1}{2}\sigma$ matrices. Hence, in $\rho_{\frac{1}{2}}^{\otimes 2}$ we have

$$\begin{aligned} S^{\pm} &= q^{s^z} \otimes s^{\pm} + s^{\pm} \otimes q^{-s^z} \\ S^z &= 1 \otimes s^z + s^z \otimes 1 \end{aligned} \tag{2.27}$$

and a straightforward calculation gives

$$\rho_1 : \quad \begin{cases} |11> = |\tfrac{1}{2}\tfrac{1}{2}> \otimes |\tfrac{1}{2}\tfrac{1}{2}> & (2.28a) \\ |10> = \left(q^{\frac{1}{2}} |\tfrac{1}{2}\tfrac{1}{2}> \otimes |\tfrac{1}{2}-\tfrac{1}{2}> + q^{-\frac{1}{2}} |\tfrac{1}{2}-\tfrac{1}{2}> \otimes |\tfrac{1}{2}\tfrac{1}{2}> \right) (q+q^{-1})^{-\frac{1}{2}} \\ |1-1> = |\tfrac{1}{2}-\tfrac{1}{2}> \otimes |\tfrac{1}{2}-\tfrac{1}{2}> \end{cases}$$

$$\rho_0 : |00> = \left(q^{-\frac{1}{2}} |\tfrac{1}{2}\tfrac{1}{2}> \otimes |\tfrac{1}{2}-\tfrac{1}{2}> - q^{\frac{1}{2}} |\tfrac{1}{2}-\tfrac{1}{2}> \otimes |\tfrac{1}{2}\tfrac{1}{2}> \right) (q+q^{-1})^{-\frac{1}{2}} \tag{2.28b}$$

In computing the norm of $|jm>$, q is treated as a fixed variable and is not conjugated. Projections on ρ_0 and ρ_1 are

$$P_0 = \frac{1}{(2)_q} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.29a}$$

$$P_1 = 1 - P_0 = \frac{1}{(2)_q} \begin{pmatrix} (2)_q & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & (2)_q \end{pmatrix} \tag{2.29b}$$

2.4. The universal $\mathcal{R}$ matrix

Since the commutation relations (2.17) are left invariant in $q \to q^{-1}$, another coproduct is $\bar{\Delta}$

$$\bar{\Delta}(S^z) = \mathbf{1}\otimes S^z + S^z\otimes\mathbf{1}$$
$$\bar{\Delta}(S^\pm) = q^{-S^z} \otimes S^\pm + S^\pm \otimes q^{S^z}. \tag{2.30}$$

Remarkably there is an operator acting in $(\mathcal{U}_q\, sl(2))^{\otimes 2}$ that intertwines Δ and $\bar{\Delta}$

$$\mathcal{R}\Delta = \bar{\Delta}\mathcal{R}. \tag{2.31}$$

It reads explicitly,

$$\mathcal{R} = q^{2(S^z\otimes S^z)} \sum_{n\geq 0} \frac{(1-q^{-2})^n}{(n)_q!} q^{\frac{-n(n-1)}{2}} q^{-n(S^z\otimes 1 - 1\otimes S^z)} (S^-)^n \otimes (S^+)^n \tag{2.32}$$

where $(n)_q! = (1)_q(2)_q \cdots (n)_q$. The existence and form of $\mathcal{R}$ are most easily understood in the framework of Hopf algebras, which is beyond the scope of these lectures. It is, however, a rather easy exercise to check (2.31) "by hand". A useful formula for this purpose is

$$[S^+,(S^-)^n] = (S^-)^{n-1}(n)_q(2S^z - n + 1)_q$$
$$= (2S^z + n - 1)_q(n)_q(S^-)^{n-1} \tag{2.33}$$

Besides (2.32) one can choose as well

$$\mathcal{P}\mathcal{R}(q^{-1})\mathcal{P} = q^{-2(S^z\otimes S^z)} \sum_{n\geq 0} \frac{(1-q^2)^n}{(n)_q!} q^{\frac{n(n-1)}{2}} q^{-n(S^z\otimes 1 - 1\otimes S^z)} (S^+)^n \otimes (S^-)^n \tag{2.34}$$

Notice that $\big(\mathcal{R}(q)\big)^{-1} = \mathcal{R}(q^{-1})$. $\mathcal{R}$ satisfies many nice properties among which we quote

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{2.35a}$$
$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.35b}$$
$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{2.35c}$$

In (2.35a), for instance, $(\Delta \otimes 1)\mathcal{R}$ denotes the operator in $\mathcal{U}_q\, sl(2)^{\otimes 3}$ obtained from (2.32) by replacing each generator that acts on the left part of $\mathcal{U}_q\, sl(2)^{\otimes 2}$ by its coproduct. Eq.

(2.35c) is the Yang-Baxter equation but without spectral parameter. If we specialize $\mathcal{R}$ to act in $\rho_{\frac{1}{2}}^{\otimes 2}$ we get

$$\mathcal{R}^{\frac{1}{2}\frac{1}{2}} = q^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 1-q^{-2} & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.36}$$

On the other hand we have seen that the 6-vertex R matrix (1.29), for $x = -i$ and $\rho = 2iq^{-\frac{1}{2}}e^{-iu}$, has a limit as $u \to -i\infty$

$$\lim_{u \to -i\infty} R(u) = \mathcal{R}^{\frac{1}{2}\frac{1}{2}}. \tag{2.37}$$

Hence the universal $\mathcal{R}$ matrix in $\rho_{\frac{1}{2}}^{\otimes 2}$ is the $u \to -i\infty$ limit of the 6-vertex R matrix introduced in lecture 1.

Also in $\rho_{\frac{1}{2}} \otimes \rho_j$

$$\mathcal{R}^{\frac{1}{2}j} = q^{2(s^z \otimes S^z)}\left(1 + (1-q^{-2})q^{(1\otimes S^z - s^z \otimes 1)}s^- \otimes S^+\right) \tag{2.38}$$

or

$$\begin{aligned} q^{\frac{1}{2}}\mathcal{R}^{\frac{1}{2}j} =& \tfrac{1}{2}q^{\frac{1}{2}}1\otimes(q^{S^z} + q^{-S^z}) + q^{\frac{1}{2}}s^z \otimes (q^{S^z} - q^{-S^z}) + (q-q^{-1})s^- \otimes S^+ \\ =& 2(q+1)s^0 \otimes S^0 + 2(q-1)s^3 \otimes S^3 + (q-q^{-1})s^- \otimes S^+ \end{aligned} \tag{2.39}$$

Hence $\mathcal{R}^{\frac{1}{2}j}$ is the $u \to -i\infty$ limit of

$$R = 2(a+b)s^0 \otimes S^0 + 2(a-b)s^3 \otimes S^3 + ce^{iu}s^- \otimes S^+ + ce^{-iu}s^+ \otimes S^- \tag{2.40}$$

for $\rho = 2iq^{-\frac{1}{2}}e^{-iu}$. Eq. (2.40) is just the operator t introduced in (2.8) after the gauge transformation.

Thus (2.32) provides the limit of the various $sl(2)$ solutions of YB we already met. It contains of course much more since $\mathcal{R}$ in $\rho_{j_1} \otimes \rho_{j_2}$ gives a matrix solution $\mathcal{R}^{j_1 j_2}$ to (2.35) for any j_1, j_2. The inverse problem — sometimes called baxterization — of finding a spectral parameter dependent family of R matrices of which $\mathcal{R}^{j_1 j_2}$ is the limit is difficult. We shall discuss it only for $j_1 = j_2 = \frac{1}{2}$.

2.5. Hecke and Temperley-Lieb algebras.

One has

$$\bar{\Delta} = P\Delta P \tag{2.41}$$

and thus, by (2.31), the operator

$$\check{\mathcal{R}} = P\mathcal{R} \tag{2.42}$$

commutes with the generators of the quantum group

$$[\check{\mathcal{R}}, \mathcal{U}_q\, sl(2)] = 0. \tag{2.43}$$

In $\rho_{\frac{1}{2}}^{\otimes 2}$, $\check{\mathcal{R}}^{\frac{1}{2}\frac{1}{2}}$ decomposes onto P_0 and P_1

$$\check{\mathcal{R}}^{\frac{1}{2}\frac{1}{2}} = q^{-\frac{1}{2}}(qP_1 - q^{-1}P_0). \tag{2.44}$$

In the following, we use the notation

$$g = q^{\frac{1}{2}}\check{\mathcal{R}}^{\frac{1}{2}\frac{1}{2}} \tag{2.45}$$

It satisfies

$$(g-q)(g+q^{-1}) = 0 \tag{2.46}$$

due to (2.44). When working in $\rho_{\frac{1}{2}}^{\otimes L}$ we denote $g_i = \mathbf{1}\otimes\cdots\otimes g\otimes\cdots\otimes\mathbf{1}$ where g acts in the i-th and the $(i+1)$-th components of the tensor product, and we find the set of relations

$$g^2 = 1 + (q - q^{-1})g \tag{2.47a}$$

$$g_i g_{i\pm1} g_i = g_{i\pm1} g_i g_{i\pm1} \tag{2.47b}$$

$$[g_i, g_j] = 0 \qquad \text{for } |i-j| \geq 2 \tag{2.47c}$$

which define the Hecke algebra. Eq. (2.47a) holds because g expands on two projectors. A similar relation would be encountered in $\mathcal{U}_q sl(N)$ as well. Here since we deal with $sl(2)$ it can be shown that the additional relation

$$1 - q^{-1}(g_i + g_{i\pm1}) + q^{-2}(g_i g_{i\pm1} + g_{i\pm1} g_i) - q^{-3} g_i g_{i\pm1} g_i = 0 \tag{2.48}$$

also holds. This relation which is stronger than (2.47b) expresses the vanishing of the (q-analogue of the) Young antisymmetrizer of order 3 acting on $\mathcal{U}_q\, sl(2)$ representations [11]. Sometimes it is more convenient to deal with the algebra of P_0 projectors, or

$$e = (q + q^{-1})P_0 \tag{2.49}$$

for which one finds

$$e^2 = (q+q^{-1})e \tag{2.50a}$$

$$e_i e_{i\pm 1} e_i = e_i \tag{2.50b}$$

$$[e_i, e_j] = 0 \qquad \text{for } |i-j| \geq 2 \tag{2.50c}$$

Equations (2.50) define the Temperley-Lieb algebra. From (2.50) it is easy to check that

$$X(u) = \sin(\gamma+u)\mathbf{1} - \sin u\, e \tag{2.51}$$

satisfies the spectral parameter dependent YB equation

$$X_i(v)X_{i+1}(u+v)X_i(u) = X_{i+1}(u)X_i(u+v)X_{i+1}(v). \tag{2.52}$$

Explicitly X reads

$$X(u) = \begin{pmatrix} \sin(\gamma+u) & 0 & 0 & 0 \\ 0 & \sin\gamma e^{iu} & \sin u & 0 \\ 0 & \sin u & \sin\gamma e^{-iu} & 0 \\ 0 & 0 & 0 & \sin(\gamma+u) \end{pmatrix} \tag{2.53}$$

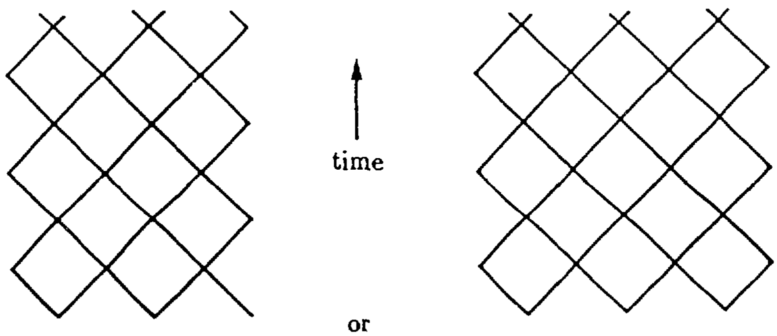
in agreement with (1.29) for $x = -i$.

3. $\mathcal{U}_q\, sl(2)$ AS A SYMMETRY OF LATTICE MODELS.

We have so far considered $\mathcal{U}_q\, sl(2)$ in connection with the YB equation, which is the "historical" point of view. It is also interesting to think of $\mathcal{U}_q\, sl(2)$ as a symmetry of lattice models.

3.1. Diagonal geometry. Commutation with $\mathcal{U}_q\, sl(2)$.

We consider again the 6-vertex model but with a transfer matrix propagating in the diagonal direction. We restrict ourselves to free boundary conditions



(depending on the parity of L). The diagonal-to-diagonal transfer matrix $\mathcal{T}^{(L)}$ acts on

$\mathcal{H} = \rho_{\frac{1}{2}}^{\otimes L}$. With the gauge transformed weights

$$a = \sin(\gamma + u) \qquad b = \sin u \qquad c_1 = \sin\gamma e^{iu} \qquad c_2 = \sin\gamma e^{-iu} \tag{3.1}$$

it reads

$$\mathcal{T}^{(L)}(u) = \prod_{i=1}^{E(L/2)} \left(\sin(\gamma+u)\mathbf{1} - \sin u\, e_{2i-1}\right) \prod_{i=1}^{E((L-1)/2)} \left(\sin(\gamma+u)\mathbf{1} - \sin u\, e_{2i}\right) \tag{3.2}$$

The $\mathcal{T}^{(L)}(u)$ do not form a family of commuting transfer matrices. On the other hand, since the Temperley-Lieb algebra generates the commutant of $\mathcal{U}_q\, sl(2)$ acting in $\mathcal{H} = \rho_{\frac{1}{2}}^{\otimes L}$, one has

$$[\mathcal{T}^{(L)}(u), \mathcal{U}_q\, sl(2)] = 0 \tag{3.3}$$

The action of $\mathcal{U}_q\, sl(2)$ is obtained by iterated applications of the coproduct formula

$$\begin{aligned} S^z &= \sum_{i=0} s_i^z \\ S^\pm &= \sum_{i=1}^{L} S_i^\pm \quad , \quad S_i^\pm = q^{s_1^z} \otimes \cdots \otimes q^{s_{i-1}^z} \otimes s_i^\pm \otimes q^{-s_{i+1}^z} \otimes \cdots \otimes q^{-s_L^z} \end{aligned} \tag{3.4}$$

and (3.3) follows simply from the fact that $e = (2)_q P_0$ is a projector.

Hence the vertex model has a (rather hidden) $\mathcal{U}_q\, sl(2)$ symmetry besides the obvious $U(1)$ symmetry due to spin conservation. It is important to notice that (3.3) would *not* hold for the standard weights (1.27). The effect of the gauge transformation can be put in boundary terms only, but these are crucial as far as symmetries and critical properties are concerned. The gauge transformation plays here the role of a charge at infinity in CFT. This is most clearly seen in the very anisotropic limit where

$$\mathcal{T}^{(L)} = \mathbf{1} - \frac{u}{\sin\gamma} \sum_{i=1}^{L-1} e_i. \tag{3.5}$$

One has

$$e_i = \tfrac{1}{4}(q+q^{-1}) - \left(s_i^+ s_{i+1}^- + s_i^- s_{i+1}^+ + (q+q^{-1}) s_i^z s_{i+1}^z + \tfrac{1}{2}(q-q^{-1})(s_i^z - s_{i+1}^z)\right) \tag{3.6}$$

hence

$$\sum_{i=1}^{L-1} e_i = \tfrac{1}{4}(q+q^{-1})(L-1) - 2\left(\sum_{i=1}^{L-1} \left(s_i^x s_{i+1}^x + s_i^y s_{i+1}^y + \cos\gamma s_i^z s_{i+1}^z\right) + \frac{i\sin\gamma}{2}(s_i^z - s_{i+1}^z)\right) \tag{3.7}$$

which differs from the standard XXZ hamiltonian (1.38) by the imaginary (for $|\Delta| \leq 1$) boundary term.

Operators $S_i^{\pm}$ in (3.4) differ from $s_i^{\pm}$ by strings on the left and right which make them non local in $\mathcal{H}$. Hence for instance,

$$S_i^{+} S_j^{+} = q^{\pm 2} S_j^{+} S_i^{+} \qquad \text{for } i > \text{ (resp. } <) \ j. \tag{3.8}$$

This may remind the reader of the Jordan-Wigner transformation used to solve the Ising model. The latter is related to $\mathcal{U}_q\, sl(2)$ for $q = i$, (which turns (3.8) into an anticommutation rule).

3.2. The generic case.

For q not a root of unity [12], the space $\mathcal{H}$ decomposes as

$$\mathcal{H} = \oplus_j w_j \otimes \rho_j \tag{3.9}$$

where w_j is a multiplicity space of dimension

$$\Gamma_j^{(L)} = \begin{pmatrix} L \\ \frac{1}{2}L - j \end{pmatrix} - \begin{pmatrix} L \\ \frac{1}{2}L - j - 1 \end{pmatrix}. \tag{3.10}$$

Let us consider the spectrum of $\mathcal{T}^{(L)}$: eigenvectors of $\mathcal{T}^{(L)}$ fall in representations ρ_j of $\mathcal{U}_q\, sl(2)$, and we denote their eigenvalues by $\lambda_j^{(\alpha)}$, $\alpha = 1, \cdots, \Gamma_j^{(L)}$. The $\mathcal{U}_q\, sl(2)$ symmetry manifest itself through the appearance of degeneracies of these eigenvalues of order $2j+1$. Since $\mathcal{T}^{(L)\dagger}$ is equivalent to $\mathcal{T}^{(L)}$ after spin relabelling, eigenvalues are real or complex conjugate by pairs. It is appropriate to introduce at this stage the "Bratteli diagram" of fig. 4. At level l the diagram displays the representations ρ_j that appear in the decomposition of $\rho_{\frac{1}{2}}^{\otimes l}$ and the embedding of level l into level $l+1$ under the tensor product by $\rho_{\frac{1}{2}}$:

$$\rho_j \otimes \rho_{\frac{1}{2}} = \rho_{j-\frac{1}{2}} \oplus \rho_{j+\frac{1}{2}}$$

The diagram is usually drawn in a descending way, which unfortunately does not agree with the conventions adopted in these lectures. We note that $\Gamma_j^{(L)}$ can be interpreted as the number of (descending) paths of L steps going from 0 to j on the Bratteli diagram.

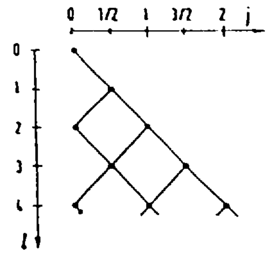


Fig. 4.

3.3. The case of q a root of unity.

The case where q is a root of unity is more involved [13]-[15]. We introduce in the following the (smallest) integer n such that

$$q^n = \pm 1. \tag{3.11}$$

A first specific property is that $(S^{\pm})$ is nilpotent

$$(S^{\pm})^n = 0. \tag{3.12}$$

It is easy to check using (2.17) that $(S^{\pm})^n$ commutes with $(S^{\pm})$ and q^{S^z}. To check that it actually vanishes is slightly more difficult. We note that since $\left(s^{\pm}\right)^2 = 0$, $(S^+)^k$, for instance, is a sum of monomials

$$S^+_{i_1} S^+_{i_2} \cdots S^+_{i_k}, \qquad i_1 \neq i_2 \cdots \neq i_k. \tag{3.13}$$

The set $i_1, \cdots, i_k$ appears in $k!$ different ways in the expansion. The different monomials must then be reordered, giving through (3.8) a phase factor $1.(1+q^2)(1+q^2+q^4)\cdots(1+q^2+\cdots+q^{2(k-1)})$. Hence for $i_1 < i_2 < \ldots < i_k$

$$\sum_{\text{permutations } P} S^+_{P(i_1)} S^+_{P(i_2)} \cdots S^+_{P(i_k)} = q^{\frac{k(k-1)}{2}} (k)_q! \; S^+_{i_1} S^+_{i_2} \cdots S^+_{i_k}. \tag{3.14}$$

Eq. (3.12) is thus a consequence of $(n)_q = 0$ when (3.11) holds. On (3.14), we also see that the ratio $(S^{\pm})^n/(n)_q!$ could be defined by analytic continuation in q. From (3.12) there is no higher weight representation of spin larger than $(n-1)/2$ [14].

Another consequence of (3.11) is that Casimir values enjoy periodicity under an "affine Weyl group" due to their trigonometric dependence on j

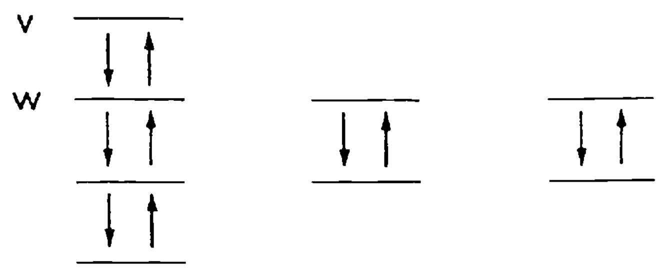
$$C_j = C_{j'} \begin{cases} \text{for } j' = j & \text{mod } n \\ \text{or } j' = -j-1 & \text{mod } n \end{cases}. \tag{3.15}$$

For later use let us note the transformation rules of q-dimensions

$$d_{j'} = \begin{cases} d_j & \text{for } j' = j \mod n \\ -d_j & \text{for } j' = -j-1 \mod n \end{cases} . \tag{3.16}$$

From (3.15), representations ρ_j, $\rho_{j'}$ with different Casimir values for generic q may get "mixed" for q satisfying (3.11) since the Casimir values become identical.

We thus expect the representation theory for q a root of unity ($\neq \pm 1$) to be different from the generic or $q = \pm 1$ cases. To illustrate the general features we now discuss the simple example of $\mathcal{H} = \rho_{\frac{1}{2}}^{\otimes 3}$. For q generic $\mathcal{H}$ splits as $\rho_{\frac{3}{2}} \oplus \rho_{\frac{1}{2}} \oplus \rho_{\frac{1}{2}}$, which it is convenient to draw as



where arrows represent the action of $S^{\pm}$. In particular, let $v = |\frac{3}{2}\frac{3}{2} >= (|\frac{1}{2}\frac{1}{2} >)^{\otimes 3}$. Suppose now $q^3 = \pm 1$. Then according to (3.15)

$$C_{\frac{3}{2}} = C_{\frac{1}{2}}. \tag{3.17}$$

Consider now $w = S^- v$ which is clearly non vanishing. We have

$$Cw = S^- S^+ w + C_{\frac{1}{2}} w \tag{3.18}$$

and also, since C commutes with S^-

$$Cw = C_{\frac{3}{2}} w. \tag{3.19}$$

Comparison of (3.17)-(3.19) implies

$$S^- S^+ w = 0. \tag{3.20}$$

If $S^+ w$ were not zero it would be proportional to v which is not annihilated by S^-. Hence we conclude

$$S^+ w = 0. \tag{3.21}$$

(Although (3.21) is more easily checked using $[S^+, S^-] = 2(S^z)_q$ it is instructive to expose its relation with (3.15).) w is actually a "zero norm state". This is established by

$$< w|w >=< S^- v|S^- v >=< v|S^+ S^- v >=< v|S^+ w >= 0$$

or by direct calculation

$$\begin{aligned} w =& q^{-1}|\tfrac{1}{2} - \tfrac{1}{2} > \otimes|\tfrac{1}{2}\tfrac{1}{2} > \otimes|\tfrac{1}{2}\tfrac{1}{2} > \\ &+|\tfrac{1}{2}\tfrac{1}{2} > \otimes|\tfrac{1}{2} - \tfrac{1}{2} > \otimes|\tfrac{1}{2}\tfrac{1}{2} > +q\,|\tfrac{1}{2}\tfrac{1}{2} > \otimes|\tfrac{1}{2}\tfrac{1}{2} > \otimes|\tfrac{1}{2} - \tfrac{1}{2} > \end{aligned} \tag{3.22}$$

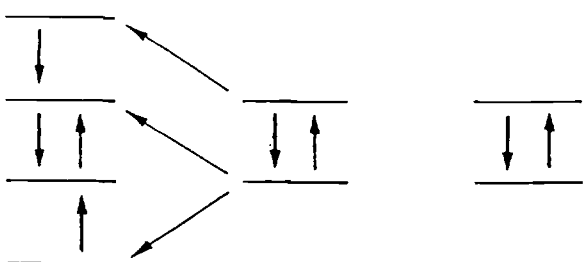
and

$$< w|w >= 1 + q^2 + q^{-2} = (3)_q = 0 \tag{3.23}$$

Moreover due to (3.12) $(S^-)^3 v = 0$. Hence we must erase two arrows from the above picture when looking at the "$\rho_{\frac{3}{2}}$ like" set of states. Notice that $|\frac{1}{2} - \frac{1}{2} >^{\otimes 3}$ can still be obtained from $|\frac{1}{2}\frac{1}{2} >^{\otimes 3}$ using the analytic continuation in q of $(S^-)^3/(3)_q!$ since

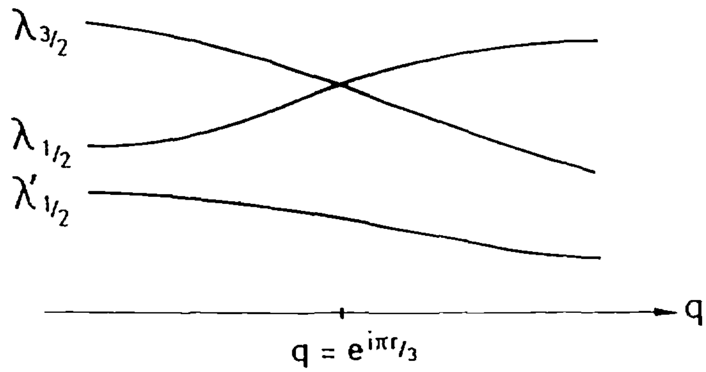
$$(S^-)^3|\tfrac{1}{2}\tfrac{1}{2} >^{\otimes 3}= (3)_q!|\tfrac{1}{2} - \tfrac{1}{2} >^{\otimes 3} \tag{3.24}$$

Now consider the weight space $V^{\frac{1}{2}}$ with $S^z = \frac{1}{2}$. In that space it is easily checked that $\mathrm{Ker}S^+$ has still dimension 2. This for q generic would have been used to build the two highest weights of the two $\rho_{\frac{1}{2}}$. Here because we already have w in $\mathrm{Ker}S^+$, any basis of $V^{\frac{1}{2}}$ must contain a state which is not in $\mathrm{Ker}S^+$, hence has an image by S^+ proportional to v. Reasoning similarly for $V^{-\frac{1}{2}}$ gives finally the diagram



Thus $\mathcal{H}$ splits into two kinds of representations. The "big one" is not irreducible but indecomposable (i.e. not fully reducible) and contains null states. We denote it $(\rho_{\frac{3}{2}}, \rho_{\frac{1}{2}})$ and call it "of type I". The "small one" is still like in the generic case, denoted $\rho_{\frac{1}{2}}$ and called of type II.

The mixing of $\rho_{\frac{3}{2}}$ and $\rho_{\frac{1}{2}}$ has important consequences for the spectrum of $\mathcal{T}^{(L)}$. For q generic we expect $\mathcal{T}^{(L)}$ to have three distinct eigenvalues $\lambda_{\frac{3}{2}}$, $\lambda_{\frac{1}{2}}$ and $\lambda'_{\frac{1}{2}}$, whereas two of them must merge when $q^3 = \pm 1$, leading to new coincidences in a numerical study. A very schematic plot of the eigenvalues against (complex!) q looks as follows:



Hence $\mathcal{U}_q\, sl(2)$ symmetry is responsible of a multiplet structure in the generic q case and of a "supermultiplet" for q a root of unity.

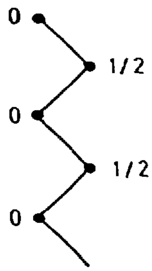
3.4. More on q a root of unity.

The above example is typical of the general case. $\mathcal{H}$ splits into type I and type II representations. Those of type II are still like generic q ones, with spins $0 \le j \le \frac{n-2}{2}$. Those of type I are either non irreducible and indecomposable and made of mixtures $(\rho_j, \rho_{j'})$ (with $j' = -j-1 \bmod n$, $|j'-j| < n$) or irreducible and like $\rho_{\frac{n-1}{2}}$. In any case they are characterized by a zero q-dimension.

One can show that the number of type II representations of spin j is given by

$$\Omega_j^{(L)} = \Gamma_j^{(L)} - \Gamma_{n-1-j}^{(L)} + \Gamma_{j+n}^{(L)} - \Gamma_{n-1-j+n}^{(L)} + \dots \tag{3.25}$$

with $\Gamma_j^{(L)}$ given in (3.10). $\Omega_j^{(L)}$ is also the number of descending paths of L steps going from 0 to j on a truncated Bratteli diagram where all spins larger than $(n-1)/2$ are deleted. For $q^3 = \pm 1$ for instance, one has



and if $L = 3$ we recover that a single type II representation of spin $\frac{1}{2}$ appears in the $\mathcal{H}$ decomposition.

As will be clear later, it is quite interesting to "truncate" $\mathcal{H}$ and keep only eigenstates of $\mathcal{T}^{(L)}$ that belong to type II representations. This is accomplished by inserting q^{2S^*} in the trace since

$$\begin{aligned} \mathrm{tr}_{\mathcal{H}}\left(q^{2S^*}\,\mathcal{T}^{(L)T}\right) &= \sum_j d_j \mathrm{tr}_{w_j}\,\mathcal{T}^{(L)T} \\ &= \sum_{\text{type II}} \mathrm{tr}\left(q^{2S^*}\,\mathcal{T}^{(L)T}\right) \end{aligned} \tag{3.26}$$

The pattern of representations of type II is very reminiscent of the one of representations of the $\widehat{SU(2)}$ Kac-Moody algebra. We recall [16] that at level $k = n - 2$, the only unitary representations that may occur in the latter are characterized by a spin j that satisfies $0 \le j \le (n-2)/2$. This parallel between representations of $\widehat{SU(2)}_k$ and of $\mathcal{U}_q\, sl(2)$, for $q = \exp i\pi/k+2$, is actually a deep and not yet fully elucidated phenomenon. It extends to the truncated tensor product or fusion of representations [17] and also to higher rank algebras [18].

4. FACE MODELS.

4.1. Generalities.

We now introduce another family of integrable models. The degrees of freedom are attached to the sites of the lattice and interact through "interactions-round-a-face" (IRF) around each plaquette. The Boltzmann weights are thus of the form

$$w(a_1, a_2, a_3, a_4) = {}^{a_4}_{a_1}\square^{a_3}_{a_2} \tag{4.1}$$

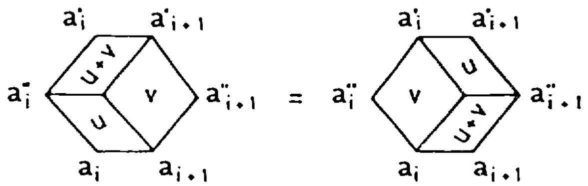
In the simplest model that we shall first consider, the a's are integers (on a finite or infinite range) that may be regarded as describing the height of a discrete fluctuating surface, and heights at neighbouring sites must differ by ± 1. Later, we shall generalize this to a more abstract situation where the a's belong to general discrete set, and are subject to some constraint. Accordingly, these models are called "height models", or SOS (solid-on-solid), or IRF (or face) models.

All the considerations of the first lecture may be repeated for these models. One seeks a one-parameter family of commuting row-to-row transfer matrices, and this may be found if the Boltzmann weights satisfy the YB equation

$$\begin{aligned}&\sum_{b''} w(a_i a_{i+1}; b'' a''_i|u) w(a''_i b''; a'_{i+1} a'_i|u+v) w(b'' a_{i+1}; a''_{i+1} a'_{i+1}|v)\\ &= \sum_{b''} w(a''_i a_i; b'' a'_i|v) w(a_i a_{i+1}; a''_{i+1} b''|u+v) w(b'' a''_{i+1}; a'_{i+1} a'_i|u) \qquad (4.2)\end{aligned}$$

which may be represented diagrammatically as in fig. 5.

Fig. 5.



One also introduces the face transfer matrix acting on configurations attached to diagonals (see fig. 6):

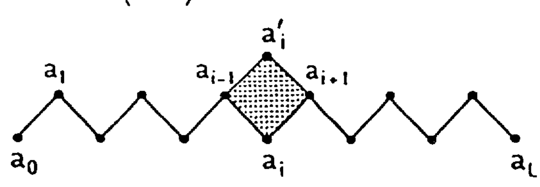
$$< a'_0 a'_1 \cdots a'_L | X_i(u) | a_0 \cdots a_L > = \prod_{j \neq i} \delta_{a_j a'_j} \, w(a_{i-1} a_i; a_{i+1} a'_i|u) \qquad (4.3)$$

and eq. (4.2) amounts to

$$X_i(u) X_{i+1}(u+v) X_i(v) = X_{i+1}(v) X_i(u+v) X_{i+1}(u) \qquad (4.4)$$

which is formally the same as (1.18).

Fig. 6.

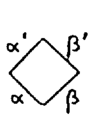


4.2. Vertex-IRF connection.

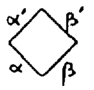
The simplest IRF model in which heights take arbitrary integer values represents a dual picture of the 6-vertex model as described in the previous lectures [1]. Instead of describing the arrows (up or down, right or left, i.e. ± 1) as we go from point to point on the lattice, we suppose we start from $j = 0$ at the boundary leftmost site, focus on the representation j that is reached at each point, and attach to it the height $\lambda = 2j+1$ [2]. This is particularly clear on the Bratteli diagram: configurations of heights along a diagonal from $\lambda_0 = 1$ to λ_L are in one-to-one correspondence with the paths on the Bratteli diagram running from the origin to spin $j_L = \frac{1}{2}(\lambda_L - 1)$ at level L.

This connection actually extends to a construction of the representation of the TL algebra for the IRF model. We recall that the TL generator $e = (q + q^{-1})P_0$ is given by eq. (2.21)

$$e = \begin{pmatrix} 0 & & & \\ & q^{-1} & -1 & \\ & -1 & q & \\ & & & 0 \end{pmatrix} = \tag{4.5}$$



in the basis $|\frac{1}{2}\alpha > \otimes |\frac{1}{2}\beta >$, $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$. If one assigns the height $\lambda_{i-1} = 2j_{i-1} + 1$ to the first site of

j_{i-1} 

then the operator to consider is

$$\sum_{m_{i-1}} |j_{i-1} m_{i-1} >< j_{i-1} m_{i-1}| \otimes P_0$$

Now the construction for the IRF model amounts to a change of basis, i.e. to a computation of (q-analogues of) Clesch-Gordan coefficients.

$$\begin{aligned} &|j_{i-1} m_{i-1} > \otimes |\tfrac{1}{2}\alpha > \otimes |\tfrac{1}{2}\beta > \\ &= \sum_{\substack{j_i m_i \\ j_{i+1} m_{i+1}}} < j_i m_i | j_{i-1} m_{i-1} \tfrac{1}{2}\alpha >< j_{i+1} m_{i+1} | j_i m_i \tfrac{1}{2}\beta > |(j_{i-1} \tfrac{1}{2})^{j_i} \tfrac{1}{2}; j_{i+1} m_{i+1} > \\ &= \sum_{\substack{JM \\ j_{i+1} m_{i+1}}} < JM | \tfrac{1}{2}\alpha \tfrac{1}{2}\beta >< j_{i+1} m_{i+1} | j_{i-1} m_{i-1} JM > |j_{i-1} (\tfrac{1}{2}\tfrac{1}{2})^J; j_{i+1} m_{i+1} > \end{aligned} \tag{4.6}$$

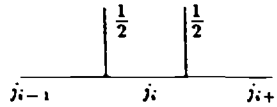
[1] There are other inequivalent ways of defining a height model dual to the six-vertex model [19].

[2] We use the notation λ for these integer heights, and reserve the notation a for the height of the generic model to be studied in lect. 5.

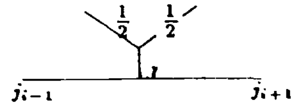
or rather of (q-analogues of) $6j$ coefficients that relate the two bases [20],[21],[17]

$$|(j_{i-1}\tfrac{1}{2})^{j_i}\tfrac{1}{2};j_{i+1}m_{i+1}>=\sum_J \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i \\ \frac{1}{2} & j_{i+1} & J \end{Bmatrix}_q |j_{i-1}(\tfrac{1}{2}\tfrac{1}{2})^J;j_{i+1}m_{i+1}> \tag{4.7}$$

(The normalizations are not the conventional ones if $q=1$.)



$$=\sum_J \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i \\ \frac{1}{2} & j_{i+1} & J \end{Bmatrix}_q$$



Comparing the two decompositions (4.6), one finds

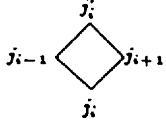
$$\begin{aligned}\sum_{JM} < JM|\tfrac{1}{2}\alpha\tfrac{1}{2}\beta> < j_{i+1}m_{i+1}|j_{i-1}m_{i-1}JM> \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i \\ \frac{1}{2} & j_{i+1} & J \end{Bmatrix}_q \\ =\sum_{m_i} < j_im_i|j_{i-1}m_{i-1}\tfrac{1}{2}\alpha>< j_{i+1}m_{i+1}|j_im_i\tfrac{1}{2}\beta>\end{aligned} \tag{4.8}$$

for any m_{i-1}, m_{i+1}, α and β, a formula which is again a q-analogue of a familiar expression in ordinary $SU(2)$ and which enables one to compute the $6j$- symbols.

The couplings of spins $j_{i-1}\otimes\frac{1}{2}\to j_i$, $j_i\otimes\frac{1}{2}\to j_{i+1}$ is what we want to describe the successive heights at sites $i-1$, i and $i+1$, whereas the coupling $\frac{1}{2}\otimes\frac{1}{2}\to J$ is more suited to use the fact that the projector P_0 forces $J=0$. A straightforward computation then leads to

$$\begin{aligned}\sum_{i-1}|j_{i-1}m_{i-1}> < j_{i-1}m_{i-1}|\otimes P_0 = \sum_{\substack{j_i,j_i'\\ j_{i+1}}} \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i \\ \frac{1}{2} & j_{i+1} & 0 \end{Bmatrix}_q \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i' \\ \frac{1}{2} & j_{i+1} & 0 \end{Bmatrix}_q \\ \times\sum_{m_{i+1}}|(j_{i-1}\tfrac{1}{2})_{j_i}\tfrac{1}{2};j_{i+1}m_{i+1}>< (j_{i-1}\tfrac{1}{2})_{j_i'}\tfrac{1}{2};j_{i+1}m_{i+1}|\end{aligned} \tag{4.9}$$

The relevant $6j$-symbols do not vanish only if $j_{i-1}=j_{i+1}$ and thus in the new basis, P_0 reads



$$=\delta_{j_{i-1}j_{i+1}} \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i \\ \frac{1}{2} & j_{i-1} & 0 \end{Bmatrix}_q \begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i' \\ \frac{1}{2} & j_{i-1} & 0 \end{Bmatrix}_q \tag{4.10}$$

Computing the Clesch-Gordan coefficients in (4.8), one finds

$$\begin{Bmatrix} j_{i-1} & \frac{1}{2} & j_i \\ \frac{1}{2} & j_{i-1} & 0 \end{Bmatrix}_q = \left(\frac{(2j_i+1)_q}{(2)_q(2j_{i-1}+1)_q}\right)^{\frac{1}{2}} \tag{4.11}$$

and therefore returning to the height variables $\lambda = 2j+1$, $e = (2)_q P_0$ reads

$$< \lambda'_0 \lambda'_1 \cdots \lambda'_L | e_i | \lambda_0 \lambda_1 \cdots \lambda_L > \prod_{j \neq i} \delta_{\lambda_j \lambda'_j} \frac{\left((\lambda_i)_q (\lambda'_i)_q\right)^{\frac{1}{2}}}{(\lambda_{i-1})_q} \delta_{\lambda_{i_1} \lambda_{i+1}} \tag{4.12}$$

By the same formula as before, this yields an IRF solution to the YB equation (4.4)

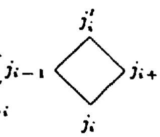
$$X_i(u) = \sin(\gamma + u)\mathbb{1} - \sin u \, e_i \tag{4.13}$$

Remark.

Note that one may also say that the original and final matrix elements of e_i are intertwined as:

$$\sum_{\alpha,\beta,m_i}$$ 

$$< j_i m_i | j_{i-1} m_{i-1} \tfrac{1}{2}\alpha >< j_{i+1} m_{i+1} | j_i m_i \tfrac{1}{2}\beta >$$

$$= \sum_{j'_i, m_i}$$ 

$$< j'_i m_i | j_{i-1} m_{i-1} \tfrac{1}{2}\alpha' >< j_{i+1} m_{i+1} | j'_i m_i \tfrac{1}{2}\beta' > \tag{4.14}$$

(a special case of the cells introduced by Ocneanu [22]). These cells satisfy the orthogonality and completeness relations of Clebsch-Gordan coefficients [20]

$$\sum_{m_{i-1},\alpha} < j_i m_i | j_{i-1} m_{i-1} \tfrac{1}{2}\alpha >< j_{i-1} m_{i-1} \tfrac{1}{2}\alpha | j'_i m'_i > = \delta_{j_i j'_i} \delta_{m_i m'_i} \tag{4.15}$$

$$\sum_{m_i,\alpha} q^{-2\alpha} < j_i m_i | j_{i-1} m_{i-1} \tfrac{1}{2}\alpha >< j'_{i-1} m'_{i-1} \tfrac{1}{2}\alpha | j_i m_i > = \frac{(2j_i+1)_q}{(2j_{i-1}+1)_q} \delta_{j_{i-1} j'_{i-1}} \delta_{m_{i-1} m'_{i-1}}$$

4.3. Restricted IRF models

As discussed in the previous lecture, whenever q is a $2n$−th root of 1, $q^n = \pm 1$, representations of $\mathcal{U}_q\, sl(2)$ split into two types and it is possible to restrict oneself to type II representations, by projecting with the modified trace $\mathrm{Tr}(..) = \mathrm{tr}(q^{2S^z} ...)$. On the Bratteli diagram, this means that spins $j \geq \frac{1}{2}(n-1)$ i.e. heights $\lambda = 2j+1 \geq n$ are discarded. One is thus led to the RSOS (restricted SOS) model in which the height varies in a *finite range.* (This was originally discovered and formulated in a different language in [23]):

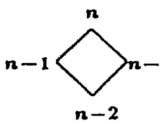
$$1 \leq \lambda \leq n-1 \tag{4.16}$$

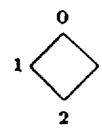
These heights may be regarded as living on the graph

$$A_{n-1} = \bullet_1 \text{———} \bullet_2 \text{———} \bullet_3 \cdots\cdots \text{———} \bullet_{n-1} \tag{4.17}$$

or alternatively, this pattern appears in two successive (generic) rows of the Bratteli diagram (for $L \geq n-1$).

It turns out that the matrix elements of e computed in the previous subsection are consistent with the truncation operation: it is easy to see that the generators e_i of the TL algebra have vanishing matrix elements



$\sim (n)_q^{\frac{1}{2}} = 0$ and 

$= 0$

and thus configurations of heights in the admissible range (4.16) do not couple to non admissible ones. In other words, the representation of the TL algebra defined on the full Bratteli diagram becomes reducible.

4.4. Modified trace.

We have introduced above a modified trace, defined in the $U_q\, sl(2)$ language by:

$$\begin{aligned} \mathrm{Tr}(x) &= \frac{\mathrm{tr}(q^{2S^z} x)}{\mathrm{tr}(q^{2S^z})} \\ &= \frac{\mathrm{tr}\left((q^{2s_1^z} \otimes q^{2s_2^z} \otimes \cdots \otimes q^{2s_L^z})x\right)}{\mathrm{tr}\left((q^{2s_1^z} \otimes q^{2s_2^z} \otimes \cdots \otimes q^{2s_L^z})\right)} \end{aligned} \tag{4.18}$$

This modified trace is normalized by $\mathrm{Tr}\mathbf{1} = 1$ and enjoys two important properties (Markov properties):

1) For two operators x and y which commute with S^z (in particular which belong to the TL algebra, commutant of $U_q\, sl(2)$)

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx) \tag{4.19}$$

2) For x belonging to the algebra generated by $\mathbf{1}, e_1, e_2, \cdots, e_{k-1}$,

$$\mathrm{Tr}(e_k x) = \tau \mathrm{Tr}(x) \tag{4.20}$$

where τ does not depend on the explicit representation of the TL algebra, but only on q:

$$\tau = \frac{1}{q+q^{-1}} = \frac{1}{(2)_q} \tag{4.21}$$

This is a non trivial property which has to be checked in each new representation. In the representation (4.5), it can be shown by direct computation. It may also be instructive to see it as a consequence of another identity, on the sum at link j:

$$\begin{aligned}\mathrm{tr}(q^{2s_j^z}e_{j-1}) &= \sum_{\beta=\pm\frac{1}{2}} q^{2\beta}e_{\alpha\beta,\alpha\beta} = q.q^{-1}\delta_{\alpha,-\frac{1}{2}} + q^{-1}.q\delta_{\alpha,\frac{1}{2}} \\ &= 1 = \tau\,\mathrm{tr}q^{2s_j^z}\end{aligned}$$

independent of α. Hence $\mathrm{Tr}e = \tau$ and it is easy to see that this extends recursively to an arbitrary chain $i_1 < i_2 < \cdots < i_k$:

$$\mathrm{Tr}(e_{i_1}e_{i_2}\cdots e_{i_k}) = \tau^k \tag{4.22}$$

which is equivalent to (4.20). By use of the defining relations of the Hecke algebra (2.47), any polynomial in the e_i's may be recast as a linear combination of expressions of the form (4.22). Thus the modified trace of any polynomial is a universal polynomial of τ.

These Markov properties are important in two different contexts:

– in knot theory, they are the key to the construction of knot invariants using the trace Tr: see lecture 6.

– in statistical mechanics, this modification is not innocent: it amounts to introducing an extra operator in the trace of x (the "sewing" term of [24]) and modifies the physics. For example the physical partition function of the original model, computed with the ordinary trace $Z = \mathrm{tr}\mathcal{T}^{(L)T}$ does not equal the modified trace $Z_{\mathrm{mod}} = \mathrm{tr}(q^{2S^z}\mathcal{T}^{(L)T})$. For large "time" T (and finite L), however, the modification becomes irrelevant: if $\lambda^{(L)}$ denotes the largest eigenvalue of $\mathcal{T}^{(L)}$, $Z \approx (2j+1)(\lambda^{(L)})^T$ whereas $Z_{\mathrm{mod}} \approx d_j(\lambda^{(L)})^T$ where j is the spin of the eigenvectors corresponding to $\lambda^{(L)}$ and provided the latter q–dimension does not vanish, the two partition functions have the same leading behaviour as $T \to \infty$. Now consider two models associated with two different representations of the TL algebra for the same value of q. Their transfer matrix $\mathcal{T}^{(L)T}$ is in the algebra generated by the e's, and thus the modified partition function is a universal function of q, independent of the representation. In the large T limit (with L fixed), the modified partition function approaches the physical ones which are thus asymptotically equal. The two models must be simultaneously critical or non-critical and in the former case, they must exhibit the same central charge (but not the same operator content: recall that the central charge describes the leading finite size effect [25], whereas the other operators control the subleading terms, sensitive to the modification).

In the height models, the representation of the TL algebra still satisfies the Markov properties by construction. In that case, the modified trace takes a simpler form. For x commuting with $\mathcal{U}_q\, sl(2)$,

$$\mathrm{Tr}(x) = \frac{\sum_j (2j+1)_q \mathrm{tr}_{w_j}(x)}{\sum_j (2j+1)_q \mathrm{tr}_{w_j}\mathbf{1}} \tag{4.23}$$

using the notations of eq. (3.9). On the Bratteli diagram, contributions to $\mathrm{tr}_{w_j}(x)$ come from configurations of heights ranging from height $\lambda_0 = 1$ at site 1 to height $\lambda_L = 2j+1$. Hence, one may rephrase (4.23) as

$$\mathrm{Tr}(x) = \frac{\sum_{\{\lambda\},\lambda_0=1} (\lambda_L)_q < \lambda_0\lambda_1 \cdots \lambda_L | x | \lambda_0\lambda_1\cdots\lambda_L >}{\sum_{\{\lambda\},\lambda_0=1} (\lambda_L)_q < \lambda_0\lambda_1 \cdots \lambda_L | \lambda_0\lambda_1\cdots\lambda_L >} \tag{4.24}$$

Moreover, the sum in the denominator of (4.23) equals $(2)_q^L (\lambda_0)_q$. Thus

$$\mathrm{Tr}(x) = \tau^L \sum_{\substack{\{\lambda\} \\ \lambda_0=1}} (\lambda_L)_q < \lambda_0\lambda_1 \cdots \lambda_L | x | \lambda_0\lambda_1\cdots\lambda_L > \tag{4.25}$$

or alternatively, denoting $\mathcal{N} = \sum_{\lambda=1}^{n-1} (\lambda)_q^2 = n/\big(2\sin^2(\frac{\pi}{n})\big)$, we can relax the condition that $\lambda_0 = 1$ and write:

$$\mathrm{Tr}(x) = \tau^L \mathcal{N}^{-1} \sum_{\{\lambda\}} (\lambda_0)_q (\lambda_L)_q < \lambda_0\lambda_1 \cdots \lambda_L | x | \lambda_0\lambda_1\cdots\lambda_L > . \tag{4.26}$$

In particular, the modified partition function may be regarded as the sum of contributions $Z_{\lambda_0\lambda_L}$, partition functions with fixed boundary conditions at the ends of the chain, weighted by $(\lambda_0)_q(\lambda_L)_q$.

5. FACE MODELS ATTACHED TO GRAPHS.

5.1. Reinterpretation of the RSOS model.

We have seen in sect. 4.3 that the heights of the RSOS model may be regarded as living on the graph

$$A_{n-1} = \bullet_1 \!-\!-\!-\! \bullet_2 \!-\!-\!-\! \bullet_3 \cdots\cdots \!-\!-\!-\! \bullet_{n-1}$$

which is the A_{n-1} Dynkin diagram. It turns out that all the properties of the RSOS model previously spelled out may be rephrased in this new language. The spectral properties of the adjacency matrix of the graph

$$A_{\lambda\mu} = \text{ number of edges between the modes } \lambda \text{ and } \mu$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{5.1}$$

play an important role. The eigenvalues of A are

$$\gamma^{(\rho)} = 2\cos\frac{\rho\pi}{n} = \frac{\sin\frac{2\rho\pi}{n}}{\sin\frac{\rho\pi}{n}} \tag{5.2}$$

where ρ runs over the same set as the heights:

$$\rho = 1, 2, \dots, m = n - 1. \tag{5.3}$$

The corresponding orthonormalized eigenvectors are

$$\phi_\lambda^{(\rho)} = \sqrt{\frac{2}{n}}\sin\frac{\lambda\rho\pi}{n} = S_{\lambda\rho} \tag{5.4}$$

i.e. equals the matrix element of the matrix S of modular transformations of characters of the $\widehat{SU(2)}_k$ Kac-Moody algebra of level $k = n - 2$. As shown by Verlinde [26], the fusion algebra of the representations of $\widehat{SU(2)}_k$ labelled by $\lambda = 2j + 1$, $0 \le j \le k/2$ i.e. $1 \le \lambda \le n - 1$ is described by the matrix

$$N_{\lambda\mu\nu} = \sum_\rho \frac{S_{\lambda\rho}}{S_{1\rho}} S_{\mu\rho} S_{\nu\rho} \tag{5.5}$$

and the matrices $N_\lambda = (N_{\lambda\mu\nu})$ satisfy themselves the fusion algebra:

$$N^\lambda N^\mu = \sum_\nu N_{\lambda\mu\nu} N^\nu. \tag{5.6}$$

In particular the fusion by the spin $\frac{1}{2}$ representation ($\lambda = 2$) is

$$\begin{aligned} N_{2\mu\nu} &= \sum_\rho \frac{S_{2\rho}}{S_{1\rho}} S_{\mu\rho} S_{\nu\rho} \\ &= \sum_\rho{}' \gamma^{(\rho)} \phi_\mu^{(\rho)} \phi_\nu^{(\rho)} = A_{\mu\nu} \end{aligned} \tag{5.7}$$

Indeed the adjacency matrix A which describes the hopping on the graph A_{n-1} also yields the fusion by the spin $\frac{1}{2}$ representation in the Kac-Moody algebra, or equivalently, as alluded to at the end of lect. 3, the truncated tensor product of (type II) representations in the $U_q\, sl(2)$ algebra by the spin $\frac{1}{2}$ representation. This in of course in agreement with the previous considerations on the Bratteli diagram: the A_{n-1} diagram is what appears on two successive rows of the truncated Bratteli diagram.

The matrix elements of the TL generators are

$$< \lambda_0 \cdots \lambda_L | e_i | \lambda'_0 \cdots \lambda'_L >= \prod_{j \neq i} \delta_{\lambda_j \lambda'_j} \frac{(\phi^{(1)}_{\lambda_i} \phi^{(1)}_{\lambda'_i})^{\frac{1}{2}}}{\phi^{(1)}_{\lambda_{i-1}}} \delta_{\lambda_{i-1} \lambda_{i+1}} \tag{5.8}$$

in terms of the components of the eigenvector $\phi^{(1)}$ of largest eigenvalue. Finally the modified trace of eq. (4.25)-(4.26) may also be rewritten as

$$\begin{aligned} \mathrm{Tr} x =& (\tau)^L \sum_{\substack{\{\lambda\} \\ \lambda_0 = 1}} \frac{\phi^{(1)}_{\lambda_L}}{\phi^{(1)}_{\lambda_0}} < \lambda_0 \cdots \lambda_L | x | \lambda_0 \cdots \lambda_L > \\ =& (\tau)^L \sum_{\{\lambda\}} \phi^{(1)}_{\lambda_0} \phi^{(1)}_{\lambda_L} < \lambda_0 \cdots \lambda_L | x | \lambda_0 \cdots \lambda_L > . \end{aligned}$$

5.2. Representations of the TL algebra on the paths of a graph.

We have just reinterpreted the representation of the TL algebra pertaining to the RSOS model as attached to the set $\mathcal{P}_{1\mu}$ of open paths running on the graph A_m from $\lambda_0 = 1$ to some given $\lambda_L = \mu$.

It is thus very natural to ask if there exist other graphs, such that their set of paths supports a representation of the TL algebra [27]-[28]. We denote the nodes of the graph by a, b, etc... The graph (assumed to be symmetric and without multiple edges) is unambiguously described by its adjacency matrix G ($G_{ab} = 0$ or 1). As the entries of the matrix are non negative, the Perron-Frobenius theorem asserts that the eigenvector $\psi^{(1)}$ of largest eigenvalue is unique and has positive components. One may then show that

$$< a'_0 \cdots a'_L | e_i | a_0 \cdots a_L >= \prod_{j \neq i} \delta_{a_j a'_j} \frac{(\psi^{(1)}_{a_i} \psi^{(1)}_{a'_i})^{\frac{1}{2}}}{\psi^{(1)}_{a_{i-1}}} \delta_{a_{i-1} a_{i+1}} \tag{5.10}$$

satisfies the TL algebra with a relation between q and the largest eigenvalue (pertaining to $\psi^{(1)}$) given by

$$\gamma^{(1)} = (2)_q = q + q^{-1}. \tag{5.11}$$

Generalizing what was seen before in the original RSOS case, we define the modified trace by

$$\mathrm{Tr}x = (\tau)^{L-1} \sum_{\{a\}} \psi^{(1)}_{a_0} \psi^{(1)}_{a_L} < a_0 \cdots a_L | e_i | a_0 \cdots a_L > \tag{5.12}$$

for x in the algebra generated by $1, e_1, \ldots, e_{L-1}$. As we have the identities

$$\sum_{a_{i+1}} < a_0 \cdots a_{i+1} | e_i | a_0 \cdots a_{i+1} > \psi^{(1)}_{a_{i+1}} = \sum_{a_{i+1}} \delta_{a_{i-1} a_{i+1}} \frac{\psi^{(1)}_{a_i}}{\psi^{(1)}_{a_{i-1}}} \psi^{(1)}_{a_{i-1}} = \psi^{(1)}_{a_i} \tag{5.13a}$$

$$\sum_{a_{i+1}} < a_0 \cdots a_{i+1} | a_0 \cdots a_{i+1} > \psi^{(1)}_{a_{i+1}} = (2)_q \psi^{(1)}_{a_i} \tag{5.13b}$$

it is easy to prove recursively that Tr satisfies the second Markov property. The first Markov property follows from the fact that the generators $e_1, \cdots, e_{L-1}$ do not affect the heights a_0 and a_L and therefore for x and y in the TL algebra:

$$< a_0 \cdots a_L | xy | a_0 \cdots a_L > = < a_0 \cdots a_L | yx | a_0 \cdots a_L > .$$

We can thus construct a representation of the TL algebra, with a Markov trace, associated with each symmetric (undirected) graph. Among all these representations, only those corresponding to q of modulus one (and $q \neq 1$) concern us here. This means that we look for graphs whose largest eigenvalue $\gamma^{(1)}$ satisfies

$$|\gamma^{(1)}| < 2. \tag{5.14}$$

This is a well known problem [29]. The only graphs with this property are :

i) either the simply laced Dynkin diagrams A, D, E

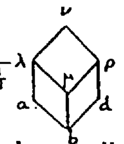
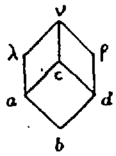
ii) or the quotients $A_{2\ell}/Z_2$

(see Table). The Z_2 quotients of $A_{2\ell}$, however, are readily seen to produce the same statistical mechanical models as the $A_{2\ell}$ and we discard them.

As a final remark, it is noteworthy that the matrix elements (5.8) and (5.10) (for the same q) may be intertwined by cells (compare with (4.14)) [30],[31]

$$\begin{aligned} &\sum_{\mu} \frac{1}{(\phi^{(1)}_{\mu} \psi^{(1)}_{b})^{\frac{1}{2}}} C\begin{pmatrix} \lambda & \mu \\ a & b \end{pmatrix} C\begin{pmatrix} \mu & \rho \\ b & d \end{pmatrix} {}_{\lambda}\overset{\nu}{\underset{\mu}{\Diamond}}{}_{\rho} \\ &= \sum_{c} \frac{1}{(\phi^{(1)}_{\nu} \psi^{(1)}_{c})^{\frac{1}{2}}} {}_{a}\overset{c}{\underset{b}{\Diamond}}{}_{d}\, C\begin{pmatrix} \lambda & \mu \\ a & c \end{pmatrix} Y\begin{pmatrix} \nu & \rho \\ c & d \end{pmatrix}. \end{aligned} \tag{5.15}$$

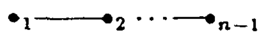
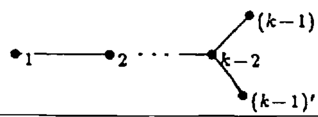
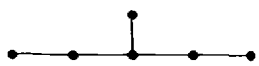
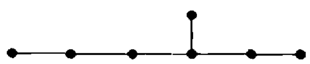
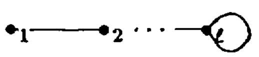
This is represented pictorially as follows

$$\sum_{\mu} \frac{1}{(\phi_\mu \psi_b)^{\frac{1}{2}}}$$  $$= \sum_{c} \frac{1}{(\phi_\nu \psi_c)^{\frac{1}{2}}}$$  (5.16)

The orthogonality relations that these cells satisfy and their explicit expressions may be found in [31].

Table

List of graphs satisfying (5.14) and of their exponents

	Graph	n	Exponents
A_{n-1}		n	$1, 2, \cdots, n-1$
D_k		$2(k-1)$	$1, 3, \cdots, 2k-3; k-1$
E_6		12	$1, 4, 5, 7, 8, 11$
E_7		18	$1, 5, 7, 9, 11, 13, 17$
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$
$A_{2\ell}/Z_2$		n	$1, 3, \cdots, 2\ell-1$

5.3. Spectral properties of the Dynkin diagrams and intertwiners.

Each Dynkin diagram G of the ADE type has a spectrum of eigenvalues of the form (5.2) where n is the Coxeter number of the Dynkin diagram and the r (= number of nodes in the diagram = rank of the corresponding algebra) values of ρ are the "exponents" of G

$$\rho \in \{\rho_1 = 1, \rho_2, \cdots, \rho_r = n-1\}$$

(see the Table). We denote $\psi_a^{(\rho)}$ a set of orthonormalized eigenvectors. As above, $\phi_\lambda^{(\rho)} = S_{\lambda\rho}$ refers to the eigenvector of the A_{n-1} diagram with the same Coxeter number n as G, for the same eigenvalue (5.2) as $\psi^{(\rho)}$.

One may wonder if there is a property analogous to (5.13) satisfied by the other eigenvectors. In fact, one has the identities [3]

$$\sum_{a_i,a_{i+1}} < a_0 \cdots a_{i+1} | e_{i-1} e_i | a_0 \cdots a_{i+1} > \psi^{(\rho)}_{a_{i+1}} = \psi^{(\rho)}_{a_{i-1}} \tag{5.13a}$$

$$\sum_{a_i,a_{i+1}} < a_0 \cdots a_{i+1} | e_i | a_0 \cdots a_{i+1} > \psi^{(\rho)}_{a_{i+1}} = \gamma^{(1)} \psi^{(\rho)}_{a_{i-1}} \tag{5.13b}$$

$$\sum_{a_{i+1}} < a_0 \cdots a_{i+1} | a_0 \cdots a_{i+1} > \psi^{(\rho)}_{a_{i+1}} = \gamma^{(\rho)} \psi^{(\rho)}_{a_i} \tag{5.13c}$$

Together with the summation over the first height, $\sum_{a_0} \psi^{(\rho)}_{a_0} \psi^{(\sigma)}_{a_0} = \delta_{\rho\sigma}$, they lead recursively to the following generalization of (4.22)

$$\sum_{\{a\}} \psi^{(\sigma)}_{a_0} \psi^{(\rho)}_{a_L} < a_0 \cdots a_L | e_{i_1} \cdots e_{i_k} | a_0 \cdots a_L > = \delta_{\rho,\sigma} f^{(\rho)}_{i_1 \ldots i_k}(q) \tag{5.18}$$

Here, $f^{(\rho)}_{i_1 \ldots i_k}(q)$ is a universal (G-independent) function of q. This is interesting because it implies the following relation between partition functions with fixed boundary conditions weighted by the various eigenvectors:

$$\begin{aligned}\sum_{a,b} \psi^{(\rho)}_a \psi^{(\sigma)}_b Z^{(G)}_{ab} &= \sum_{\lambda,\mu} \phi^{(\rho)}_\lambda \phi^{(\sigma)}_\mu Z^{(A)}_{\lambda\mu} \\ &= \delta_{\rho,\sigma} \sum_\lambda \frac{\phi^{(\rho)}_\lambda}{\phi^{(\rho)}_1} Z^{(A)}_{1\lambda}.\end{aligned} \tag{5.19}$$

Using the orthonormality of the eigenvectors, this may be inverted as

$$Z^{(G)}_{ab} = \sum_\lambda V^\lambda_{ab} Z^{(A)}_{1\lambda} \tag{5.20}$$

[3] These arguments have been developped in collaboration with N. Sochen [36]

where we have introduced the set of numbers

$$V_{ab}^{\lambda} = \sum_{\substack{\rho \\ \text{exponents of } G}} \frac{\phi_{\lambda}^{(\rho)}}{\phi_{1}^{(\rho)}} \psi_a^{(\rho)} \psi_b^{(\rho)} \tag{5.21}$$

Formula (5.21) is an extension of (5.5) to which it reduces for $G = A$.

The numbers V_{ab}^{λ} have the following properties:

i) $\sum_{\mu} A_{\lambda\mu} V_{ab}^{\mu} = \sum_c V_{ac}^{\lambda} G_{cb}$, i.e. for fixed a, the rectangular matrix V_{ab}^{λ} intertwines between the matrices A and G: $AV = VG$.

ii) $V_{ab}^{1} = \delta_{ab}$ $\qquad V_{ab}^{2} = G_{ab}$

iii) V_{ab}^{λ} are non negative integers.

iv) The square matrices $V^{\lambda} = (V_{ab}^{\lambda})$ satisfy

$$V^{\lambda} V^{\mu} = \sum_{\nu} N_{\lambda\mu\nu} V^{\nu} \tag{5.22}$$

Only property iii) is non trivial to prove: it may be checked by inspection of the various cases D or E (A is already proved by (5.5)). Property iv), on the other hand, means that these matrices form a representation of the $\widehat{SU(2)}_{n-2}$ or of the $\mathcal{U}_q\, sl(2)$, $q = e^{\frac{i\pi}{n}}$, fusion algebra for which the fusion coefficients $N_{\lambda\mu\nu}$ form the regular representation (see (5.6)).

Conversely one may wonder what are the most general representations of the $\widehat{SU(2)}$ fusion algebra on matrices with non negative entries. In this case of $SU(2)$, the problem may be shown to be equivalent to condition (5.14) and therefore leads again to ADE.

Why are the coefficients V_{ab}^{λ} integers and what is the algebraic interpretation of these numbers? We have found representations of the TL algebra on the space of paths $\mathcal{P}_{ab}^{(L)G}$ of length L running from a to b on the graph G. Among these representations, call them $\mathcal{R}_{ab}^{(L)G}$, the $\mathcal{R}_{1\lambda}^{(L)A}$ are the only irreducible ones [13], whereas *traces* on the others may be decomposed according to

$$\mathrm{tr}_{\mathcal{R}_{ab}^{(L)G}}(.) = \sum_{\lambda} V_{ab}^{\lambda}\, \mathrm{tr}_{\mathcal{R}_{1\lambda}^{(L)A}}(.) \tag{5.23}$$

with integer multiplicities. This implies (5.20).

The whole discussion of representations of the TL algebra may be extended to closed paths. One has to impose the condition that also $e_L = e_0$ satisfies eq. (2.50b). Then one finds that irreducible representations $\mathcal{R}_{ss'}^{\mathrm{per}}$ are labelled by a pair of integers s, s', $1 \le s, s' \le n-1$ [13].

5.4. *Continuum limit.*

These ADE models are critical and, according to an argument presented before, exhibit a central charge c which depends only on q. For $q = e^{\frac{i\pi}{n}}$ one may show that

$$c = 1 - \frac{6}{n(n-1)} \tag{5.24}$$

with a spectrum of conformal weights to be found in the Kac table (see Verlinde's lectures at this school)

$$h_{rs} = \frac{\left(rn - s(n-1)\right)^2 - 1}{4n(n-1)} \quad \text{with} \quad \begin{cases} 1 \le r \le n-2 \\ 1 \le s \le n-1 \end{cases} \tag{5.25}$$

When dealing with fixed boundary conditions, the partition function of a conformal field theory is a *linear* form in the characters [32] and one shows [33],[13] that

$$Z_{1\lambda}^{(A)} = \chi_{1,\lambda} \tag{5.26}$$

while eq. (5.20) transposes into a similar equation in the continuum limit

$$Z_{ab}^{(G)} = \sum_{\lambda} V_{ab}^{\lambda} \chi_{1,\lambda}. \tag{5.27}$$

The reason why the coefficients of the linear form have to satisfy the fusion algebra has been explained in the A case and in the continuum limit in [34].

Among the conformal weights (5.25), not all appear necessarily in the primary fields of the theory. The actual spectrum of the theory is described by the modular invariant partition function of (A,A), (A,D) or (A,E) type [35]

$$Z_{(A,G)} = \tfrac{1}{2} \sum_{r=1}^{n-2} \Big(\sum_{s \text{ exponent of } G} |\chi_{rs}|^2 + \sum_{s \neq s'}{}' \chi_{rs} \chi_{rs'}^* \Big) \tag{5.28}$$

where we have written explicitly only the diagonal terms which display the exponents of the algebra. In fact one may prove [13] that the contribution of the representation $\mathcal{R}_{ss'}^{\rm per}$ of the periodic TL algebra gives in the continuum limit

$$\sum_{r=1}^{n-2} \chi_{rs} \chi_{rs'}^* \tag{5.29}$$

and the coefficients of the quadratic form in (5.28) describe the decomposition of the reducible representations of the periodic TL algebra attached to G in terms of the $\mathcal{R}_{ss'}^{per}$.

Notice that the (D, A_n) and (E, A_n) unitary models have not yet received an integrable realization (but of course no theorem guarantees that there should exist such a realization).

All the discussion of the previous lectures may and must be generalized to higher rank algebras, in particular to $SU(N)$. One follows the same steps: construction of a vertex model, in which the degrees of freedom living on links are weights of the fundamental representation $\square$ of $SU(N)$; study of the quantum algebra $\mathcal{U}_q sl(N)$, of its commutant in $\square^{\otimes 2}$, described by a factor of the Hecke algebra (2.47) and of its representations, in particular for q a root of unity; reformulation of the vertex model as a face model in which the degrees of freedom are now fundamental weights of $SU(N)$; truncation of the weight lattice whenever q is a root of unity, and possibility of other restrictions, attached to graphs; continuum limit in which these restricted models are described by coset theories $SU(N)_{k-1} \otimes SU(N)_1 / SU(N)_k$. This is a vast program in which no complete classification has been achieved yet, in contrast with $SU(2)$. For references, we refer the reader to [31],[37].

5.5. More on the continuum limit.

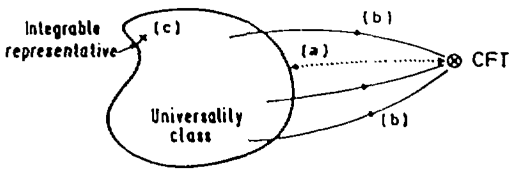
By lack of space, we do not discuss further the connections between integrable lattice models and conformal field theories (CFT). We shall content ourselves with giving a list of topics and references for the interested reader.

a) At a formal level first, there are many algebraic structures that appear in a similar way in both fields. In this respect for instance, the six-vertex model behaves as the Feigin-Fuchs free field [13], $\mathcal{U}_q sl(2)$ playing the role of the algebra of screening operators, TL the role of the Virasoro algebra [13],[38], while the truncation of the Bratteli diagram corresponds to discarding null states in CFT. Also the fusion algebra of CFT may be described by quantum algebras [17], the braiding of conformal fields provides braid group representations [39] and $6-j$ coefficients are related to the braiding and fusing matrices [40]. Modular invariance itself can be used for finite integrable systems [13]. For other related topics, see [41]. The study of these common algebraic stuctures is far from complete and remains somehow mysterious.

b) At a more physical level, it is of course of importance to take the continuum limit of integrable critical lattice models and to identify the associated CFT. Rather well understood techniques have been devised for this purpose: Bethe Ansatz and scaling analysis [42], mappings on the Coulomb gas [43]...

c) Most common systems turn out to have an integrable model in the same universality class. This allows to get predictions of experimentally observable properties, for example in polymer physics [44].

The relation between critical models and CFT may be summarized in the diagram



Full arrows (*b*) represent the continuum limit, the dotted arrow (*a*) refers to the existence of algebraic structures similar to those of the CFT but present for finite systems *before* the continum limit. Whether there is always an integrable realization of a CFT, and how its Boltzmann weights can be explicitly built from the properties of this CFT are still open questions.

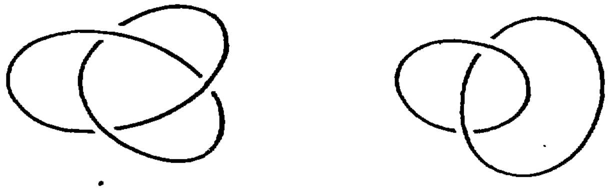
6. YANG-BAXTER EQUATION, BRAID GROUP AND LINK POLYNOMIALS.

This last lecture is a short introduction to link polynomials in relation with YB equation. Knot theory [45] is a theme common to various subjects addressed to in this school: integrable systems [46], conformal theories (via the "braiding" of operators[47][39]) and topological field theories [48].

6.1. Definitions.

By abstraction of everyday's experience, a **link** is an embedding of a collection of circles in three-dimensional euclidean space. Two links are **ambient isotopic** if they can be continuously deformed into each other. We are primarily interested into equivalence classes of ambient isotopy.

A link is conveniently represented in the plane by a **link diagram**



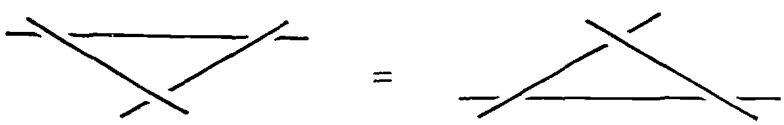
Trefoil

A one-component link that is not isotopic to the circle is called a **knot**. The circle is the **unknot**.

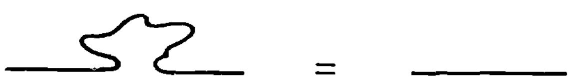
Reidemeiter moves. 1.

2.

3.

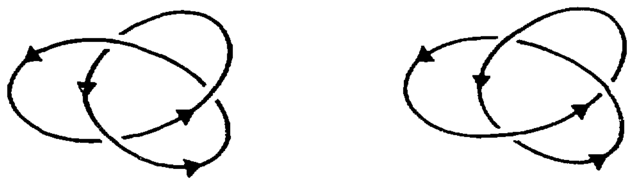


"0".



Reidemeister moves represent deformations of link diagrams that correspond to ambient isotopic links. More precisely, two link diagrams represent ambient isotopic links if and only if they can be transformed into one another by a finite sequence of Reidemeiter moves.

Oriented link and its mirror image



T $\quad$ T^*

Question: are T and T^* ambient isotopic?

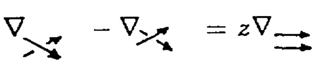
6.2. Alexander-Conway polynomial.

The Alexander-Conway polynomial is defined by the following axioms:

1. To each oriented link K, there is an associated polynomial $\nabla_K(z) \in Z(z)$. Equivalent links yield identical polynomials.

2. $\nabla_{\bigcirc} = 1.$

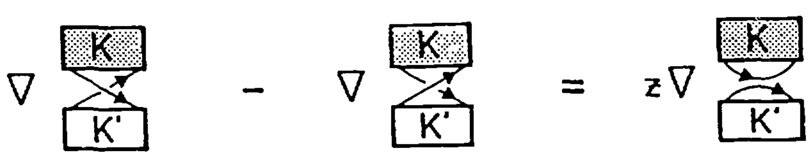
3. Skein relations



(6.1)

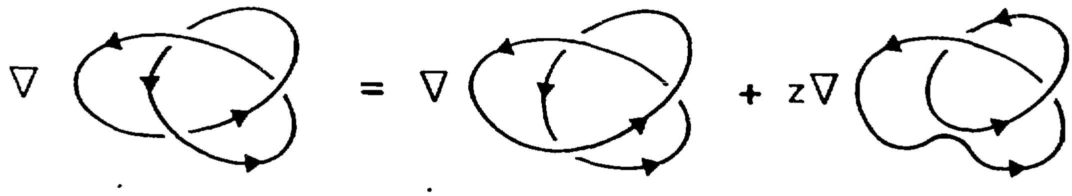
Examples.

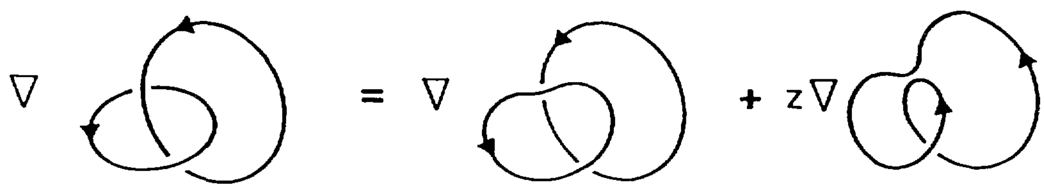
For a split link, $\nabla = 0$:



The first two links are ambient isotopic by a type 1 Reidemeister move, hence have the same ∇. Hence the polynomial of the split link of the r.h.s. vanishes. For the trefoil,

$\nabla_T = 1 + z^2$.

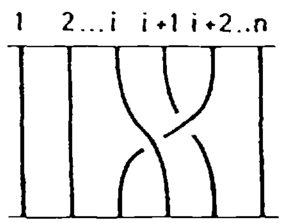




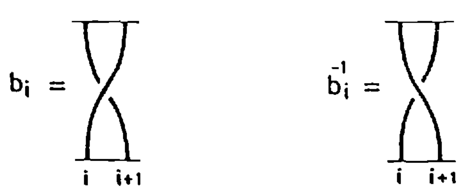
$$\nabla_T = 1 + z(0 + z) = 1 + z^2 = \nabla_{T^*}.$$

6.3. Braid group.

A braid is made of n strings joining n points on a horizontal line to n points on a parallel line.

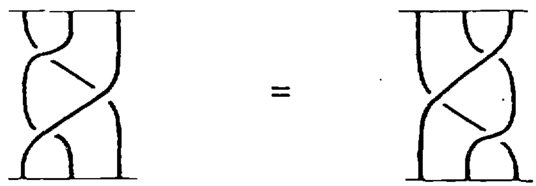


The braid group B_n is the group formed by ambient isotopy classes of braids, with the obvious concatenation operation. It has $n-1$ generators b_i, $i = 1, \ldots, n-1$:



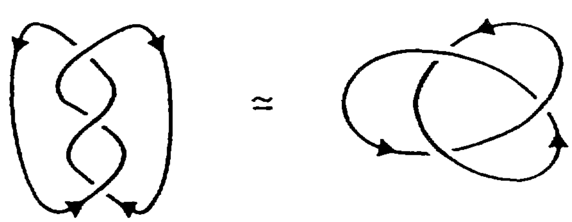
Notice that the index i does not label an individual string but refers to the relative location of the strings. These generators b_i satisfy the following set of constraints:

$$[b_i, b_j] = 0 \qquad |i - j| \geq 2$$
$$b_i b_{i\pm1} b_i = b_{i\pm1} b_i b_{i\pm1} \tag{6.2}$$



The latter equation is nothing else than the YB equation without spectral parameter, and the similarity with defining equations of the Hecke algebra will not escape the reader. (In fact any representation of the Hecke algebra yields a representation of the braid group.)

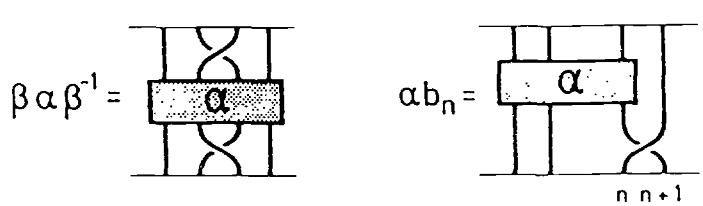
Time closure of a braid α gives an oriented link $\hat{\alpha}$:



and conversely any oriented link is the closure of some braid (Alexander theorem). This representation, however, is far from unique and it is an important question to identify which braids give equivalent links. Let us define the Markov moves:

Type I: $\alpha \in B_n \to \beta\alpha\beta^{-1}$ where $\beta \in B_n$

Type II: $\alpha \in B_n \to \alpha b_n^{\pm 1}$ where $b_n^{\pm 1} \in B_{n+1}$.



Two braids $\alpha \in B_n$ and $\beta \in B_m$ have isotopic closures if and only if there is a finite sequence of Markov moves taking α to β.

6.4. Markov trace and link polynomials. Homfly polynomial.

Given a representation of the braid group B_n in some vector space, a **Markov trace** is a linear functional Tr(.) on that space such that (we denote the representative of the element

α of B_n by the same letter α)

$$
\begin{aligned}
\mathrm{Tr}(\beta\alpha\beta^{-1}) &= \mathrm{Tr}(\alpha) \qquad , \qquad \alpha, \beta \in B_n \\
\mathrm{Tr}(\alpha b_n) &= \tau \mathrm{Tr}(\alpha) \qquad , \qquad \alpha \in B_n, b_n \in B_{n+1} \\
\mathrm{Tr}(\alpha b_n^{-1}) &= \bar{\tau} \mathrm{Tr}(\alpha)
\end{aligned}
\tag{6.3}
$$

Then let $P_{\hat{\alpha}}$ be

$$
P_{\hat{\alpha}} = (\tau\bar{\tau})^{-\frac{1}{2}(n-1)} \left(\frac{\bar{\tau}}{\tau}\right)^{\frac{1}{2}e(\alpha)} \mathrm{Tr}(\alpha) \tag{6.4}
$$

where $e(\alpha)$ is the sum of the exponents appearing in the expression of α in terms of the generators b_i. $P_{\hat{\alpha}}$ is invariant under Markov moves: it is a link invariant.

Example: Jones polynomial

We choose as a representative of b_n the $\check{R}$ matrix in the spin $\frac{1}{2}$ representation:

$$
\check{R}^{\frac{1}{2}\frac{1}{2}} = b = q^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^{-2} & q^{-1} & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.5}
$$

We then use the Markov trace of lect. 4 and compute

$$
\begin{aligned}
\mathrm{Tr}(\alpha) &= \frac{1}{(2)_q^n} \mathrm{tr}\left(q^{2S^z}\alpha\right) \qquad \alpha \in B_n \\
\mathrm{Tr}(b) &= \frac{1}{(2)_q^2} q^{\frac{1}{2}}\left(q^2 + 1 - q^{-2} + q^{-2}\right) = \frac{q^{\frac{3}{2}}}{(2)_q} \\
\mathrm{Tr}(b^{-1}) &= \frac{q^{-\frac{3}{2}}}{(2)_q}
\end{aligned}
\tag{6.6}
$$

The trefoil is the closure of b^3

$$
b^3 = q^{\frac{3}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^{-2}+q^{-4}-q^{-6} & * & 0 \\ 0 & * & q^{-2}-q^{-4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

hence

$$
\mathrm{Tr}(b^3) = \frac{q^{\frac{3}{2}}}{(2)_q^2}\left(q^2 + 1 - q^{-6} + q^{-2}\right)
$$

In this case, $P_{\hat{\alpha}}$ of (6.4) is a Laurent polynomial in q, q^{-1} called the Jones polynomial $P_{\hat{\alpha}}^{\text{Jones}}$. For the trefoil T

$$
\begin{aligned}
P_T^{\text{Jones}} =&(\tau\bar{\tau})^{-\frac{1}{2}}\left(\frac{\bar{\tau}}{\tau}\right)^{\frac{3}{2}}\operatorname{Tr}(b^3)\\
=&(2)_q q^{-\frac{9}{2}}\frac{q^{\frac{3}{2}}}{(2)_q^2}\left(1+q^2+q^{-2}-q^{-6}\right)\\
=&q^{-3}\frac{1+q^2+q^{-2}-q^{-6}}{q+q^{-1}} = q^{-2}+q^{-6}-q^{-8}
\end{aligned}
$$

The mirror image T^* of the trefoil is the closure of b^{-3}.

$$
P_T^{\text{Jones}} = q^{-2}+q^{-6}-q^{-8} \qquad\qquad P_{T^*}^{\text{Jones}} = q^{2}+q^{6}-q^{8} \tag{6.7}
$$

Hence P^{Jones} discriminates T from T^*.

From $(b-q^{\frac{1}{2}})(b+q^{-\frac{3}{2}})=0$ or $b = q^{\frac{1}{2}} - q^{-\frac{3}{2}} + q^{-1}b^{-1}$, we get

$$
P^{\text{Jones}}_{\nearrow\!\!\!\!\nwarrow} = (q^{\frac{1}{2}} - q^{-\frac{3}{2}})\left(\frac{\bar{\tau}}{\tau}\right)^{\frac{1}{2}} P^{\text{Jones}}_{\rightrightarrows} + q^{-1}\left(\frac{\bar{\tau}}{\tau}\right)P^{\text{Jones}}_{\nwarrow\!\!\!\!\nearrow} \tag{6.8}
$$

Besides this algebraic approach, the Jones polynomial may also be defined axiomatically, as the Alexander-Conway polynomial. Axiom (6.1) in particular is replaced by

$$
q^2 P^{\text{Jones}}_{\nearrow\!\!\!\!\nwarrow} - q^{-2} P^{\text{Jones}}_{\nwarrow\!\!\!\!\nearrow} = (q-q^{-1})P^{\text{Jones}}_{\rightrightarrows} \tag{6.9}
$$

$\mathcal{U}_q sl(N)$ polynomials.

Take $\mathcal{U}_q sl(N)$, $\check{R}^{\square,\square}$ provides a representation of the braid group. The analogues of the above Markov trace allow to define a link polynomial in $Z(q,q^{-1})$ that satisfies

$$
q^k P^{(N)}_{\nearrow\!\!\!\!\nwarrow} - q^{-k} P^{(N)}_{\nwarrow\!\!\!\!\nearrow} = (q-q^{-1})P^{(N)}_{\rightrightarrows} \tag{6.10}
$$

Formally the Alexander-Conway polynomial is reproduced by letting $k=0$. It is actually associated to the Z_2 graded quantum algebra $\mathcal{U}_q sl(1,1)$ [49].

Homfly polynomial [50].

The $P^{(N)}$ are specializations of the two-variable Homfly polynomial which satisfies in particular

$$
aH_{\nearrow\!\!\!\!\nwarrow} - a^{-1}H_{\nwarrow\!\!\!\!\nearrow} = zH_{\rightrightarrows} \tag{6.11}
$$

For the trefoil:

$$
\begin{aligned}
H_T =&(2a^{-2}-a^{-4})+z^2a^{-2}\\
H_{T^*} =&(2a^{2}-a^{4})+z^2a^{2}.
\end{aligned} \tag{6.12}
$$

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