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COMPUTATION OF THE TOPOLOGICAL SUSCEPTIBILITY FOR THE 2D $\mathbb{CP}^3$ MODEL ON A SPHERICAL LATTICE

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i) During the last years a lot of efforts have been made to investigate the topological vacuum structure of lattice gauge theories [1]. As an indicator, probing the contribution of topologically non-trivial excitations to the quantum vacuum usually serves the topological susceptibility

$$\chi_t = \langle Q_t^2 \rangle / V^{(d)} \tag{1}$$

where Q_t represents the topological charge of the gauge field and $V^{(d)}$ the volume in the d-dimensional space-time. Various methods have been invented to define Q_t on the lattice (see Ref. [1]). For the quantized fields generated numerically by Monte Carlo (MC) simulations they lead typically to different Q_t values event by event and to somewhat different χ_t estimates at accessible couplings in the scaling region. Surely, the disagreement is due to short-range fluctuations and possibly to lattice artifacts, which are taken into account or are suppressed in a different manner [2]. However, a priori, it is very difficult to say in as far short-range fluctuations with $Q_t \neq 0$ should be irrelevant in the continuum limit or whether instanton-like semi-classical background fields alone determine the topological properties of the vacuum state.

In this letter we are going to discuss this question within the framework of the two-dimensional CP^{n-1} model, which has a lot of similarities with the Yang-Mills theory in four dimensions. The latter fact concerns also the small volume limit for both theories formulated in the continuum on spheres S^2 and S^4, respectively. It has been shown for the Yang-Mills theory by Lüscher [3] and afterwards for the CP^{n-1} model by Schwab [4] that in this limit χ_t tends to zero and becomes dominated by the one-instanton contribution, provided the semi-classical approximation makes sense at all.

We want to see, whether the same will happen within the formulation of the theory on a latticized sphere, i.e. under the same boundary conditions as in the continuum case.

ii) The CP^{n-1} model we are considering is defined in a curved continuum space by the action

$$S = \frac{1}{f} \int d^2x \sqrt{g}\, g^{\mu\nu} (D_\mu z^a)(D_\nu z^a)^* \quad , \quad \mu,\nu = 1,2 \tag{2}$$

where the complex vectors $z^a = z^a(x)$, $a=1,2,\ldots,n$ satisfy the normalization condition $\sum_a z^a (z^a)^* = 1$. $g^{\mu\nu}$ denotes the metric tensor on S^2 and $d^2x\sqrt{g}$ represents the invariant volume element. The covariant derivative $D_\mu \equiv \partial_\mu - iA_\mu$ contains the Abelian "gauge" field A_μ which can be easily expressed by the z-field using the equations of motion. Here we want to treat it as a dynamical field which allows to read off the topological charge as a sum over winding numbers

$$Q_t = \frac{1}{2\pi} \sum_{\mathcal{E}} \oint_{\mathcal{E}} dx_\mu A_\mu \tag{3}$$

in a straightforward manner.

The theory is approximated on a triangular lattice. In principle, the latter could be a random one, but we prefer to generate an almost regular lattice. It can be obtained by starting from a regular tetraeder in $\mathbb{R}^3$ the corners of which are placed on the sphere S^2 . Then successively finer lattices enumerated by $N = 1,2,\ldots$ are provided by dividing all the edges into two parts each and projecting the midpoints onto the sphere. Connecting the newly projected sites with each other by new links gives us the next finer lattice. In this way the lattices produced contain $P = 2^{2N+1}+2$ sites and $L = 3(P-2)$ links. Every site has six nearest neighbours except the original corner points of the tetraeder which have three. This lattice construction has the advantage easily to be mapped onto the two-dimensional plane by cutting the lattice along three of the original edges of the tetraeder. We take the simplices of the lattice to be plane ones in $\mathbb{R}^3$. Already for $N = 2$ the area sum of all simplices represents 88% of the area of the sphere.

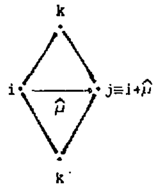
The lattice formulation of the model (2) looks as follows. We approximate the covariant derivative at lattice site x_i in the standard way

$$D_\mu z^a(x_i) \longrightarrow \left[z^a_{i+\hat\mu}\, U_{i,i+\hat\mu} - z^a_i \right] / l_{i,i+\hat\mu} \tag{4}$$

where l_{ij} is the length of the link vector $\vec{l}_{ij}$ connecting the neighbour sites x_i , x_j . The link variable is defined by

$$U_{i,i+\hat\mu} \equiv \exp(-iA_\mu(x_i) l_{i,i+\hat\mu}) \equiv \exp(-i\varphi_{i,i+\hat\mu}) \in U(1)$$

The metric tensor at the flat simplex (i,j,k)



is given by [5]

$$g^{\sigma\rho} = \frac{l_{ij}^2\, l_{ik}^2}{4(\Delta_{ijk})^2} \begin{pmatrix} 1 & -\dfrac{\vec{l}_{ij}\,\vec{l}_{ik}}{l_{ij}\, l_{ik}} \\ -\dfrac{\vec{l}_{ij}\,\vec{l}_{ik}}{l_{ij}\, l_{ik}} & 1 \end{pmatrix}_{\rho\sigma} \tag{5}$$

with the simplex area

$$\Delta_{ijk} = \frac{1}{4}\left(2l_{ij}^2 l_{jk}^2 + 2l_{ij}^2 l_{ik}^2 + 2l_{jk}^2 l_{ik}^2 - l_{ij}^4 - l_{jk}^4 - l_{ki}^4 \right)^{1/2} .$$

The lattice action can be cast into a form of a sum over links joining the nearest neighbours i, j

$$S_L = \beta \sum_{(i,j)} s_{(i,j)} \quad , \quad \beta = \frac{1}{f} \tag{6}$$

where $s_{(i,j)}$ gets contributions of both adjacent simplices (i,j,k) and (i,j,k')

$$s_{(i,j)} = z_i^{\alpha}(U_{ij}\Gamma_{ij} - U_{ik}U_{kj}\Omega_{ij} - U_{ik'}U_{k'j}\Omega'_{ij})(z_j^{\alpha})^{*} + \text{c.c.} - 3\Gamma_{ij} \tag{7}$$

$$\left.\begin{aligned} &\text{with the weights} \quad \Gamma_{ij} = \frac{l_{ij}^2 - l_{ik}^2 - l_{kj}^2}{12\Delta_{ijk}} + \frac{l_{ij}^2 - l_{ik'}^2 - l_{k'j}^2}{12\Delta_{ijk'}} \\ &\Omega_{ij} = \vec{l}_{ki} \cdot \vec{l}_{kj} / 12\Delta_{ijk} \quad , \qquad \Omega'_{ij} = \vec{l}_{k'i} \cdot \vec{l}_{k'j} / 12\Delta_{ijk'} \end{aligned}\right\} \tag{8}$$

Our lattice action is manifestly locally $U(1)$ gauge invariant. In contrast to CP^{n-1} lattice actions invented in earlier papers [6,7,8] it is a non-local one containing products of two $U(1)$ link variables. In the following we restrict ourselves to the CP^3 model, i.e. $n=4$.

iii) The model is quantized according to the functional integral

$$\int \prod_{i=1}^{P} \prod_{\alpha=1}^{4} d^2 z_i^\alpha \, \delta\Big(\sum_{\alpha=1}^{4} |z_i^\alpha|^2 - 1\Big) \int_{-\pi}^{\pi} \prod_{(j,k)}^{L} \frac{d\varphi_{jk}}{2\pi} \exp\Big[-S_L(\{z_i\},\{\varphi_{jk}\})\Big] \tag{9}$$

The quantum fields z *and* φ *have been generated in the MC simulation* acc. to (9) by the heatbath and the standard Metropolis algorithms, respectively. We tried as well different Metropolis update codes for the z-fields, but found the configurations to be strongly correlated from iteration to iteration.

Our MC runs were carried out for lattices with divisions $N = 2, 3$ and 4. Usually we made 400 thermalization sweeps, after which we started measurements during 5000 sweeps (except for $N = 2$, $\beta = 4.75, 5.00$ where we have run 50000 sweeps). We checked our MC code by comparing the numerical results for $\langle z_i U_{ij} z_j^* \rangle$ with a corresponding strong coupling expansion up to order β^2.

iv) The topological charge on the lattice for the CP^{n-1} model is defined by

$$Q_t = \frac{1}{2\pi} \sum_{\sigma} \left[\sum_{[i,j]\in\partial\sigma} \varphi_{ij} \right] \tag{10}$$

where the first sum runs over all simplices σ and the second one over the corresponding links (the latter one implies to take the angles φ_{ij} in the right orientation). The brackets mean the reduction to the interval

$$-\pi < \left[\sum \varphi_{ij} \right] \equiv \sum \varphi_{ij} + 2\pi\nu_\sigma \leq +\pi \ , \qquad \nu_\sigma \in \mathbb{Z} \ . \tag{11}$$

By Monte Carlo simulation we measure $\bar{l}^2 \chi_t$ acc. to Eq. (1), where the average lattice scale $\bar{l} = \sum_{[i,j]} l_{ij} / L$ should behave in accordance with the renormalization group

$$\Lambda_L \bar{l} = \left(\frac{\beta\pi}{2} \right)^{1/2} \exp\left(-\frac{\beta\pi}{2} \right) \tag{12}$$

First topological investigations of the lattice CP^1 model (which is identical with the non-linear O(3) σ-model) showed strong scaling violations for $l^2 \chi_t$ due to the dominance of dislocations with an action $S_L(\mathrm{disl}) < \pi$ [7]. It has been argued that scaling should be restored for $n \geq 4$. Moreover, Petcher and Lüscher [8] have constructed a modified ("ferromagnetic") lattice action for which in the $n=3$ case the MC data were in agreement with scaling. In our case, the CP^3 model, we have made a rough check, whether dislocations are expected to spoil scaling. For a single simplex σ_o with an arbitrary position we have chosen the surrounding links with $|\varphi_{ij}| = \pi/3 + \epsilon$, $\epsilon \ll 1$, such that ν_{σ_o} acc. to (11) became just ∓ 1. All other φ's were put equal to zero and the z's equal to $(1,0,0,0)$. For this kind of dislocation one finds

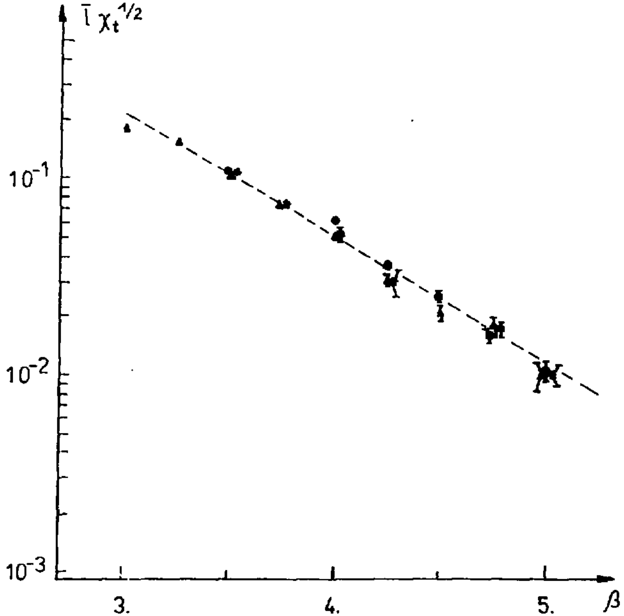
$$\bar{S}_L(\mathrm{disl}) = 3.29\,\beta \ > \ \pi\beta \tag{13}$$

after averaging over all positions of the simplex σ_o. So this kind of exceptional configuration does not cause danger in the continuum limit of the theory.

v) The topological charge has been measured every 10-th MC sweep. In this way correlations between subsequent measurements were under control.

Our results for the topological susceptibility $\bar{l}\,\chi_t^{1/2}$ are presented in the Figure for all lattice sizes considered. The errors indicated, for simplicity, are the pure statistical ones for Q_t^2. The straight line corresponds to the scaling behaviour acc. to Eq. (12). We see that all the data points very well fit to this behaviour. At the given accuracy we do not see any scaling *violation in the range* $3.25 \lesssim \beta \lesssim 5.0$. Very surprisingly, at least for us, the points for $N = 2,3$ and 4 lie on the same universal curve, i. e. there is not any finite size effect visible! Obviously, the topological susceptibility at large β is dominated by short-range fluctuations only. For the N=2 lattice, which contains 34 lattice sites only, we found 36 and 13 events with $|Q_t| = 1$ among the 5000 measured charges at $\beta = 4.75$ and 5.0, respectively. These non-trivial events were isolated, they disappeared immediately. Nevertheless, there is a region, where Q_t-values happen to be "frozen" over up to O(100) sweeps. For $N = 4$ this has been observed at $4.0 \lesssim \beta \lesssim 4.5$. One would like to interpret this phenomenon by the existence of long-range fluctuations, what has been reported for the CP^2 model as well [8]. If one would like to insist in suppressing short-range fluctuations in Q_t measurements, e. g. by "cooling" (see Ref. [1]), then finite size effects became visible for $N = 4$ at $\beta \approx 4.5$. It is only in this case that the semi-classical finite-volume picture [3,4] could be established as well on the lattice.

vi) Our findings very much resemble lattice results obtained in SU(2) Yang-Mills theory, where Q_t has been determined on MC equilibrium configurations by the geometric method of Phillips and Stone [9]. There is seen a beautiful scaling behaviour [10], where at the same time short-range fluctuations are present. Finite size effects are weak compared with those seen after cooling [11]. Of course, it would be worthwhile to study the finite volume effects in the 4D Yang-Mills case on the sphere in the same way as it has been presented here for the CP^3 model.



Topological susceptibility as function of β. Dots, triangles and stars correspond to lattice sizes with N = 4, 3 and 2, respectively. The dashed line shows the renormalization group behaviour Eq. (12).

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Йозефини Б., Мюллер-Пройскер М., Шультка Н. E2-89-607

Исследование топологической вакуумной структуры двумерной CP^3-модели на сферической решетке

С помощью метода Монте-Карло определяется топологическая восприимчивость на симилексной решетке аппроксимирующей сферу S^2. Полученные данные хорошо согласуются с ренормгрупповым поведением. В пределе маленького объема не наблюдается уменьшение плотности топологических флуктуаций в отличие от предсазанной инстантонных вычислений.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

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Computation of the Topological Susceptibility for the 2D CP^3 Model on a Spherical Lattice

Using Monte Carlo simulations we calculate the topological susceptibility for the CP^3 model on a simplicial lattice approximating the sphere S^2.. Our data exhibit the right scaling behaviour but do not show a suppression of topologically relevant fluctuations in the small volume limit.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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