DYNAMICAL PARASUPERSYMMETRIES IN QUANTUM MECHANICS

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Supersymmetric field theories have the distinctive feature of being invariant under transformations that mix bosonic and fermionic variables. Reduction to 0+1 dimensions yields mechanical models with an analogous invariance. In this case, the Grassmannian variables are interpreted as describing (classically) the spin degrees of freedom of the particles involved. After canonical quantization, the corresponding quantities obey the standard anticommutation relations of fermionic creation and annihilation operators [1].

It is known that paraquantization offers alternatives to the usual quantization scheme. In this framework, one can expect that it is possible to construct parasupersymmetric theories, that is, theories which are invariant under transformations between bosonic and parafermionic variables. As a matter of fact, Rubakov and Spiridonov have recently [2] shown how the parasupersymmetric generalization of supersymmetric Quantum Mechanics proceeds. In this case, the fermionic creation and annihilation operators obey paracommutation relations.

The applications of supersymmetric Quantum Mechanics are many. One might hope that its parasupersymmetric generalization will be as useful. The elaboration of parasupersymmeric Quantum Mechanics moreover has led to new mathematical constructs; indeed, the symmetry generators realize algebras involving products of degree higher than 2. In view of the increasing role that algebraic methods are playing in Physics, these structures clearly deserve investigation.

This set of lectures is aimed at giving an introduction to parasupersymmetric Quantum Mechanics and parasuperalgebras. Their contents are given below.

Lecture 1: Parasupersymmetric quantum mechanics

1.1 Paraquantization

1.2 Supersymmetric quantum mechanics

1.3 Parasupersymmetric quantum mechanics

Lecture 2: Conformal parasupersymmetries in quantum mechanics

2.1 Conformal symmetry

2.2 Conformal supersymmetry

2.3 Conformal parasupersymmetry

2.4 Representations of the parasuperconformal algebra

In Lecture 1, we mainly summarize the work of Rubakov and Spiridonov. In Lecture 2, we introduce the second order parasuperconformal algebra which generalizes the $OSp(2,1)$ superalgebra. An explicit realization will be given in terms of the symmetry generators of a quantum mechanical example. The relevant unitary representations will be constructed, allowing for an algebraic determination of the energy spectrum and wave functions.

LECTURE 1: PARASUPERSYMMETRIC QUANTUM MECHANICS

1.1-Paraquantization

Following Ohnuki and Kamefuchi [3], we shall illustrate the philosophy of paraquantization with two simple examples. Let us consider first a Bose oscillator whose classical Lagrangian is given by

$$L = \frac{1}{2}(\dot{q}^2 - q^2). \tag{1.1}$$

The Euler-Lagrange (i.e. classical) equation of motion is

$$\ddot{q} + q = 0. \tag{1.2}$$

Introducing the conjugate momentum $p = \dot{q}$, the associated Hamiltonian is the usual

$$H = \frac{1}{2}(p^2 + q^2), \qquad p = \dot{q}. \tag{1.3}$$

When the system is quantized, the Heisenberg (i.e. quantum) equations of motion

$$i\frac{dq}{dt} = [q, H] \quad and \quad i\frac{dp}{dt} = [p, H] \tag{1.4}$$

are required to be compatible with $p = \dot{q}$ and with the Euler-Lagrange equation (1.2). Hence, one demands

$$[q, H] = ip, \tag{1.5a}$$

$$[p, H] = -iq. \tag{1.5b}$$

The paraquantization point of view consists in asking what kind of commutation relations between q and p should we have for equations (1.5) to be verified. We know, of course, that the canonical commutation relation $[q, p] = i$ does the job, but as shown by Wigner [4], it is not the only possibility.

Let us now introduce the standard creation and annihilation operators

$$a \equiv \frac{1}{\sqrt{2}}(q + ip), \qquad a^\dagger \equiv \frac{1}{\sqrt{2}}(q - ip). \tag{1.6}$$

In terms of these operators, the Hamiltonian (1.3) becomes

$$H = \frac{1}{2}(a^\dagger a + aa^\dagger) \tag{1.7}$$

and Eqs. (1.5) take the following form:

$$[a, H] = a, \qquad [a^\dagger, H] = -a^\dagger. \tag{1.8}$$

One possible way of satisfying (1.8) is by imposing

$$[a, a^\dagger] = 1 \tag{1.9}$$

which, of course, is equivalent to $[q,p]=i$. But, as already mentioned, there are other solutions. Rewrite the first of the relations (1.8) as

$$[a, [a^\dagger, a]_+] = 2a. \tag{1.10}$$

Equation (1.10) will be regarded as the fundamental commutation relation to be satisfied in the parabose quantization procedure. Let us show with an example that it admits solutions other than (1.9). Suppose that we write a and $a^\dagger$ in the form

$$a = a^{(1)} + a^{(2)}, \qquad a^\dagger = a^{(1)^\dagger} + a^{(2)^\dagger}, \tag{1.11}$$

and that we require the "components" to satisfy

$$[a^{(\alpha)}, a^{(\alpha)^\dagger}] = 1, \qquad \alpha, \beta = 1, 2 \tag{1.12a}$$

$$[a^{(\alpha)}, a^{(\beta)^\dagger}]_+ = 0, \quad [a^{(\alpha)}, a^{(\beta)}]_+ = 0, \quad \alpha \neq \beta. \tag{1.12b}$$

It is easy to see that

$$[a^\dagger, a]_+ = [a^{(1)^\dagger}, a^{(1)}]_+ + [a^{(2)^\dagger}, a^{(2)}]_+. \tag{1.13}$$

Since

$$[a^{(1)}_{(2)}, [a^{(2)^\dagger}_{(1)}, a^{(2)}_{(1)}]_+] = 0, \tag{1.14}$$

we have using (1.13)

$$\begin{aligned} [a, [a^\dagger, a]_+] &= [a^{(1)}, [a^{(1)^\dagger}, a^{(1)}]_+] + [a^{(2)}, [a^{(2)^\dagger}, a^{(2)}]_+] \\ &= 2(a^{(1)} + a^{(2)}) = 2a, \end{aligned} \tag{1.15}$$

and indeed, a solution to (1.10) which is clearly inequivalent to (1.9).

We shall now turn to Fermi-like oscillators. In this case we take for the Hamiltonian

$$H = \frac{1}{2}(a^\dagger a - a a^\dagger) \equiv \frac{1}{2}[a^\dagger, a]_-, \tag{1.16}$$

while keeping the same equations of motion, that is (1.8). Substituting for H the expression (1.16), the equations of motion now amount to

$$[a,[a^\dagger,a]_-] = 2a, \qquad [a^\dagger,[a^\dagger,a]_-] = -2a^\dagger. \tag{1.17}$$

The above are the fundamental fermionic paracommutation relations. The standard anticommutation* relations

$$\{a,a^\dagger\} = 1, \qquad a^2 = a^{\dagger 2} = 0, \tag{1.18}$$

associated to the canonical quantization of fermionic variables are the simplest conditions that we can impose on the creation and annihilation operators so that eqs. (1.17) be verified. There are however, other possibilities in this case also. This may be seen as follows. Define

$$J_3 = \frac{1}{2}[a^\dagger,a] = H, \tag{1.19a}$$

$$J_+ = a^\dagger, \qquad J_- = a. \tag{1.19b}$$

As the notation suggests, these operators obey the $SU(2)$ commutation relations:

$$[J_+,J_-] = 2J_3, \qquad [J_3,J_\pm] = \pm J_\pm. \tag{1.20}$$

They therefore admit representations in spaces of dimension $p = 1,2,3,\dots$. Accordingly, the spectrum of $J_3 = H$ is $-p/2, -p/2+1, \dots, p/2-1, p/2$. The parameter p is called the *order* of paraquantization.

When $p=1$ we have in terms of the Pauli matrices: $J_3 = \frac{1}{2}\sigma_3$, $a^\dagger = \sigma_+$ and $a = \sigma_-$. In this case, a and $a^\dagger$ clearly satisfy the commutation relations (1.18) of ordinary Fermi oscillators. When $p = 2$ we have

$$J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{1.21a}$$

$$a = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad a^\dagger = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.21b}$$

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* We shall use indifferently the notation $[\ ,\ ]_+$ or $\{\ ,\ \}$ for the anticommutator and $[\ ,\ ]$ or $[\ ,\ ]_-$ for the commutator.

We see that a and $a^\dagger$ do not satisfy anymore the anticommutation relations (1.18); one can quickly check that they instead verify

$$aa^\dagger a = 2a, \qquad a^3 = 0, \tag{1.22a}$$

$$a^2a^\dagger + a^\dagger a^2 = 2a. \tag{1.22b}$$

As in the Bose case, one can also obtain a realization of (1.17) through a decomposition of the form (1.11). Here, one must impose the following relations on the components:

$$[a^{(\alpha)}, a^{(\alpha)\dagger}]_+ = 1, \qquad (a^{(\alpha)})^2 = 0, \tag{1.23a}$$

$$[a^{(\alpha)}, a^{(\beta)\dagger}]_- = 0, \quad [a^{(\alpha)}, a^{(\beta)}]_- = 0, \quad \alpha \neq \beta. \tag{1.23b}$$

The a and $a^\dagger$ so defined, are actually of second order since they obey the relations (1.22) which in turn imply the equations (1.17). This representation however, contrary to the 3-dimensional one given in (1.21), is reducible. This can be shown as follows. Let $|0\rangle$ be the "vacuum state" defined by the conditions $a^{(\alpha)}|0\rangle = 0$, $\alpha = 1,2$. We may take for basis of the representation space the set $\{\,|0\rangle, a^{(1)\dagger}|0\rangle, a^{(2)\dagger}|0\rangle, a^{(1)\dagger}a^{(2)\dagger}|0\rangle\,\}$. It then can straightforwardly be checked that $a^{(1)\dagger}|0\rangle - a^{(2)\dagger}|0\rangle$ spans an invariant 1-dimensional space since it belongs to the kernel of both a and $a^\dagger$.

Let us conclude this section by saying that all that has been discussed so far can be generalized to situations (like field theory) where there are many oscillators [3]. The general paracommutation relations read

$$[a_k, [a_l^\dagger, a_m]_\mp] = 2\delta_{kl}a_m, \tag{1.24a}$$

$$[a_k, [a_l^\dagger, a_m^\dagger]_\mp] = 2\delta_{kl}a_m^\dagger \mp 2\delta_{km}a_l^\dagger, \tag{1.24b}$$

$$[a_k, [a_l, a_m]_\mp] = 0. \tag{1.24c}$$

The upper and lower signs respectively correspond to the parafermi and parabose cases.

A realization of the above relations is provided by the so-called Green representation which is in general reducible. Examples of this representation have already been given. For paravariables of order p, it is constructed by taking

$$a_k = \sum_{\alpha=1}^{p} a_k^{(\alpha)}, \qquad a_k^\dagger = \sum_{\alpha=1}^{p} a_k^{(\alpha)\dagger}, \tag{1.25}$$

with the components $a_k^{(\alpha)}$ satisfying

$$[a_k^{(\alpha)}, a_l^{(\alpha)\dagger}]_\pm = \delta_{kl}, \qquad [a_k^{(\alpha)}, a_l^{(\alpha)}]_\pm = 0, \tag{1.26a}$$

$$[a_k^{(\alpha)}, a_l^{(\beta)\dagger}]_\mp = [a_k^{(\alpha)}, a_l^{(\beta)}]_\mp = 0 \qquad (\alpha \neq \beta). \tag{1.26b}$$

1.2-Supersymmetric quantum mechanics [1]

We shall review in this Section the main features of ordinary supersymmetric Quantum Mechanics before we discuss its parasupersymmetric generalization.

A quantum mechanical system is said to be supersymmetric if there exist supercharges Q and $Q^\dagger$ which together with the Hamiltonian H realize the superalgebra

$$[Q, Q^\dagger]_+ = 2H, \tag{1.27a}$$

$$Q^2 = Q^{\dagger 2} = 0, \tag{1.27b}$$

$$[H, Q]_- = [H, Q^\dagger]_- = 0. \tag{1.27c}$$

In one dimension, such supercharges can be constructed with the help of the fermionic creation and annihilation operators f and $f^\dagger$ that satisfy

$$[f, f^\dagger]_+ = 1, \qquad f^2 = f^{\dagger 2} = 0. \tag{1.28}$$

These operators are irreducibly represented by the following 2×2 matrices

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad f^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.29}$$

Let $p = -i\partial_x$ and Q, $Q^\dagger$ be of the form

$$Q = (p + iW)f, \qquad Q^\dagger = (p - iW)f^\dagger, \tag{1.30}$$

with $W(x)$ an arbitrary function. Substitution into (1.27a) gives the following Hamiltonian

$$H = \frac{1}{2}\left(p^2 + W^2 + W'[f^\dagger, f]_-\right). \tag{1.31}$$

As usual W' stands for $dW(x)/dx$. Note that $[f^\dagger, f]_- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \equiv \sigma_3$. The Hamiltonian H governs the one-dimensional motion of a "spin-$\frac{1}{2}$" particle in the potential $\frac{1}{2}W^2$ and the "magnetic field" $\frac{1}{2}W'$ along the third axis.

Let us record that the structure relations (1.27) are replaced by

$$\{Q_i, Q_j\} = 2\delta_{ij}H, \quad [H, Q_i] = 0, \qquad i,j = 1,2\,, \tag{1.32}$$

if one uses the hermitian basis

$$Q_1 = \frac{1}{\sqrt{2}}(Q + Q^\dagger), \qquad Q_2 = \frac{i}{\sqrt{2}}(Q - Q^\dagger). \tag{1.33}$$

1.3-Parasupersymmetric quantum mechanics [2]

The second-order parasupersymmetric generalization of supersymmetric Quantum Mechanics is achieved through the replacement of the fermionic operators f and $f^\dagger$ by the parafermionic operators a and $a^\dagger$ defined in Section 1.1. Recall that these creation and annihilation operators satisfy the trilinear relations (1.22) of which a 3-dimensional representation has been given in (1.21b). In analogy with the supersymmetric Quantum Mechanics construction, we look for parafermionic charges $Q, Q^\dagger$ such that

$$Q^3 = Q^{\dagger^3} = 0. \tag{1.34a}$$

The simplest generalisation of (1.30) which satisfies the condition (1.34a) is

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ p + iW_1 & 0 & 0 \\ 0 & p + iW_2 & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & p - iW_1 & 0 \\ 0 & 0 & p - iW_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.35}$$

with $W_1(x)$ and $W_2(x)$ for now arbitrary functions of x. We shall refer to the parafermionic charges Q and $Q^\dagger$ as the parasupertranslation generators. At this point, we want to determine the associated Hamiltonian H and the algebra which is realized when it is adjoined to Q and $Q^\dagger$. One can check that bilinear combinations of Q and $Q^\dagger$ do not yield expressions of the form p^2+$U(x)$ with $U(x)$ a 3×3 matrix depending only on x. We therefore look for trilinear combinations of Q and $Q^\dagger$. For the Hamiltonian H given by

$$4H = 2p^2 + W_1^2 + W_2^2 + \begin{pmatrix} 3W_1' + W_2' & 0 & 0 \\ 0 & W_2' - W_1' & 0 \\ 0 & 0 & -W_1' - 3W_2' \end{pmatrix} \tag{1.36}$$

and provided $W_1(x)$ and $W_2(x)$ satisfy

$$(W_2^2 - W_1^2)' + (W_2 + W_1)'' = 0, \tag{1.37}$$

we find the following algebra:

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4QH, \qquad (1.34b)$$

$$Q^{\dagger 2}Q + Q^\dagger QQ^\dagger + QQ^{\dagger 2} = 4Q^\dagger H, \qquad (1.34c)$$

$$[H,Q] = [H,Q^\dagger] = 0. \qquad (1.34d)$$

These relations, when realized, define what we shall call a parasupersymmetric system. Using a and $a^\dagger$, Q, $Q^\dagger$ and H can be written as follows:

$$Q = \frac{1}{2\sqrt{2}}\Big((p+iW_1)a^\dagger a^2 + (p+iW_2)a^2a^\dagger\Big), \qquad (1.38a)$$

$$Q^\dagger = \frac{1}{2\sqrt{2}}\Big((p-iW_1)a^{\dagger 2}a + (p-iW_2)aa^{\dagger 2}\Big), \qquad (1.38b)$$

$$H = \frac{1}{2}\Big(p^2 + \tfrac{1}{2}(W_1^2 + W_2^2) + \tfrac{3}{2}(W_1' - W_2') + W_2'a^\dagger a - W_1'aa^\dagger\Big). \qquad (1.38c)$$

In terms of the hermitian charges $Q_1 = (Q^\dagger + Q)/2$ and $Q_2 = (Q^\dagger - Q)/2i$, the relations (1.34*abcd*) can be cast in the form

$$[H, Q_i] = 0, \qquad (i,j = 1,2) \qquad (1.39a)$$

$$\begin{aligned} Q_i(\{Q_j,Q_k\} - 2\delta_{jk}H) + Q_j(\{Q_k,Q_i\} - 2\delta_{ki}H) \\ + Q_k(\{Q_i,Q_j\} - 2\delta_{ij}H) = 0. \end{aligned} \qquad (1.39b)$$

Note how the relations that define the ordinary supersymmetry algebra are factored in (1.39*b*).

Let us now look at a simple example: the parasupersymmetric harmonic oscillator. Take $W_1(x)$, $W_2(x)$ to be

$$W_1 = W_2 = x. \qquad (1.40)$$

When we substitute for these functions in (1.38), we find that the parasupercharges can be written as

$$Q = b^\dagger a, \qquad Q^\dagger = ba^\dagger, \qquad (1.41a)$$

with

$$b = \frac{1}{\sqrt{2}}(p - ix), \qquad b^\dagger = \frac{1}{\sqrt{2}}(p + ix). \qquad (1.41b)$$

The Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2}(p^2 + x^2) + \frac{1}{2}[a^\dagger, a] \\ &= \frac{1}{2}[b^\dagger, b]_+ + \frac{1}{2}[a^\dagger, a]. \end{aligned} \tag{1.42}$$

It is thus simply the sum of the Hamiltonians associated on the one hand to an ordinary Bose oscillator and on the other hand to a second order para-Fermi oscillator. Using the fact that $[b, b^\dagger] = 1$ and the definition of J_3, we may equivalently write

$$H = b^\dagger b + J_3 + \tfrac{1}{2}. \tag{1.43}$$

The eigenvalues of this operator clearly are

$$E_{n,s} = n + s + \tfrac{1}{2}, \qquad s = 0, \pm 1\,, \quad n = 0, 1, 2, \ldots . \tag{1.44}$$

Note that the ground state has negative energy and that levels above the second one are three-fold degenerate.

We can use this example to illustrate the nature of parasupersymmetry transformations. Since these transformations mix bosonic and parafermionic variables, their parameters must be some second order generalization of the usual Grassmann numbers of supersymmetry. Once therefore introduces a para-Grassmann algebra generated by elements θ_i which satisfy

$$\theta_i \theta_j \theta_k + \theta_k \theta_j \theta_i = 0. \tag{1.45}$$

These numbers are further require to obey the following commutation relations with a and $a^\dagger$:

$$[a, [a^\dagger, \theta_i]] = 2\theta_i, \tag{1.46a}$$

$$[a, [a, \theta_i]] = [\theta_i, [a, \theta_j]] = 0. \tag{1.46b}$$

To the parasupercharges $Q^\dagger$ and Q, we therefore associate two such numbers, say θ and $\bar{\theta}$. The corresponding infinitesimal transformation of a dynamical variable ϕ is then given by

$$\delta\phi = [[\bar{\theta}, Q] + [\theta, Q^\dagger], \phi]. \tag{1.47}$$

For the oscillator, we get

$$\begin{aligned} \delta a &= b^\dagger[[\bar{\theta}, a], a] + b[[\theta, a^\dagger], a] \\ &= 2\theta b, \end{aligned} \tag{1.48a}$$

$$\begin{aligned}\delta b &= [b^\dagger[\bar{\theta}, a], b] + [b[\theta, a^\dagger], b] \\ &= [a, \bar{\theta}].\end{aligned} \tag{1.48b}$$

We may then directly check the invariance of H under these transformations. Indeed

$$\begin{aligned}\delta H &= \delta b^\dagger b + b^\dagger \delta b + \frac{1}{2}[\delta a^\dagger, a] + \frac{1}{2}[a^\dagger, \delta a] \\ &= -[a^\dagger, \theta] b + b^\dagger [a, \bar{\theta}] + \frac{1}{2}[2\bar{\theta} b^\dagger, a] + \frac{1}{2}[a^\dagger, 2\theta b] \\ &= 0.\end{aligned} \tag{1.49}$$

LECTURE 2: CONFORMAL PARASUPERSYMMETRIES IN QUANTUM MECHANICS

Conformal symmetries are playing an important role in many of the recent developments in theoretical physics. In problems where they are present, they typically enable one to solve the dynamics. We shall here discuss a case in parasupersymmetric Quantum Mechanics where this is true.

In what follows, we shall identify the parasuperalgebra that generalizes the $OSp(2,1)$ superalgebra. To this end, we shall successively examine the supersymmetric and parasupersymmetric extensions of a simple conformal invariant system.

2.1-Conformal symmetry [5]

Consider the following operators:

$$H = \frac{1}{2}p^2, \qquad D = -\frac{1}{4}(xp + px), \qquad K = \frac{1}{2}x^2. \tag{2.1}$$

It is immediate to check that they satisfy the $O(2,1)$ commutation relations:

$$[H,K] = 2iD, \qquad [D,K] = iK, \qquad [D,H] = -iH. \tag{2.2}$$

The same algebra is realized if the Hamiltonian H is replaced by

$$H \longrightarrow H = \frac{1}{2}\left(p^2 + \frac{\lambda^2}{x^2}\right) \tag{2.3}$$

with λ an arbitrary real parameter. The above operators give rise to an *algebra* of *conserved* charges:

$$A(t) = e^{-iHt}Ae^{iHt}, \qquad A \in \{H, D, K\}, \tag{2.4a}$$

$$\frac{dA(t)}{dt} = \frac{\partial A(t)}{\partial t} + i[H, A(t)] = 0. \tag{2.4b}$$

For $\psi(t)$ a solution of the Schrödinger equation $i\partial\psi/\partial t = H\psi$, it follows that $A(t)\psi(t)$ is also a solution. The Hilbert space of state vectors $\psi(t)$'s therefore support a *representation* of $O(2,1)$. Note that it suffices to know the action of the charges A at $t=0$ to determine how they transform states at any given time.

Please observe that we have the same $O(2,1)$ symmetry algebra if we take the Hamiltonian to be

$$H' = H + K = \frac{1}{2}\left(p^2 + \frac{\lambda^2}{x^2} + x^2\right) \qquad (2.5)$$

instead of H. In this case however, the conserved charges are given by

$$A(t) = e^{-iH't} A_{(t=0)} e^{iH't}. \qquad (2.6)$$

Again, it suffices to know the action of these operators at $t=0$ in order to obtain the symmetries of the corresponding Schrödinger equation.

2.2-Conformal supersymmetry [6,7]

Let us now consider the supersymmetric extension of the system introduced in the preceding section. Substituting $W = \lambda/x$ in (1.30) and (1.31) gives the following supercharges and Hamiltonian:

$$Q = (p + i\frac{\lambda}{x})f, \qquad Q^\dagger = (p - i\frac{\lambda}{x})f^\dagger, \qquad (2.7a)$$

$$H = \frac{1}{2}\left(p^2 + \frac{\lambda^2}{x^2} - \frac{\lambda}{x^2}\sigma_3\right) = \frac{1}{2}\{Q, Q^\dagger\}. \qquad (2.7b)$$

The above Hamiltonian comprises a pair of Hamiltonians with $1/x^2$ potentials. If we combine this operator with the generator D and K of the previous section, we clearly get again a realization of $O(2,1)$. Since H is supersymmetric, there are actually still more symmetries. Indeed, computing the commutator of Q and K we find

$$[Q, K] = [p, \tfrac{1}{2}x^2]f = i(-xf) \equiv iS. \qquad (2.8)$$

The supertranslation generator Q being conserved, we thus get two new symmetry operators, namely

$$S = -xf, \qquad S^\dagger = -xf^\dagger. \qquad (2.9)$$

These charges generate superconformal transformations as is seen from the fact that

$$\{S, S^\dagger\} = x^2 = 2K. \qquad (2.10a)$$

Evaluating the anticommutators between the supercharges, we further obtain another bosonic constant of motion. We get

$$\{Q, S\} = \{Q^\dagger, S^\dagger\} = 0 \qquad (2.10b)$$

and

$$\{Q, S^\dagger\} = 2D - 2iY, \tag{2.10c}$$

with

$$Y = \frac{1}{2}\left(\frac{\sigma_3}{2} + \lambda\right). \tag{2.10d}$$

This last operator generates rotations in internal space. At this point a complete algebraic set has been obtained. The fermionic charges Q,$Q^\dagger$,S,$S^\dagger$ and the bosonic operators H,D,K,Y, in fact, close under the graded Lie product to realize the $OSp(2,1)$ superalgebra. If one uses the hermitian charges

$$Q_1 = \frac{1}{\sqrt{2}}(Q + Q^\dagger), \qquad Q_2 = \frac{i}{\sqrt{2}}(Q - Q^\dagger), \tag{2.11a}$$

$$S_1 = \frac{1}{\sqrt{2}}(S + S^\dagger), \qquad S_2 = \frac{i}{\sqrt{2}}(S - S^\dagger), \tag{2.11b}$$

one may check that the following structure relations are obeyed ($i,j = 1,2$):

$$\begin{aligned} &[D,H] = -iH, \quad [K,H] = -2iD, \quad [D,K] = iK, \\ &[Y,H] = 0, \qquad [Y,D] = 0, \qquad [Y,K] = 0, \end{aligned} \tag{2.12a}$$

$$\begin{aligned} &[H,Q_i] = 0, && [H,S_i] = iQ_i, \\ &[D,Q_i] = -\frac{i}{2}Q_i, && [D,S_i] = \frac{i}{2}S_i, \\ &[K,Q_i] = -iS_i, && [K,S_i] = 0, \\ &[Y,Q_i] = \frac{i}{2}\epsilon_{ij}Q_j, && [Y,S_i] = \frac{i}{2}\epsilon_{ij}S_j, \end{aligned} \tag{2.12b}$$

$$\begin{aligned} \{Q_i, Q_j\} = 2\delta_{ij}H, \qquad & \{S_i, S_j\} = 2\delta_{ij}K, \\ \{Q_i, S_j\} = 2\delta_{ij}&D - 2\epsilon_{ij}Y. \end{aligned} \tag{2.12c}$$

2.3-Conformal parasupersymmetry [8]

We now come to the parasupersymmetric extension of the $1/x^2$-potential which we shall use to abstract the structure of the second-order parasuperconformal algebra.

Let $W_1 = \lambda/x$ and $W_2 = (\lambda + 1)/x$, these functions satisfy condition (1.37). Substitution into (1.36) and (1.38) yields the following paracharges and Hamiltonian:

$$Q = \frac{1}{\sqrt{2}}\left((p + \frac{i\lambda}{x})a + \frac{i}{2x}a^2 a^\dagger\right), \tag{2.13a}$$

$$Q^\dagger = \frac{1}{\sqrt{2}}\left((p - \frac{i\lambda}{x})a^\dagger - \frac{i}{2x}aa^{\dagger 2}\right), \tag{2.13b}$$

$$\begin{aligned} H &= \frac{1}{2}\left(p^2 + \frac{(\lambda^2+\lambda+2)}{x^2} - \frac{\lambda}{x^2}[a^\dagger, a] - \frac{1}{x^2}a^\dagger a\right) \\ &= \frac{p^2}{2} + \frac{1}{2x^2}\begin{pmatrix} \lambda(\lambda-1) & 0 & 0 \\ 0 & \lambda(\lambda+1) & 0 \\ 0 & 0 & (\lambda+1)(\lambda+2)\end{pmatrix}. \end{aligned} \tag{2.13c}$$

Here again H, $D = -\frac{1}{4}\{x,p\}$ and $K = \frac{1}{2}x^2$ form an $O(2,1)$ algebra and, like in the supersymmetric case, commuting Q with K reveals the existence of the parasuperconformal generators:

$$S = -\frac{1}{\sqrt{2}}xa, \qquad S^\dagger = -\frac{1}{\sqrt{2}}xa^\dagger. \tag{2.14}$$

The set of symmetries is completed with the addition of the internal rotation generator Y given by

$$Y = \tfrac{1}{2}\left(J_3 + \lambda + \tfrac{1}{2}\right), \tag{2.15}$$

with

$$J_3 = \frac{1}{2}[a^\dagger, a]. \tag{2.16}$$

Our task is now to find the algebraic relations which are obeyed by this set of charges. It is not difficult to see that the commutation relations involving at least one bosonic generator are identical to those obtained in the supersymmetric case. (See the previous Section.) The fermionic products however are drastically altered. Instead of the anticommutation relations characteristic of superalgebras, one finds that the fermionic generators satisfy trilinear relations among themselves. These relations are explicitly given below.

$$Q^3 = Q^{\dagger 3} = S^3 = S^{\dagger 3} = 0, \tag{2.17a}$$

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4QH, \tag{2.17b}$$

$$S^2S^\dagger + SS^\dagger S + S^\dagger S^2 = 4SK, \tag{2.17c}$$

$$Q^2S^\dagger + QS^\dagger Q + S^\dagger Q^2 = 4Q(D - iY), \tag{2.17d}$$

$$S^2Q^\dagger + SQ^\dagger S + Q^\dagger S^2 = 4S(D + iY), \tag{2.17e}$$

$$Q^2S + QSQ + SQ^2 = 0, \tag{2.17f}$$

$$S^2Q + SQS + QS^2 = 0, \tag{2.17g}$$

$$Q\{S, Q^\dagger\} + S\{Q^\dagger, Q\} + Q^\dagger\{Q, S\} = 4Q(D + iY) + 4SH, \tag{2.17h}$$

$$S\{Q, S^\dagger\} + Q\{S^\dagger, S\} + S^\dagger\{S, Q\} = 4S(D - iY) + 4QK, \tag{2.17i}$$

(plus the hermitian conjugated relations).

It is instructive to reexpress the above relations in terms of the hermitian charges

$$Q_1 = \frac{1}{2}(Q + Q^\dagger), \qquad Q_2 = \frac{i}{2}(Q - Q^\dagger), \tag{2.18a}$$

$$S_1 = \frac{1}{2}(S + S^\dagger), \qquad S_2 = \frac{i}{2}(S - S^\dagger). \tag{2.18b}$$

One then finds $(i, j, k = 1, 2)$:

$$\begin{aligned} Q_k(\{Q_i, Q_j\} - 2\delta_{ij}H) + Q_i(\{Q_j, Q_k\} - 2\delta_{jk}H) \\ + Q_j(\{Q_k, Q_i\} - 2\delta_{ki}H) = 0, \end{aligned} \tag{2.19a}$$

$$\begin{aligned} S_k(\{S_i, S_j\} - 2\delta_{ij}K) + S_i(\{S_j, S_k\} - 2\delta_{jk}K) \\ + S_j(\{S_k, S_i\} - 2\delta_{ki}K) = 0, \end{aligned} \tag{2.19b}$$

$$\begin{aligned} Q_k(\{Q_i, S_j\} - 2\delta_{ij}D + 2\epsilon_{ij}Y) + Q_i(\{S_j, Q_k\} - 2\delta_{jk}D - 2\epsilon_{jk}Y) \\ + S_j(\{Q_k, Q_i\} - 2\delta_{ki}H) = 0, \end{aligned} \tag{2.19c}$$

$$\begin{aligned} S_k(\{Q_i, S_j\} - 2\delta_{ij}D + 2\epsilon_{ij}Y) + Q_i(\{S_j, S_k\} - 2\delta_{jk}K) \\ + S_j(\{S_k, Q_i\} - 2\delta_{ki}D - 2\epsilon_{ki}Y) = 0. \end{aligned} \tag{2.19d}$$

Observe the factorization of the expressions which, if set equal to zero, would represent the odd-odd part of the structure relations of $OSp(2,1)$.

In the next section, we shall be concerned with another Hamiltonian this one gotten by taking for the superpotentials

$$W_1 = \frac{\lambda}{x} + \omega x, \qquad W_2 = \frac{(\lambda + 1)}{x} + \omega x, \tag{2.20}$$

with ω an arbitrary real constant. In this case, equation (1.38a) gives for the parasupertranslation generator

$$\tilde{Q} = \frac{1}{\sqrt{2}}\left((p + i\omega x + \frac{i\lambda}{x})a + \frac{i}{2x}a^2 a^\dagger\right) = Q - i\omega S, \tag{2.21}$$

while correspondingly (1.38c) leads to the following Hamiltionian:

$$\tilde{H} = \frac{1}{2}\Big(p^2 + \frac{(\lambda^2+\lambda+2)}{x^2} - \frac{\lambda}{x^2}[a^\dagger, a] - \frac{1}{x^2}a^\dagger a + \omega^2 x^2 + \omega[a^\dagger, a] + \omega(2\lambda+1)\Big). \tag{2.22}$$

It is readily seen that $\tilde{H}$ can be expressed as a linear combination of some of our previous generators:

$$\tilde{H} = H + \omega^2 K + 2\omega Y. \tag{2.23}$$

Comparing with H, we see that $\tilde{H}$ possesses additional harmonic and constant terms.

2.4-Representations of the parasuperconformal algebra [8]

The Hamiltonian $\tilde{H}$ shares with H the same dynamical symmetries. We know therefore that its eigenstates belong to a representation space of the parasuperconformal algebra defined in Section 2.3. We shall now determine the spectrum and the wave functions of $\tilde{H}$ by constructing the relevant unitary representations of this algebra.

A complete set of quantum numbers is provided by the eigenvalues of Y (or equivalently of J_3) and of

$$R = \frac{1}{2\omega}H + \frac{\omega}{2}K, \tag{2.24}$$

the compact generator of $O(2,1)$. Let $m = 1, 0, -1$ denote the eigenvalues of J_3. In the present realization, the $O(2,1)$ Casimir operator $C = \frac{1}{2}(HK+KH) - D^2$ is given by

$$C = \tfrac{1}{4}\left[\lambda^2 + \lambda - 2\lambda m + m(m-1) - \tfrac{3}{4}\right]. \tag{2.25}$$

(In obtaining this expression we have used the fact that $a^\dagger a = J_+ J_- = J^2 - J_3^2 + J_3 = 2 - m(m-1)$.) For each value of m, the eigenstates of $\tilde{H}$ will therefore span a definite irreducible representation of $O(2,1)$. It is not difficult to see that the spectrum of $\tilde{H}$ is bounded from below. If we set the eigenvalues of C in the form $\Delta_m(\Delta_m - 1)$, it then follows from the representation theory of $O(2,1)$ that the eigenvalues of R are given by $\Delta_m + n$ with $n = 0, 1, 2, \ldots$. From (2.25), we get two solutions for Δ_m:

$$\Delta_m^\pm = \tfrac{1}{2} \pm \tfrac{1}{2}\left(\lambda - m + \tfrac{1}{2}\right). \tag{2.26}$$

Our basis states $|n,m\rangle$ are thus characterized by the following eigenvalue equations:

$$R|n,m\rangle = (\Delta_m + n)|n,m\rangle, \tag{2.27a}$$

$$Y|n,m\rangle = \tfrac{1}{2}(m + \lambda + \tfrac{1}{2})|n,m\rangle. \tag{2.27b}$$

At this point, it is convenient to introduce the ladder operators:

$$B_\pm = \frac{1}{2\omega}H - \frac{\omega}{2}K \mp iD, \tag{2.28}$$

$$F^L_\pm = \frac{1}{2\sqrt{2\omega}}Q \mp \frac{i\sqrt{2\omega}}{4}S, \tag{2.29a}$$

$$F^R_\pm = \frac{1}{2\sqrt{2\omega}}Q^\dagger \mp \frac{i\sqrt{2\omega}}{4}S^\dagger = (F^L_\mp)^\dagger. \tag{2.29b}$$

Note that F^L_+ is up to a normalization factor the parasupertranslation generator associated to $\tilde{H}$, that is $F^L_+ = 1/(2\sqrt{2\omega})\tilde{Q}$. In this Cartan-type basis the stucture relations of the parasuperconformal algebra read:

$$\begin{gathered}[R, B_\pm] = \pm B_\pm, \qquad [B_+, B_-] = -2R, \\ [Y, R] = [Y, B_\pm] = 0,\end{gathered} \tag{2.30}$$

$$\begin{gathered}[R, F^{L,R}_\pm] = \pm\tfrac{1}{2}F^{L\ R}_\pm, \\ [Y, F^L_\pm] = -\tfrac{1}{2}F^L_\pm, \qquad [Y, F^R_\pm] = \tfrac{1}{2}F^R_\pm, \\ [B_\pm, F^{L,R}_\pm] = 0, \qquad [B_\pm, F^{L,R}_\mp] = \mp F^{L,R}_\pm,\end{gathered} \tag{2.31}$$

$$\begin{gathered}(F^{L,R}_\pm)^3 = 0, \\ (F^L_\pm)^2 F^R_\mp + F^L_\pm F^R_\mp F^L_\pm + F^R_\mp (F^L_\pm)^2 = F^L_\pm (R \pm Y), \\ (F^L_\pm)^2 F^R_\pm + F^L_\pm F^R_\pm F^L_\pm + F^R_\pm (F^L_\pm)^2 = F^L_\pm B_\pm, \\ (F^L_\pm)^2 F^L_\mp + F^L_\pm F^L_\mp F^L_\pm + F^L_\mp (F^L_\pm)^2 = 0, \\ F^L_\pm(\{F^L_\mp, F^R_\mp\} - B_\mp) + F^L_\mp(\{F^R_\mp, F^L_\pm\} - R \mp Y) + F^R_\mp\{F^L_\pm, F^L_\mp\} = 0,\end{gathered} \tag{2.32}$$

plus the hermitian conjugated relations.

The action of the $O(2,1)$ raising and lowering operators is well known; one has

$$B_\pm\ |n,m\rangle = \sqrt{(\Delta_m + n)(\Delta_m + n \pm 1) - \Delta_m(\Delta_m - 1)}\ |n \pm 1, m\rangle. \tag{2.33}$$

The transformation law of the basis states under $F_{\pm}^{L,R}$ depends on the solution which is taken for Δ_m. Let us take first $\Delta_m = \Delta_m^+ = \frac{1}{2}(\lambda - m + \frac{3}{2})$. From the commutation relation of $F_{\pm}^L$ with Y we find

$$\begin{aligned} Y\left(F_{\pm}^L\,|n,m\rangle\right) &= F_{\pm}^L\left(Y - \tfrac{1}{2}\right)|n,m\rangle \\ &= \tfrac{1}{2}\left(m - 1 + \lambda + \tfrac{1}{2}\right)\left(F_{\pm}^L\,|n,m\rangle\right). \end{aligned} \tag{2.34}$$

We can therefore conclude that

$$F_{\pm}^L\,|n,m\rangle = c_{n,m}\,|n',m-1\rangle. \tag{2.35}$$

Similarly, from the commutation relations of $F_{\pm}^L$ with R we get

$$\begin{aligned} R\left(F_{\pm}^L\,|n,m\rangle\right) &= F_{\pm}^L(R \pm \tfrac{1}{2})\,|n,m\rangle \\ &= \left(\Delta_m^+ + n \pm \tfrac{1}{2}\right)F_{\pm}^L\,|n,m\rangle. \end{aligned} \tag{2.36}$$

From (2.35), we also have

$$R\left(F_{\pm}^L\,|n,m\rangle\right) = \left(\Delta_{m-1}^+ + n'\right)F_{\pm}^L\,|n,m\rangle. \tag{2.37}$$

Equating the right-hand-side of (2.36) and (2.37), we immediately solve for n' to find that

$$F_{\pm}^L\,|n,m\rangle = c_{n,m}\,|n - \tfrac{1}{2} \pm \tfrac{1}{2}, m-1\rangle. \tag{2.38}$$

Since the basis states are assumed to be orthonormalized,

$$|c_{n,m}|^2 = \langle n,m|\,F_{\mp}^R F_{\pm}^L\,|n,m\rangle. \tag{2.39}$$

The following identity

$$F_{\mp}^R F_{\pm}^L = \frac{1}{4}a^\dagger a\left[R \mp Y \pm (\lambda + 1)\right] \tag{2.40}$$

can be shown to hold in the present realization through a straightforward computation. It therefore determines the normalization factor $c_{n,m}$ up to a phase which we can consistently choose to be unity. In the end, one has

$$\begin{aligned} &F_{\pm}^L\,|n,m\rangle = \\ &\frac{1}{2}\sqrt{\left[2 - m(m-1)\right]\left[\Delta_m^+ + n \pm \tfrac{1}{2}(\lambda + \tfrac{3}{2} - m)\right]}\;|n - \tfrac{1}{2} \pm \tfrac{1}{2}, m-1\rangle. \end{aligned} \tag{2.41a}$$

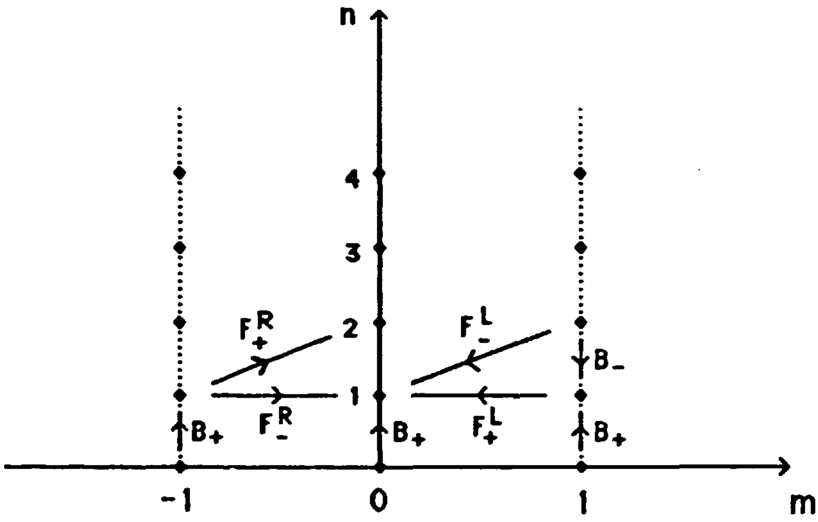

Fig. 1. The action of the parasupercharges $F_\pm^{L,R}$ which connect the towers of states that belong to the three $O(2,1)$ irreducible representations labelled by m is illustrated for the case $\Delta_m = \Delta_m^+$.

The action of $F_\pm^R$ is obtained by repeating the same steps and one finds

$$F_\pm^R\,|n,m\rangle = \frac{1}{2}\sqrt{[2-m(m+1)]\left[\Delta_m^+ + n \mp \tfrac{1}{2}(\lambda - \tfrac{1}{2} - m)\right]}\;|n+\tfrac{1}{2}\pm\tfrac{1}{2}, m+1\rangle. \qquad (2.41b)$$

The above actions are depicted in fig. 1. From (2.41), we see that there is a unique state, namely $|0,1\rangle$, which is annihilated by F_-^L and F_-^R:

$$F_-^L\,|0,1\rangle = F_-^R\,|0,1\rangle = 0. \qquad (2.42)$$

Repeated application of F_+^L and F_+^R on this state will then allow to reach all the other states. The spectrum of $\tilde{H}$ on this representation space is immediately derived from (2.23), one obtains

$$E_n = 2\omega(n + \lambda + 1), \qquad n = 0,1,2,\ldots . \qquad (2.43)$$

Since it is independent of m, all levels (including the ground state) exhibit a three-fold degeneracy. This is easily understood by recalling that F_+^L, which here does not change n but decreases m by one unit, commutes with $\tilde{H}$. Parasupersymmetry is thus spontaneously broken in this representation. The shape of the spectrum given by formula (2.43) is illustrated in fig. 2*a*.

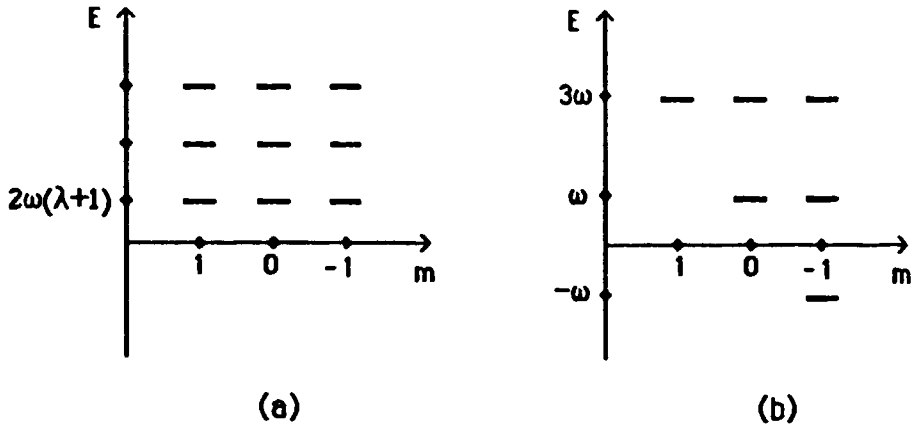

Fig. 2. Energy levels associated to the series (a) $\Delta_m = \Delta_m^+$ and (b) $\Delta_m = \Delta_m^-$.

If the other solution for Δ_m is adopted, that is, if $\Delta_m = \Delta_m^- = \frac{1}{2}(m - \lambda + \frac{1}{2})$, equations (2.41) should be replaced by

$$F_\pm^L \, |n,m\rangle = \frac{1}{2}\sqrt{[2 - m(m-1)]\,\left[\Delta_m^- + n \pm \tfrac{1}{2}(\lambda + \tfrac{3}{2} - m)\right]}\; |n + \tfrac{1}{2} \pm \tfrac{1}{2}, m - 1\rangle, \tag{2.44a}$$

$$F_\pm^R \, |n,m\rangle = \frac{1}{2}\sqrt{[2 - m(m+1)]\,\left[\Delta_m^- + n \mp \tfrac{1}{2}(\lambda - \tfrac{1}{2} - m)\right]}\; |n - \tfrac{1}{2} \pm \tfrac{1}{2}, m + 1\rangle. \tag{2.44b}$$

One would illustrate these actions by exchanging the roles of F^L and F^R in fig.1. In this case, $|0,-1\rangle$ is the state which is annihilated by F_-^R and F_-^L. As for the spectrum of $\tilde{H}$ on this module, it is given by

$$E_{n,m} = 2\omega(n + m + \tfrac{1}{2}). \tag{2.45}$$

Its form is sketched in fig. 2*b*. Here, parasupersymmetry is unbroken, the ground state has negative energy and a three-fold degeneracy is observed only above the second level.

To conclude we wish to obtain the wave functions. This is done by returning to the coordinate realization and setting

$$\psi_{n,m}(x) = \langle x | n, m\rangle. \tag{2.46}$$

Let us focus first on the case where $\Delta_m = \Delta_m^+$. From $\langle x|F_-^L|0,1\rangle = 0$, we get for $\psi_{0,1}(x)$ the following equation:

$$\left(\frac{d}{dx} + \omega x - \frac{\lambda}{x}\right)\psi_{0,1}(x) = 0, \tag{2.47}$$

whose normalized solution is

$$\psi_{0,1}(x) = \sqrt{\frac{\omega^{\lambda+\frac{1}{2}}}{\Gamma(\lambda+\frac{1}{2})}}\; x^{\lambda}\, e^{-\omega x^2/2}. \tag{2.48}$$

One may now easily determine $\psi_{0,0}(x)$ and $\psi_{0,-1}(x)$ with the help of (2.41*a*) by acting respectively once and twice on $\psi_{0,1}(x)$ with F_+^L; one thus finds

$$\psi_{0,m}(x) = \frac{1}{\sqrt{x\Gamma(2\Delta_m^+)}}\;(\omega x^2)^{\Delta_m^+}\, e^{-\omega x^2/2}, \qquad m = 1,0,-1. \tag{2.49}$$

The wave function $\psi_{n,m}(x)$ for an arbitrary state is then obtained by applying n times the raising operator B_+ to $\psi_{0,m}(x)$. Proceeding inductively one shows that

$$\begin{aligned}\psi_{n,m}(x) &= \frac{1}{\sqrt{(2\Delta_m^+ + n - 1)n}}\; B_+\,\psi_{n-1,m}(x)\\ &= \frac{1}{\sqrt{n!\,\Gamma(2\Delta_m^+ + n)}}\;\left(\frac{y}{\omega}\right)^{-1/4} y^{\Delta_m^+}\, e^{-y/2}\\ &\qquad\cdot\left[e^{y}\, y^{-(2\Delta-1)}\,\frac{d^n}{dy^n}\, e^{-y}\, y^{(2\Delta+n-1)}\right]\end{aligned} \tag{2.50}$$

with $y = \omega x^2$. Using Rodrigues' formula for the generalized Laguerre polynomials L_n^α, i.e.

$$n!\,L_n^\alpha(y) = e^{y}\, y^{-\alpha}\,\frac{d^n}{dy^n}\left(e^{-y} y^{n+\alpha}\right), \tag{2.51}$$

we finally get for $\psi_{n,m}(x)$:

$$\psi_{n,m}(x) = \sqrt{\frac{n!}{x\Gamma(2\Delta_m^+ + n)}}\;(\omega x^2)^{\Delta_m^+}\, e^{-\omega x^2/2}\, L_n^{2\Delta_m^+ - 1}(\omega x^2). \tag{2.52}$$

Upon repeating appropriately the above analysis, it is seen that the wave functions for $\Delta_m = \Delta_m^-$ can be obtained by substituting Δ_m^- for Δ_m^+ in (2.52).

$m \backslash \lambda$	$-\infty$		$-\frac{5}{2}$		$-\frac{3}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$		$\frac{3}{2}$		∞
1		$-$		$-$		$-$		$\pm$		$\pm$		$+$	
0		$-$		$-$		$\pm$		$\pm$		$+$		$+$	
-1		$-$		$\pm$		$\pm$		$+$		$+$		$+$	

Fig. 3. The values of λ and m for which the $O(2,1)$ series are unitary are given. The entry "+" indicates that Δ^+ is positive for the corresponding value of m and range of λ, the entry "−" similarly indicates that Δ^- is positive.

Let us add the following comments. The $O(2,1)$ representations arising in our problem will be unitary if Δ is positive. This condition is easily derived from the requirement that the eigenvalues of B_-B_+ be positive definite [9]. It can also be inferred from the fact that the Laguerre polynomials L_n^α are defined for $\alpha > -1$. For values of λ and m such that $\Delta_m^+ > 0$ and $\Delta_m^- > 0$ simultaneously, both sets of $O(2,1)$ representations provide admissible quantum states. The values of λ and m for which either series should be retained are given in Fig. 3. It should be remarked that Δ_m^+ and Δ_m^- cannot be positive at the same time for all values of m. It is clear however that the three $(m = 0, \pm 1)$ $O(2,1)$ irreducible representations associated to Δ^+ or Δ^- are needed to construct a unitary representation of the full parasuperconformal algebra. It therefore follows that for $-\frac{5}{2} < \lambda < \frac{3}{2}$, parasupersymmetry is at best realized only on a subset of the physical states. It is interesting to relate these observations with the self-adjointness properties of H. By computing $Ker(H \pm i)$, one can show that H is self-adjoint provided $|\lambda - m + \frac{1}{2}| \geq 1$ (see Ref. [10]). This condition is satisfied for all m when we have $\lambda \geq \frac{3}{2}$ or $\lambda \leq -\frac{5}{2}$. We thus note that H is self-adjoint in the cases where parasupersymmetry is unitarily implemented on all physical states.

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References

[1] E. Witten, *Nucl. Phys.* **B185** (1981) 513.

[2] V. A. Rubakov and V. P. Spiridonov, *Mod. Phys. Lett.* **A3** (1988) 1337.

[3] Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics* (Springer-Verlag, New York, 1982).

[4] E. P. Wigner, *Phys. Rev.* **77** (1950) 711.

[5] U. Niederer, *Helv. Phys. Acta* **45** (1972) 802;
V. DeAlfaro, S. Fubini and G. Furlan, *Nuovo Cimento* **A34** (1976) 569.

[6] E. D'Hoker, *Phys. Rev.* **D28** (1983) 1346;
S. Fubini and E. Rabinovici, *Nucl. Phys.* **B245** (1984) 17.

[7] E. D'Hoker, V. A. Kostelecky and L. Vinet, "Spectrum Generating Superalgebras" in *Dynamical Groups and Spectrum Generating Algebras*, edited by A. Barut, A. Bohm and Y. Ne'eman (World Scientific, Singapore, 1989).

[8] S. Durand and L. Vinet, "Conformal Parasupersymmetry in Quantum Mechanics", UdeM preprint (1989).

[9] B. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974).

[10] E. D'Hoker and L. Vinet, *Commun. Math. Phys.* **97** (1985) 391.