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REGULARITY OF EXPONENTIALLY HARMONIC FUNCTIONS

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1. The Theorems

Here is an example of a strictly (not uniformly) elliptic variational problem whose minima are smooth.

Let $u : M \to \mathbb{R}$ be a function defined on a smooth Riemannian manifold. The *exponential energy density* is the function $e(u) : M \to \mathbb{R}$ given by

(1.1) $$e(u)(x) = \exp |Du(x)|^2/2,$$

where $|Du(x)|^2 = g^{ij}(x)\, D_i u(x)\, D_j u(x)$ with (g_{ij}) representing the metric of M, $(g^{ij}) = (g_{ij})^{-1}$; and $D_i = \partial/\partial x^i$.

(1.2) We are interested in extrema of the functional

(1.3) $$\mathbb{E}(u) = \int e(u) \sqrt{\det g_{ij}}\, dx^1 \dots dx^m$$

on compact domains of M. Let

$$\mathbb{W}(M) = \{u \in W^{1,2}(M) : \mathbb{E}(u) < \infty\}.$$

Say that $u \in \mathbb{W}(M)$ *is a local* $\mathbb{E}$*-minimum* if for every $v \in \mathbb{W}(M)$ there is $\varepsilon > 0$ such that

(1.4) $$\mathbb{E}(u) \le \mathbb{E}(u + t(v - u)) \quad \text{for all } t \in [0, \varepsilon].$$

Here are our main results:

(1.5) **Theorem.** *Every local* $\mathbb{E}$*-minimum is in* $C^\infty(M)$.

(1.6) **Theorem.** *Suppose that* M *is compact and has smooth boundary* ∂M. *If* $\varphi \in \mathcal{W}(M)$, *then there is a unique* $\mathbb{E}$*-minimum* $u \in \mathcal{W}(M, \varphi) = \{w \in \mathcal{W}(M) : w = \varphi$ *on* $\partial M\}$. *Furthermore,* $u \in C^\infty(M \backslash \partial M)$.

(1.7) The Euler-Lagrange operator formally associated with $\mathbb{E}$ is the quasi-linear strictly elliptic operator

$$\Delta u = \mathrm{div}\,(e(u)Du) = g^{ij}\,\nabla_j(e(u)D_i u) \tag{1.8}$$

$$= e(u)Q^{ij}(u)\nabla_j\, D_i u,$$

where

$$Q^{ij}(u) = g^{ij} + g^{ip}\, g^{iq}\, D_p u\, D_q u$$

and ∇_j denotes the covariant derivative; thus $\nabla_j D_i u = D_{ij}u - \Gamma^k_{ij} D_k u$. A C^2 function $u : M \to \mathbb{R}$ *is exponentially harmonic* if $\Delta u \equiv 0$.

(1.9) In the context of Theorem 1.6, it is well known (Theorem 11.9 in [GT, p. 289]) that *if* $\varphi \in C^3(M)$ *then* $u \in C^3$ *is a solution of the Dirichlet problem*

$$\Delta u = 0 \ \ \textit{with} \ \ u = \varphi \ \ \textit{on} \ \ \partial M \tag{1.10}$$

iff u *is the unique* $\mathbb{E}$*-minimum in*

$$\{w \in C^3(M) : w = \varphi \ \text{on} \ \partial M\}.$$

Clearly that problem is equivalent to solving

$$Q^{ij}(u)\,\nabla_j D_i u = 0 \ \text{ with } \ u = \varphi \ \text{ on } \ \partial M \ \text{ in } \ C^3(M). \tag{1.11}$$

If the metric on M is flat ($g_{ij} = \delta_{ij}$ in suitable charts), then standard methods provide a solution of (1.11). That has been verified by Eells-Lemaire, using [S, Theorem 1, p. 452]. In general, however, in following the basic existence/regularity programme of [S, §2] we face a serious complication: The integrand of $\mathbb{E}$ involves the domain variable, so we cannot appeal to the maximum principle to obtain interior gradient estimates from those on the boundary.

Nonetheless, *we do establish a key result (Theorem 2.11) on the boundedness of positive subsolutions of certain non-uniformly elliptic equations.* That is used to obtain the required interior gradient estimates from which Theorems 1.5 and 1.6 follow by standard regularity methods.

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2. Boundedness of subsolutions

Throughout this section M denotes an open subset of $\mathbb{R}^n$, with induced flat metric. Let $K(\rho)$ be a disc of radius ρ contained in M. Take $u \in W^{1,m}(M)$ with $n > m > 1$, and let $A(k, \rho) = \{x \in K(\rho) : u(x) > k\}$; and

$\lambda[A(k, \rho)]$ its Lebesgue measure.

(2.1) **Lemma.** *For some* $\hat{k}$ *and* $\sigma_0 < 1$, *take* ρ, σ *such that* $\rho_0 - \sigma_0\rho_0 \le \rho - \sigma\rho <$ $\rho \le \rho_0$. *Assume that for any concentric discs* $K(\rho), K(\rho - \sigma\rho), K(\rho_0)$ *and* $k > \hat{k}$ *the function* $u \in W^{1,m}(M)$ *satisfies*

$$\int_{A(k,\,\rho-\sigma\rho)} |Du|^m\, dx \le \gamma \left\{ (\sigma\rho)^{-m} \left[\int_{A(k,\,\rho)} (u-k)^{m\delta}\, dy \right]^{1/\delta} + \rho^{-\varepsilon n}\, k^\alpha\, \lambda\big[A(k,\rho)\big]^{1+\varepsilon-m/n} \right\} \tag{2.2}$$

where γ, δ, α *and* ε *are positive constants with* $\varepsilon \le m/n$, $1 \le \delta < n/(n-m)$, $m \le \alpha < \varepsilon m + m$. *Then* $\|u\|_{L^\infty(K(\rho_0 - \sigma_0\rho_0))}$ *is bounded by a constant depending only on* $\sigma_0, \hat{k}, n, m, \gamma, \delta, \varepsilon, \alpha$ *and the average*

$$a\ \rho_0^{\,n} \int_{A(\hat{k},\rho_0)} (u(x) - \hat{k})^{m}\, dx.$$

Proof. If $\delta = 1$ that is just Lemma 5.4 of [LU, p. 76]. We assume $\delta > 1$ and make the necessary technical adjustments.

We can suppose $\rho_0 = 1$, and transform (2.2) into

$$(2.3)\qquad \int_{A(k,\rho-\sigma\rho)} |Du|^m dy \le \gamma \left\{ \sigma^{-m} \left[\int_{A(k,\rho)} (u-k)^{m\delta} dy \right]^{1/\delta} + k^{\alpha} \lambda \left[A(k,\rho)\right]^{1+\varepsilon-m/n} \right\}$$

for $1 - \sigma_0 \le \rho - \sigma\rho < \rho \le 1$. Take $k_0 > \max(\hat{k}, 1)$ and define the sequences

$$\rho_h = 1 - \sigma_0 + \sigma_0/2^h \quad \text{and} \quad k_h = 2k_0 - k_0/2^h$$

for integers $h \ge 0$. Set

$$J_h = \left[\int_{A(k_h,\rho_h)} (u-k_h)^{m\delta} dy \right]^{1/\delta}$$

and

$$J_h(y) = \zeta(2^{h+1}(|y| - 1 + \sigma_0)),$$

where ζ is a non-increasing smooth function on $\mathbb{R}$ with $\zeta(t) \equiv 1$ for $t \le \sigma_0$ and $\zeta(t) \equiv 0$ for $t \ge 3\sigma_0/2$. Thus $\zeta_h(K(\rho_{h+1})) = 1$, $\zeta_h(\mathbb{R}^n \setminus K(\bar{\rho}_h)) = 0$, where $\bar{\rho}_h = (\rho_h + \rho_{h+1})/2 \ge \rho_{h+1}$; and $A(k, \rho_{h+1}) \subset A(k, \bar{\rho}_h)$.

In the following estimates $c_1, c_2, \ldots$ denote positive constants depending only on $\sigma_0, \hat{k}, n, m, \gamma, \delta, \varepsilon, \alpha$, and a. By the Sobolev inequality (2.12 in [LU, p. 45]),

$$(2.4)\qquad J_{h+1} \le \left[\int_{A(k_{h+1},\bar{\rho}_h)} (u(y) - k_{h+1})^{m\delta} \zeta_h^{m\delta} dy \right]^{1/\delta}$$

$$\le c_1 \lambda \left[A(k_{h+1},\bar{\rho}_h)\right]^{1/\delta-(n-m)/n} \int_{A(k_{h+1},\bar{\rho}_h)} \left[|Du|^m \zeta_h^m + (u-k_{h+1})^m |D\zeta_h|^m \right] dy$$

$$\le c_1 \lambda \left[A(k_{h+1},\bar{\rho}_h)\right]^{1/\delta-1+m/n} \left[\int_{A(k_{h+1},\bar{\rho}_h)} |Du|^m dy \right.$$

$$\left. + \lambda \left[A(k_{h+1},\bar{\rho}_h)\right]^{1-1/\delta} \left(\int_{A(k_{h+1},\bar{\rho}_h)} (u-k_{h+1})^{m\delta} |D\zeta_h|^{m\delta} dy \right)^{1/\delta} \right]$$

$$\le c_1 \lambda \left[A(k_{h+1},\bar{\rho}_h)\right]^{1/\delta-1+m/n} \left\{ \int_{A(k_{h+1},\bar{\rho}_h)} |Du|^m dy \right.$$

$$\left. + c_2 2^{mh} \lambda \left[A(k_{h+1},\bar{\rho}_h)\right]^{1-1/\delta} J_h \right\},$$

where $c_2 = \max\left(|\zeta'(t)|^m : t \in [\sigma_0, 3\sigma_0/2] \right)$.

Next, putting $k = k_{h+1}$, $\rho = \rho_h$, $\rho - \sigma\rho = \bar{\rho}_h$ into (2.3) gives

$$(2.5)\qquad \int_{A(k_{h+1},\bar{\rho}_h)} |Du|^m dy \le$$

$$\gamma \left\{ 2^{m(h+3)} \left[\int_{A(k_{h+1},\rho_h)} (u-k_{h+1})^{m\delta} dy \right]^{1/\delta} + k_{h+1}^{\alpha} \lambda \left[A(k_{h+1},\rho_h)\right]^{1+\varepsilon-m/n} \right\}$$

$$\le c_3 \left\{ 2^{mh} J_h + k_0^{\alpha} \lambda \left[A(k_{h+1},\rho_h)\right]^{1+\varepsilon-m/n} \right\}.$$

On the other hand,

$$(2.6)\qquad J_h \ge \left[\int_{A(k_{h+1},\rho_h)} (u-k_h)^{m\delta} \right]^{1/\delta} \ge (k_{h+1}-k_h)^m \lambda\left[A(k_{h+1},\rho_h)\right]^{1/\delta}$$

$$\ge 2^{-m(h+1)} k_0^m \lambda[A(k_{h+1},\rho_h)]^{1/\delta}.$$

By (2.4), (2.5) and (2.6) we get

$$J_{h+1} \le c_1 \left(2^{m(k+1)} k_0^{-m} J_h\right)^{1-\delta+m\delta/n} \Big\{ c_3\, 2^{mh} J_h$$

$$+ c_3 k_0^{\alpha} \left(2^{m(h+1)} k_0^{-m} J_h\right)^{\delta+\delta\varepsilon-m\delta/n} + c_2\, 2^{mh}\left(2^{m(h+1)} k_0^{-m} J_h\right)^{\delta-1} J_h \Big\}$$

$$\le c_4 \Big\{ 2^{2mh-mh\delta} k_0^{-m+m\delta-m^2\delta/n} J_h^{2-\delta+m\delta/n}$$

$$+ 2^{mh\delta-m^2h\delta/n+mh\delta} k_0^{\alpha-m-m\delta\varepsilon} J_h^{\delta-m\delta/n+\delta\varepsilon}$$

$$+ 2^{mh\delta} k_0^{-m\delta+m} J_h^{\delta} \Big\}$$

$$\le c_5^h k_0^{-\alpha_1} \left\{ J_h^{2-\delta+m\delta/n} + J_h^{\delta-m\delta/n+\delta\varepsilon} + J_h^{\delta} \right\}$$

where $\alpha_1 = \min(m - m\delta + m^2 + m^2\delta/n, -\alpha + m + m\delta\varepsilon, m\delta - m) > 0$.
Since every $J_h \le a$ and

$$\min(2-\delta+m\delta/n,\ \delta - m\delta/n + \delta\varepsilon,\ \delta) > 1,$$

there is $\alpha_2 > 0$ for which every

$$J_{h+1} \le c_6^h k_0^{-\alpha_1} J_h^{1+\alpha_2}.$$

We are now in a position to apply Lemma 4.7 of [LU, p. 66]: For k_0 sufficiently large (depending on σ_0, k, n, m, γ, δ, ε, α, a), $J_{h+1} \to 0$ as $h \to \infty$; consequently

$$\| u \|_{L^\infty(K(\rho_0-\sigma_0))} \le 2k_0. \qquad \square$$

(2.7) **Lemma.** *For any* $1 \le i, j \le n$ *let* A_{ij}, B_j, C *and* θ *be measurable functions on* M. *Assume that* $B_j^2, C, \theta^2 \in L^p(M)$ *for some* $p > n/2$, *and that*

$$(2.8)\qquad \sum_{j=1}^n \xi_j^2 \le \sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \le (1+\theta^2(x)) \sum_{j=1}^n \xi_j^2$$

for every $x \in M$ *and* $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$. *Let* w *be a nonnegative function in* $C^2(M)$ *such that*

$$(2.9)\qquad \sum_{i,j=1}^n D_j(A_{ij} D_i w) + \sum_{j=1}^n B_j D_j w + Cw \ge 0$$

on M. *Then for any open relatively compact subset* M_1 *of* M, $\| w \|_{L^\infty(M_1)}$ *depends only on* M_1, $\|\theta\|_{2p}$, $\|B_j\|_{2p}$, $\|C\|_p$ *and* $\|w\|_{L^1(M_1)}$.

Proof. Take a disc $K(\rho)$ in M_1 with $\lambda[K(\rho)] < 1$, and $\zeta \in C_0^\infty(K(\rho), [0,1])$. For $k \ge 0$ put $A(k) = \{x \in M : w(x) > k\}$. Multiplying (2.9) by $-\zeta^2 \max(w-k, 0)$ and integrating gives

$$\int_{A(k)} A_{ij}\zeta^2 D_j w\, D_i w\, dx \le -2 \int_{A(k)} A_{ij}\, \zeta (w-k) D_j w\, D_i\zeta\, dx$$

$$+ \int_{A(k)} B_j \zeta^2 (w-k) D_j w\, dx + \int_{A(k)} C\zeta^2\, w(w-k)\, dx.$$

Let q be the conjugate index of p. Then for any $\varepsilon > 0$ we apply the Cauchy-Schwartz and Hölder inequalities to obtain

$$2 \int_{A(k)} |\zeta(w-k) A_{ij} D_j w D_i\zeta| \, dx$$

$$\le \varepsilon \int_{A(k)} \zeta^2 A_{ij} D_i w D_j w \, dx + \frac{1}{\varepsilon} \int_{A(k)} (w-k)^2 A_{ij} D_i\zeta D_j\zeta \, dx$$

$$\le \varepsilon \int_{A(k)} \zeta^2 A_{ij} D_i w D_j w \, dx + \frac{1}{\varepsilon} \int_{A(k)} (1+\theta^2)(w-k)^2 |D\zeta|^2 \, dx$$

$$\le \varepsilon \int_{A(k)} \zeta^2 A_{ij} D_i w D_j w \, dx + \frac{1}{\varepsilon} \left[\int_{A(k)} (1+\theta^2)^p \, dx \right]^{1/p} \left[\int_{A(k)} (w-k)^{2q} |D\zeta|^{2q} \, dx \right]^{1/q}.$$

Also,

$$\int_{A(k)} B_j \zeta^2 (w-k) D_j w \, dx \le \frac{\varepsilon}{2} \int_{A(k)} \zeta^2 |Dw|^2 \, dx + \frac{1}{2\varepsilon} B^2 \zeta^2 (w-k)^2 \, dx;$$

$$\int_{A(k)} |C| \zeta^2 w(w-k) dx \le \int_{A(k)} |C| \zeta^2 w^2 \, dx.$$

With (2.9) and these three estimates we can argue in a manner similar to the proof of Theorem 13.1 in [LU, pp. 197–199]: For any sufficiently small $\varepsilon > 0$ we have

$$\int_{A(k)} |Dw|^2 \zeta^2 \, dx \le c_0 \int_{A(k)} \zeta^2 A_{ij} D_j w D_j w \, dx$$

$$\le c_1 \left\{ \left[\int_{A(k)} (w-k)^{2q} |D\zeta|^{2q} \, dx \right]^{1/q} + \int_{A(k)} (w-k)^2 B^2 \zeta^2 \, dx + \int_{A(k)} |C| \zeta^2 w^2 \, dx \right\},$$

where $c_0, c_1, \ldots$ denote positive constants depending only on M_1, $\|\theta\|_{2p}$, $\|B_j\|_{2p}$ and

$\|C\|_p$. With $A(k,\rho) = A(k) \cap K(\rho)$ we again apply Hölder's inequality.

(2.10)
$$\int_{A(k,\rho)} |Dw|^2 \zeta^2 \, dx$$

$$\le c_2 \left\{ \left[\int_{A(k,\rho)} (w-k)^{2q} |D\zeta|^{2q} \, dx \right]^{1/q} + (k^2+1)\,\lambda\left[A(k,\rho)\right]^{1/q} \right\}.$$

For any $\sigma \in (0,1)$ choose ζ so that $\zeta(K(\rho - \sigma\rho)) = 1$ and $|D\zeta| \le c/\sigma\rho$. Now because $p > n/2$ and $\lambda[A(k,\rho)] < 1$, from (2.10) we obtain

$$\int_{A(k,\rho)} |Dw|^2 \, dx \le$$

$$c_4 \left\{ (\sigma\rho)^{-2} \left[\int_{A(k,\rho)} (w-k)^{2q} \, dx \right]^{1/q} + \rho^{-n\varepsilon} (k^2+1)\,\lambda\left[A(k,\rho)\right]^{1+\varepsilon-2/n} \right\}.$$

We see that $1 < q < n/(n-2)$, and for sufficiently large k we can choose $\varepsilon \in (0, 2/n)$, and $\alpha \in [2, 2+2\varepsilon]$ for which $k^2 + 1 \le k^\alpha$. Therefore w satisfies the hypotheses of Lemma 2.1 with $m = 2$, $\delta = q$. The conclusion of Lemma 2.7 now follows at once. □

Finally, arguing as in the proof of Theorem 13.1 in [LU, p. 199], we obtain

(2.11) **Theorem.** *If* $w \in C^0(M) \cap C^2(M \backslash \partial M)$ *is a nonnegative function satisfying* (2.9), *then* $\|w\|_\infty$ *depends only on* M, $\|\theta\|_{2p}$, $\|B_j\|_{2p}$, $\|C\|_p$, $\|w\|_1$ *and* $\|w|_{\partial M}\|_\infty$.

(2.12) **Remark.** Similar arguments produce analogues of Lemma 2.7 and Theorem 2.11 for weak solutions $w \in W^{1,2}(M)$ of the inhomogeneous form

$$\sum_{i,j=1}^{n} D_j(A_{ij} D_i w) + \sum_{j=1}^{n} B_j D_j w + Cw = f$$

of (2.9), where $f \in L^p(M)$ and $p > n/2$.

3. Our Auxiliary Equation (3.6)

In this section M denotes a compact Riemannian manifold with smooth boundary.

(3.1) **Lemma.** *If* $u \in C^0(M) \cap C^3(M \backslash \partial M)$ *is a solution of the Dirichlet problem* (1.11) *with boundary values* $\varphi \in C^3(M)$, *then* $\| Du|_{\partial M} \|_\infty$ *is bounded by a constant depending only on* $\|\varphi\|_{C^3(M)}$.

Proof. We use the method of barrier functions [S, Theorem 1, p. 432]. For $t > 0$ and $\eta > 0$, set $\theta(t) = \eta \log(1+t)$; thus

$$\theta'(t) = \frac{\eta}{1+t} \quad \text{and} \quad \theta''(t) = \frac{-\theta'(t)}{1+t}.$$

We shall consider u in a tubular neighbourhood of ∂M in M, and work in a chart in which x^1 is the distance $d(x)$ of x to ∂M. Then

$$D_i\theta(d(x)) = \theta'(x^1)\delta_i^1 \quad \text{and} \quad D_{ij}\theta(d(x)) = \frac{-\theta'(x^1)}{1+x^1}\delta_i^1\delta_j^1.$$

Set $w_\pm(x) = \pm\theta(d(x)) + \varphi(x)$; then

$$D_i w_\pm(x) = \frac{\pm\eta}{1+x^1}\delta_i^1 + D_i\varphi(x)$$

$$D_{ij}w_\pm(x) = \frac{\mp\eta}{(1+x^1)^2}\delta_i^1\delta_j^1 + D_{ij}\varphi(x).$$

Write (1.8) in the form $Au/e(u) = Q(u)$ and substitute $w_\pm$ for u to obtain

$$Q(w_\pm) = A_3\eta^3 + A_2\eta^2 + A_1\eta + A_0,$$

where the coefficients are functions of g^{ij}, Γ^k_{ij}, $D_i\varphi$ and $D_{ij}\varphi$; in fact,

$$A_3 = \mp g^{i1}g^{j1}(\delta_i^1\delta_j^1 + \Gamma^1_{ij}(1+x^1))/(1+x^1)^4$$

$$= \mp 1/(1+x^1)^4 \quad \text{because} \quad g^{i1} = \delta^{i1} \text{ and } \Gamma^1_{11} = 0;$$

that has constant sign in a sufficiently thin tube.

We can choose $\eta > 0$ depending on $\|\varphi\|_{C^3(M)}$ so that

$$Qw_- > 0 = Qu > Qw_+.$$

Because $w_- \le u = \varphi \le w_+$ on ∂M, we can apply the comparison principle [GT, p. 263] to conclude that $w_- \le u \le w_+$ in a neighbourhood of ∂M. From that it follows that for any point $y \in \partial M$, $|Du(y)| \le \max(|Dw_-(y)|, |Dw_+(y)|) \le \eta + |D\varphi(y)|$. That concludes the proof of the lemma. □

(3.2) **Lemma.** *If* $u \in C^0(M) \cap C^3(M \backslash \partial M)$ *is a solution of* (1.11), *then* $\|Du\|_\infty$ *is bounded by a constant depending only on* $\|\varphi\|_{C^3(M))}$ *and* $\mathbb{E}(u)$; *for any open relatively compact subset* M_1 *of* M, $\|Du\|_{L^\infty(M_1)}$ *depends only on* M_1 *and* $\mathbb{E}_{M_1}(u)$.

Here $\mathbb{E}_{M_1}(u)$ denotes the integral (1.3) evaluated on M_1.

Proof. We shall abbreviate D_pu by u_p and the covariant derivative $\nabla_i D_p u$ by $u_{p,i}$. Set $v = e(u)$. Then

$$v_i = vg^{pq}u_{p,i}u_q; \qquad \text{and}$$

(3.3)

$$v_{i,j} = g^{pq}(vu_{p,ij}u_q + vu_{p,i}u_{q,j} + v_j u_{p,i}u_q).$$

Step 1. From (1.10) we obtain

$$0 = g^{ij}(vu_{i,j} + v_j u_i). \tag{3.4}$$

And applying ∇_s to both sides of (1.11) gives

$$0 = Q^{ij}(u)u_{i,\, js} + 2g^{ip}\, g^{iq}\, u_p\, u_{q,\, s}\, u_{i,\, j}.$$

Multiply that by $vg^{rs}u_r$ and apply (3.3):

(3.5) $$vQ^{ij}(u)g^{rs}\, u_r\, u_{i,\, js} = -2v^{-1}\, g^{js}v_j\, v_s.$$

Step 2. Next we compute

$$\nabla_j(Q^{ij}(u)v_i) = Q^{ij}(u)g^{rs}(vu_{r,\, ij}\, u_s + vu_{r,\, i}\, u_{s,\, j} + v_j\, u_{r,\, i}\, u_s)$$

$$+\, g^{ir}\, g^{js}\, u_{r,\, j}\, u_s\, v_i + g^{ir}\, g^{js}\, u_r\, u_{s,\, j}\, v_i$$

$$= T_1 + \ldots + T_5\, .$$

We calculate each of these terms separately; for T_1 we use the commutation formula

$$u_{i,\, js} - u_{i,\, sj} = u_k\, R^k{}_{ijs},$$

where R denotes the curvature tensor of g; and $u_{i,\, j} = u_{j,\, i}$. Consequently, by (3.5)

$$T_1 = vQ^{ij}(u)g^{rs}\, u_{r,\, ij}\, u_s$$

$$= -2v^{-1}\, g^{js}\, v_j\, v_s - vQ^{ij}(u)g^{rs}\, u_r\, u_k\, R^k{}_{ijs}.$$

Also, using (3.3) repeatedly,

$$T_2 = vQ^{ij}(u)g^{rs}\, u_{r,\, i}\, u_{s,\, j}$$

$$= vg^{ij}\, g^{rs}\, u_{r,\, i}\, u_{s,\, j} + v^{-1}\, g^{rs}\, v_r\, v_s.$$

$$T_3 = v^{-1}\, g^{ij}\, v_i\, v_j + v^{-1}\, g^{ip}\, g^{jq}\, v_i\, v_j\, u_p\, u_q.$$

$$T_4 = v^{-1}\, g^{ir}\, v_i\, v_r.$$

$$T_5 = -\, v^{-1}\, g^{ir}\, g^{js}\, u_r\, u_s\, v_i\, v_j, \text{ using (3.4).}$$

Thus

$$\nabla_j(Q^{ij}(u)v_i) = T_1 + \ldots + T_5$$

$$= -\, vQ^{ij}(u)g^{rs}u_r u_k\, R^k{}_{ijs} + vg^{ij}\, g^{rs}\, u_{r,\, i}\, u_{s,\, j} + v^{-1}\, g^{rs}\, v_r v_s.$$

Step 3. Rewrite that *as an equation in* v:

(3.6) $$D_j(Q^{ij}(u)v_i) + Q^{kh}(u)\Gamma^j_{h\, j}\, v_k + Q^{ij}(u)\, g^{rs}\, u_r u_s R^k{}_{ijs} v$$

$$= v\,|\,\nabla Du\,|^2 + v^{-1}\,|\,Dv\,|^2.$$

The right member is non-negative so the left has the form (2.9). The hypotheses of Lemma 3.1 and Theorem 2.11 are satisfied. We conclude that $\|\, v\,\|_\infty$ depends only on $\|\,\varphi\,\|_{C^3(M)}$ and $\mathbb{E}(u)$; and the same for $\|\, Du\,\|_\infty$.

Similarly for M_1: By Lemma 2.7, $\|\, v\,\|_{L^\infty(M_1)}$ and hence $\|\, Du\,\|_{L^\infty(M_1)}$ depend only on M_1 and $\mathbb{E}_{M_1}(u)$. □

4. Proof of the Theorems

We begin with two standard results, in the context of Section 1.

(4.1) **Lemma.** *Let* $u \in C^0(M) \cap C^3(M\backslash\partial M)$ *be a solution of the Dirichlet problem* (1.11). *Then* $u \in C^{1,\alpha}(M)$ *for some* $\alpha > 0$. *Furthermore,* α *and* $\|\, u\,\|_{C^{1,\alpha}(M)}$ *depend only on* $\|\,\varphi\,\|_{C^3(M)}$. *If* M_1 *is any open relatively compact subset of* M *then* $u \in C^{1,\beta}(M_1)$, *where* β *and* $\|\, u\,\|_{C^{1,\beta}(M_1)}$ *depend on* M_1 *and* $\mathbb{E}_{M_1}(u)$.

Proof. Equation (3.6) satisfies the hypotheses of Theorem 7.2 in [LU, p. 290]. We conclude that $v \in C^\alpha(M\backslash\partial M)$ for an α depending only on $\|\,\varphi\,\|_{C^3(M)}$. Using that in (1.10), we can now apply standard regularity theory to verify each assertion in the lemma. □

(4.2) **Proposition.** *For any* $\varphi \in C^3(M)$ *the Dirichlet problem* (1.10) *has a unique solution* $u \in C^0(M) \cap C^3(M\backslash\partial M)$; *moreover,* u *is the unique* $\mathbb{E}$*-minimum in* $\mathcal{W}(M, \varphi)$. *Also, for any open relatively compact subset* M_1 *of* M *there is* $\alpha > 0$ *such that* α *and* $\| u \|_{C^{1,\alpha}(M_1)}$ *depend only on* M_1 *and* $\mathbb{E}_{M_1}(\varphi)$.

This is an application of the fixed point method described in Theorem 11.8 in [GT, p. 287], using (1.9) and Lemma 4.1.

(4.3) **Proof of Theorem 1.5.** Let u be a local $\mathbb{E}$-minimum. We can find a sequence $(u_k)_{k \geq 1} \subset C^3(M)$ which converges to $u \in W^{1,2}(M)$, and

$$\lim_{k \to \infty} \mathbb{E}(u_k) = \mathbb{E}(u).$$

Take a small geodesic disc M_0 in M and let

$$\mathcal{W}(M_0, u_k) = \{w \,|\, _{M_0} : w \in \mathcal{W}(M) \text{ and } w = u_k \text{ on } \partial M_0\}.$$

By Proposition 4.2 there is a unique $\mathbb{E}_{M_0}$–minimum $w_k \in \mathcal{W}(M_0, u_k)$ such that $w_k \in C^{1,\alpha}(M_1)$ for any relatively compact M_1 in M_0, where α and $\| w_k \|_{C^{1,\alpha}(M_1)}$ depend only on dist $(M_1 \partial M_0)$ and $\mathbb{E}(u_k)$.

Therefore we can find a subsequence of (w_k) – still called (w_k) – which converges weakly to some $w \in W^{1,p}(M_0)$ for each $p > 1$; and for relatively compact M_1 in M_0 there is $\beta > 0$ such that $(w_k \,|\, _{M_1})$ converges to $w \,|\, _{M_1}$ in $C^{1,\beta}(M_1)$. Hence

$$\mathbb{E}_{M_0}(w) \leq \liminf \mathbb{E}_{M_0}(w_k),$$

so w is an $\mathbb{E}_{M_0}$-minimum in $\mathcal{W}(M_0, u)$.

Since $w \in \bigcap \left\{W^{1,p}(M_0) : p > 1\right\}$ and $u \in \bigcap \left\{W^{1,p}(M) : p > 1\right\}$, we see that

$$w = u \text{ on } \partial M_0, \; w \in C^{0,\alpha}(\overline{M}_0), \; u \in C^{0,\alpha}(M).$$

Consequently the functions

$$v_1(x) = \begin{cases} w(x) & \text{if } x \in M_0 \\ u(x) & \text{if } x \in M \setminus M_0 \end{cases}$$

$$v_2(x) = \begin{cases} u(x) + \varepsilon(v_1(x) - u(x)) & \text{if } x \in M_0 \\ u(x) & \text{if } x \in M \setminus M_0 \end{cases}$$

are both in $\mathcal{W}(M)$, where ε is taken from the definition (1.4) of u as a local $\mathbb{E}$-minimum.

Clearly $e(v_1) \in C^{0,\alpha}(\overline{M}_0)$ and

(4.4) $$\mathbb{E}(v_1) \leq \mathbb{E}(u).$$

On the other hand, strict convexity of the exponential function insures that

$$e(v_2) \leq (1 - \varepsilon)\, e(u) + \varepsilon e(v_1)$$

at every point of M; and that inequality is strict if $| Du(x) |^2 \neq | Dv_1(x) |^2$. Taking (1.4) and (4.4) together gives

(4.5) $$e(u) = e(v_1) \text{ a.e. on } M.$$

Therefore, the solution of the Dirichlet problem

(4.6) $$\operatorname{div}(e(v_1)Du) = 0 \text{ with } u = \varphi \text{ on } \partial M_0$$

is smooth. We conclude that our local $\mathbb{E}$-minimum $u \in C^\infty(M)$. □

(4.7) **Proof of Theorem 1.6.** Take $\varphi \in \mathcal{W}(M)$ and $(u_k) \subset C^0(M) \cap C^3(M\backslash\partial M)$ a minimizing sequence in $\mathcal{W}(M, \varphi)$. Thus (u_k) is bounded in every $W^{1,p}(M\backslash\partial M)$; and we can suppose that (u_k) converges weakly to u there. It follows that u is an $\mathbb{E}$-minimum in $\mathcal{W}(M)$, by Serrin's theorem [M, p. 22]. The argument proceeds as in the proof of Theorem 1.5. □

(4.8) **Remark.** It is a straightforward task to retrace the steps in the proofs of Lemmas 3.1, 3.2, 4.1, and Proposition 4.2, to see how the estimates depend on (g_{ij}) and (Γ^k_{ij}).

(4.9) **Remark.** In the early stages of this work, John Ball established (at our request) that if $\varphi \in C^3(M)$, any $\mathbb{F}$-minimum $u \in W^{1,1}(M)$ with $u = \varphi$ on ∂M is a weak solution of (1.10); and furthermore, that $|Du|^2 e(u) \in L^1_{loc}(M)$.

(4.10) **Remark.** Our main results are valid for a more general class of equations of the form $\operatorname{div}(\rho(|Du|^2)Du) = 0$, where $\rho : M \times \mathbb{R} \to \mathbb{R}^+$ is a positive smooth density. That is the Euler-Lagrange equation of the functional

$$F(u) = \frac{1}{2} \int \int_0^{|Du(x)|^2} \rho(x, \xi)d\xi \sqrt{\det g_{ij}(x)}\ dx^1 \dots dx^m .$$

We require at least strict ellipticity, which can be expressed by

$$0 < A \le \frac{d(\xi \rho^2(\xi))}{d\xi}$$

for some constant A; however, our proof of Lemma 3.1 requires the stronger condition of strict monotonicity of ρ, as well.

By way of contrast, for the minimal graph equation [GT, p.1] we have $\rho(\xi) = (1+\xi)^{-1/2}$; in this case we have

$$0 < \frac{d(\xi \rho^2(\xi))}{d\xi} \le B < \infty ;$$

i.e., elliptic, but not strictly so.

For flat domains $M \subset \mathbb{R}^2$ the minimal graph equation takes the form

$$(1 + |D_2u|^2)D_{11}u - 2D_1u\, D_2u\, D_{12}u + (1 + |D_1u|^2)D_{22}u = 0,$$

which is the adjugate of the exponentially harmonic equation

$$(1 + |D_1u|^2)D_{11}u + 2D_1u\, D_2u\, D_{12}u + (1 + |D_2u|^2)\, D_{22}u = 0.$$

(Incidentally, that latter is cited in [S, p. 431] as an example of a non-uniformly elliptic equation which is regularly elliptic (in Serrin's sense)).

(4.11) **Remark.** Theorems 1.5 and 1.6 are first steps in the study of exponentially harmonic maps $M \to N$ between Riemannian manifolds - a programme undertaken in collaboration with L. Lemaire. They are valid in case $N = \mathbb{R}^n$, a significant extension because of the highly coupled nature of the defining system; the proof requires a generalization of Lemma 3.1 based on induction on n.

5. Representation by Differential Forms

In this section M denotes a compact oriented Riemannian manifold without boundary. The following result is in the context of the main theorem of [SS, p. 59]; however, our density ρ is not admissible in their sense.

(5.1) **Proposition.** *Let* $\rho(\xi) = \exp(\xi/2)$. *Then every real 1-dimensional cohomology class of* M *is represented by a unique smooth 1-form* ω *such that*

(5.2) $$d\omega = 0 \quad \textit{and} \quad d^*(\rho(|\omega|^2)\omega) = 0.$$

Here d denotes the exterior differential operator; and d^* its adjoint.

Proof. Firstly, we construct a weak solution. As in (1.12) we set

$$f(q) = \frac{1}{2}\int_0^q \rho(\xi)d\xi = e^{q/2} - 1;$$

then

$$\frac{|p|^2}{2} \le f(|p|^2) \text{ and } |p|^2 \le \frac{\partial^2 f(|p|^2)}{\partial p_i\, \partial p_j}\, p_i p_j.$$

That convexity insures that the functional

$$F(\omega) = \int_M f(|\omega|^2)\sqrt{\det g_{ij}}\; dx^1 \dots dx^m$$

is weakly lower semi-continuous on the Hilbert space $\mathcal{P}$ of square integrable 1-forms on M.

Let γ be a smooth closed 1-form representing a given cohomology class. Then $\gamma + dW^{1,2}(M)$ is a closed - hence weakly closed - affine subspace of $\mathcal{P}$ [M, §7.4]; therefore F achieves its minimum ω on $\gamma + dW^{1,2}(M)$. Such minima are just the weak solutions of the equations (5.2). Indeed, for any $u \in W^{1,2}(M)$

$$\frac{d}{d\varepsilon}F(\omega + \varepsilon\, du)\,|_{\varepsilon = 0} = \langle \rho(|\omega|^2)\omega, du\rangle,$$

the brackets denoting the L^2-inner product on $\mathcal{P}$. But the left member vanishes for all u iff $d^*(\rho(|\omega|^2)\omega) = 0$ weakly. Uniqueness of ω is elementary.

It remains to show that ω is smooth, which we do now: In any chart U we can write $\omega = dv$ for some function $v \in W^{1,2}(U)$; explicitly, we can take

$$v(x) = \int_{\gamma_x} \omega$$

where γ_x is any smooth path in U from a fixed point of U to x. Because $E(v) \le F(\omega)$ + Volume (M) $< \infty$, we see that $v \in W'(U)$. Smoothness follows upon application of Theorem 1.5. □

(5.3) There is a canonical isomorphism of the integral 1-dimensional cohomology group

of M with the group $[M, S^1]$ of homotopy classes of M into the circle S^1. (That is described and applied in [ES, §4D].) Say that a smooth map $M \to S^1$ *is exponentially harmonic* if it is locally an exponentially harmonic function. Then Proposition 5.1 has the

(5.4) **Corollary.** *Every homotopy class in* $[M, S^1]$ *has an exponentially harmonic representative.*

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