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**A NEW TYPE OF HYBRID DISTRIBUTION AND  
APPLICATIONS**

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A New Type of Hybrid Distribution and Applications

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A new type of hybrid distribution proposed by Uppuluri in 1986 was reviewed and studied on some features of the distribution and applied to actual data for the purpose of health risk assessment to low-level ionizing radiation. This hybrid distribution  $F(w)$  was derived from the exponential distribution  $Z \sim \text{Ex}(\lambda)$  by the transformation of variate  $Z = -\ln W - (1-W)$ ,  $0 < w < 1$ , in analogy with the hybrid lognormal distribution  $Z \sim N(0, 1)$  by the transformation of variate  $Z = (\ln \rho X + \rho X - \mu) / \sigma$ . In this memorandum we present the definition of this hybrid distribution, its main statistical properties, a mechanism of generating the distribution proposed by Uppuluri and Gröer, maximum likelihood estimation of the parameters, figures and tables of features of the distribution, and some applications to data of breeding values by selection and of latent period of leukemias and bone cancers to ionizing radiation.

Keywords: Hybrid Distribution, Exponential Distribution, Health Risk Assessment, Leukemia, Latent Period

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新しいタイプのハイブリッド分布とその応用

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(1991年2月4日受理)

本報告書は、1986年に Uppuluri が提案した新しいタイプのハイブリッド分布を理論的に検討し、この分布の種々の特徴を調べて、電離放射線に伴う健康リスク評価を目的とする実際のデータに適用したものである。このハイブリッド分布  $F(w)$  は、熊澤・沼宮内が1980年に正規分布  $Z \sim N(0, 1)$  において変数  $Z = (\ln \rho X + \rho X - \mu) / \sigma$  と変換して導いた混成対数正規分布と類似して、指数分布  $Z \sim \text{Ex}(\lambda)$  において変数  $Z = -(\ln W + 1 - W)$  と変換して導かれた。本報では、このハイブリッド分布の定義、その主な統計量、Uppuluri & Gröer が示した本分布を発生させる機構、本分布のパラメータの最尤推定法及び淘汰による育種価データ並びに電離放射線による白血病・骨がんの潜伏期間データへの適用例を示す。

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## 1. INTRODUCTION

Risk assessment is one of the focal points of the BEIR-V committee Report<sup>1)</sup>. In the discussion on the methods used in Risk Assessment, the Committee urges the need for the development of simpler statistical models. So far, the proportional hazard model is heavily used in estimating risks in the literature. For dose-response models, we wish to propose the distribution that has a well defined upper bound on the dose beyond which all persons will have a hundred per cent mortality. Thus, it is reasonable to study distributions, which are defined on bounded doses. In general, the existing statistical dose distributions, when they are fitted to actual data, fit well only at the lower tail or upper tail, but rarely at both the tails. The hybrid distributions have the feature, that they can be tailored to fit at both the tails. The distribution that is introduced and studied in detail in this memorandum depends only on one parameter and is simple in nature, which could be used as a dose-response model. We would like to propose the following form for the incidence rate  $F(D)$ , for a given dose  $D$ :

$$F(D) = \left(\frac{D}{b}\right)^{\lambda b} \exp[\lambda (b-D)] , \quad 0 < D < b .$$

Some of the basic properties of this type of distribution are presented in this technical memorandum, and some graphs showing the basic properties of this family of functions are enclosed.

## 2. REVIEW

Let  $W$  be a random variable with  $0 < w < 1$  and  $Z$  be a transformation of  $W$  given by

$$z = -\ln W - (1-W) , \quad 0 < z < \infty .$$

In analogy with the hybrid lognormal theory<sup>2)</sup>, one can ask the question about the probability distribution of  $W$ , given that  $Z$  has an exponential distribution with parameter  $\lambda$ <sup>3)</sup>. In other words, given

$$P[Z \leq z] = 1 - \exp(-\lambda z), \quad z > 0,$$

what is the distribution of  $W$ ?

The probability density function of  $Z > 0$  is given by

$$f(z) dz = \lambda \exp(-\lambda z) dz, \quad 0 < z < \infty.$$

Therefore the probability density function of  $W$  is given by

$$\begin{aligned} f(w) \left| \frac{dz}{dw} \right| dw &= \lambda \exp[-\lambda \{-\ln w - (1-w)\}] \left| -\frac{1-w}{w} \right| dw \\ &= \lambda w^{\lambda-1} (1-w) \exp[\lambda(1-w)] dw, \quad 0 < w < 1. \end{aligned}$$

One can also verify, that given the probability density function of  $W$  as  $f_{\lambda}(w)$ ,  $0 < w < 1$ , then  $Z$  will have an exponential distribution with parameter  $\lambda$ . In a sense, the exponential distribution induces a hybrid distribution by the nonlinear transformation

$$Z = -\ln W - (1-W), \quad 0 < w < 1, \quad 0 < z < \infty.$$

Next, we shall consider some basic properties of this probability density function. First we note that this probability density function of this bounded random variable  $W$ , has an exponential part and a power part in the probability density function. It is unusual to see an exponential part in the probability density function of a bounded variable. This makes this probability density function a bit intriguing and may have the potential for several new applications. The cumulative distribution function of this variable  $0 < w < 1$  is given by

$$F_{\lambda}(w) = w^{\lambda} \exp[\lambda(1-w)], \quad 0 < w < 1.$$

We note that  $F_{\lambda}(0) = 0$  and  $F_{\lambda}(1) = 1$ , and  $\frac{dF_{\lambda}(w)}{dw} = f_{\lambda}(w)$ .

The median of this distribution is obtained by solving the following equation for  $w$ :



$$\ln w + (1-w) = -\frac{\ln 2}{\lambda}.$$

The mode of this distribution is given by

$$\begin{cases} 1-\lambda^{-1/2} & \text{for } \lambda > 1, \\ 0 & \text{for } \lambda \leq 1. \end{cases}$$

The mean value of this distribution is given by

$$\begin{aligned} E[W] &= \int_0^1 w f_{\lambda}(w) dw = \int_0^1 [1 - F_{\lambda}(w)] dw \\ &= \lambda \int_0^1 w^{\lambda} (1-w) \exp[\lambda(1-w)] dw \\ \text{or} \quad &= 1 - \int_0^1 w^{\lambda} \exp[\lambda(1-w)] dw. \end{aligned}$$

The variance of this distribution, denoted by  $\text{Var}[W]$  is given by

$$\text{Var}[W] = E[W^2] - \{E[W]\}^2.$$

An alternate form for  $\text{Var}[W]$  in terms of  $E[W]$  is given by the following proposition.

**Proposition:**

$$\text{Var}[W] = \frac{2\lambda + 1}{\lambda^2} - \left\{ E[W] - \frac{\lambda + 1}{\lambda} \right\}^2$$

**Proof:** We will use the following relationship between  $f_{\lambda}(w)$  and  $F_{\lambda}(w)$  in deriving this result.

$$f_{\lambda}(w) = \lambda \frac{1-w}{w} F_{\lambda}(w).$$

Consider

$$\begin{aligned} \frac{d}{dw}\{w^2 f_{\lambda}(w)\} &= \frac{d}{dw}\{w \cdot w f_{\lambda}(w)\} \\ &= w f_{\lambda}'(w) + w \frac{d}{dw}\{w f_{\lambda}(w)\} \\ &= w f_{\lambda}'(w) + w \frac{d}{dw}\{\lambda(1-w) F_{\lambda}(w)\} \\ &= w f_{\lambda}'(w) + \lambda w \{-F_{\lambda}'(w) + (1-w) f_{\lambda}'(w)\} \\ &= w f_{\lambda}'(w) - \lambda w F_{\lambda}'(w) + \lambda w f_{\lambda}'(w) - \lambda w^2 f_{\lambda}'(w) \\ &= w f_{\lambda}'(w) + \{w f_{\lambda}'(w) - \lambda F_{\lambda}'(w)\} + \lambda w f_{\lambda}'(w) - \lambda w^2 f_{\lambda}'(w). \end{aligned}$$

Rewriting this equation, we obtain

$$w^2 f_{\lambda}'(w) = \frac{2+\lambda}{\lambda} w f_{\lambda}'(w) - F_{\lambda}'(w) - \frac{1}{\lambda} \frac{d}{dw}\{w^2 f_{\lambda}'(w)\}.$$

Integrating both sides between the limits 0 and 1, and observing that the last term on the right side integrates to zero, we obtain

$$\begin{aligned} E[W^2] &= \int_0^1 w^2 f_{\lambda}'(w) dw = \frac{2+\lambda}{\lambda} \int_0^1 w f_{\lambda}'(w) dw - \int_0^1 F_{\lambda}'(w) dw \\ &= \frac{2+\lambda}{\lambda} E[W] - 1 + \int_0^1 \{1 - F_{\lambda}(w)\} dw \\ &= \left(\frac{2+\lambda}{\lambda} + 1\right) E[W] - 1 \end{aligned}$$

$$= \frac{2(1+\lambda)}{\lambda} E[W] - 1$$

Using the variance formula, we obtain

$$\begin{aligned} \text{Var}[W] &= E[W^2] - \{E[W]\}^2 \\ &= \frac{2(\lambda+1)}{\lambda} E[W] - 1 - \{E[W]\}^2 \end{aligned}$$

Adding and subtracting  $(\frac{\lambda+1}{\lambda})^2$  on the right side, we obtain

$$\text{Var}[W] = \frac{2\lambda+1}{\lambda^2} - \left\{ E[W] - \frac{\lambda+1}{\lambda} \right\}^2.$$

In the special case  $\lambda=1$ , the probability density function is given by

$$f_1(w) = (1-w) \cdot \exp(1-w), \quad 0 < w < 1.$$

In this case, the moments are given by

$$E[W] = 3 - e, \quad E[W^2] = 11 - 4e, \quad \text{and} \quad \text{Var}[W] = 3 - (e-1)^2.$$

### 3. A MECHANISM THAT GENERATES THIS HYBRID DISTRIBUTION

The following mechanism that generates this hybrid distribution with a simple parameter  $\lambda > 0$ , was recently proposed by V.R.R. Uppuluri and Peter Gröer<sup>4)</sup>. Since this was motivated by seismic phenomena, it would be formulated in that framework. Let  $f(x)$  denote the probability density function of earthquakes of magnitude  $x$ , which is assumed to be a bounded variable such that  $0 < x < b$ , and

$$\int_0^b f(x) dx = 1.$$

Let  $F(x)$  denote the cumulative distribution function of  $x$  given by

$$F(x) = \int_0^x f(t) dt, \quad 0 < x < b.$$

Definition: We define the average density function,  $\bar{f}(x)$  by

$$\bar{f}(x) = \frac{F(x)}{x} = \frac{1}{x} \int_0^x f(t) dt, \quad 0 < x < b.$$

In the context of earthquakes, one has lots of information about earthquakes of small magnitude given by small values of  $x$ , along with their frequencies. Thus for any value of  $x$ , one can get good estimates of the average density function. Generally, we would like to know the probability of an earthquake of magnitude larger than a given value  $x$  like 6.5 or 7.0 etc. It seems to be reasonable to assume that the probability density function at  $x_0$ , as proportional to the average density function at  $x_0$ . If we further assume that this probability density function at  $x_0$ , namely  $f(x_0)$ , is also proportional to  $(b-x_0)$ , (or the distance away from the worst possible earthquake), then it characterizes the frequency distribution of earthquakes. We shall state this result as a theorem.

Theorem (Uppuluri and Gröer)<sup>4)</sup>:

Let  $0 < X < b$ , be a bounded random variable with probability density function  $f(x)$ , cumulative distribution function  $F(x)$  and the average density function  $\bar{f}(x) = \frac{F(x)}{x}$ .

If the probability density function  $f(x)$  is proportional to  $f(x)$  and  $(b-x)$ , then

$$f(x) = \lambda \left(\frac{x}{b}\right)^{\lambda b - 1} \frac{b-x}{b} \exp\{\lambda(b-x)\}, \quad 0 < x < b.$$

Proof:

$$f(x) = \lambda \bar{f}(x) (b-x) , \quad 0 < x < b$$

$$= \lambda \frac{F(x)}{x} (b-x)$$

$$\text{or } \frac{f(x)}{F(x)} = \lambda \frac{b}{x} - \lambda$$

$$\frac{d}{dx} \{ \ln F(x) \} = \lambda b \frac{d}{dx} \{ \ln x \} - \lambda$$

$$\therefore \ln F(x) = \lambda b \ln x - \lambda x + \text{const.}$$

For  $x=b$ ,  $F(b)=1$ . Then  $\text{const.} = \lambda b - \lambda b \ln b$  or

$$\ln F(x) = \lambda b \ln \left( \frac{x}{b} \right) + \lambda (b-x).$$

$$\therefore F(x) = \left( \frac{x}{b} \right)^{\lambda b} \exp[\lambda (b-x)] , \quad 0 < x < b .$$

thus, we have a mechanism which characterized the hybrid distribution discussed earlier.

#### 4. MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS

First we shall discuss the maximum likelihood estimate (MLE) of the single parameter  $\lambda$  of Uppuluri's model when the range of the variable  $W$  is between 0 and 1. In the case we have data  $w_1, w_2, \dots, w_n$  which are independent identically distributed with probability density function

$$f_{\lambda}(w) = \lambda w^{\lambda-1} (1-w) \exp[\lambda (1-w)] , \quad 0 < w < 1 .$$

The likelihood function is given by

$$L(w_1, w_2, \dots, w_n) = \prod_{i=1}^n f_{\lambda}(w_i)$$

$$= \lambda^n \prod_{i=1}^n \{w_i^{\lambda-1} (1-w_i)\} \exp[\lambda \sum_{i=1}^n (1-w_i)] .$$

The logarithm of the likelihood function is given by

$$\ell_n L(w_1, w_2, \dots, w_n) = n \ell_n \lambda + (\lambda - 1) \sum_{i=1}^n \ell_n w_i + \sum_{i=1}^n \ell_n (1-w_i) + \lambda \sum_{i=1}^n (1-w_i)$$

We obtain the maximum likelihood estimate of  $\lambda$ , by differentiating the above function with respect to  $\lambda$  and equating to it zero

$$\frac{\partial \ell_n L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \ell_n w_i + \sum_{i=1}^n (1-w_i) = 0$$

$$\text{or } \frac{n}{\lambda} + \sum_{i=1}^n \ell_n w_i + n(1-\bar{w}) = 0 \quad \text{where } n\bar{w} = \sum_{i=1}^n w_i$$

$$\text{or } \hat{\lambda} = - \frac{1}{\frac{1}{n} \sum_{i=1}^n \ell_n w_i + (1-\bar{w})}$$

Now, we shall consider the three-parameter form of Uppuluri's model, where the variable  $X$  takes the values between the parameters  $a$  and  $b$ , or  $a < X < b$ . The probability density function of  $X$  is easily obtained from the one parameter probability density function of  $W$  by the substitution

$$W = \frac{X - a}{b - a} \quad \text{or} \quad X = a + W(b - a)$$

Thus

$$f(w) dw = \lambda w^{\lambda-1} (1-w) \exp[\lambda(1-w)] dw, \quad 0 < w < 1$$

$$\text{becomes, by noting } \frac{dw}{dx} = \frac{1}{b-a},$$

$$f_{\lambda}(x) dx = \lambda \left( \frac{x-a}{b-a} \right)^{\lambda-1} \frac{b-x}{b-a} \exp\left[ \lambda \frac{b-x}{b-a} \right] \frac{1}{b-a} dx$$

Alternatively, this can also be seen from the cumulative distribution function of  $W$  and  $a < X < b$  as follows:

$$\begin{aligned} F_{\lambda}(x) &= P[X \leq x] = P[a + (b-a)W \leq x] \\ &= P[(b-a)W \leq x-a] \\ &= P\left[W \leq \frac{x-a}{b-a}\right] \\ &= \left(\frac{x-a}{b-a}\right)^{\lambda} \exp\left[\lambda \left\{1 - \frac{x-a}{b-a}\right\}\right] \\ &= \left(\frac{x-a}{b-a}\right)^{\lambda} \exp\left[\lambda \frac{b-x}{b-a}\right] \end{aligned}$$

By differentiating with respect to  $x$ , on both sides, we have

$$\begin{aligned} f_{\lambda}(x) &= \lambda \left(\frac{x-a}{b-a}\right)^{\lambda-1} \frac{1}{b-a} \exp\left[\lambda \frac{b-x}{b-a}\right] \\ &\quad - \left(\frac{x-a}{b-a}\right)^{\lambda} \frac{\lambda}{b-a} \exp\left[\lambda \frac{b-x}{b-a}\right] \\ &= \lambda \left(\frac{x-a}{b-a}\right)^{\lambda-1} \frac{1}{b-a} \exp\left[\lambda \frac{b-x}{b-a}\right] \left\{1 - \frac{x-a}{b-a}\right\}, \quad a < X < b. \end{aligned}$$

Thus we see that these two expressions are the same.

Let  $x_1, x_2, \dots, x_n$  be a set of  $n$  independent identically distributed data, such that  $a < x_1, x_2, \dots, x_n < b$  with the probability density function

$$f(x) = \lambda \left( \frac{x-a}{b-a} \right)^{\lambda-1} \frac{b-x}{b-a} \frac{1}{b-a} \exp\left[ \lambda \frac{b-x}{b-a} \right], \quad a < X < b.$$

Given this data, the likelihood function is given by

$$L(x_1, x_2, \dots, x_n) = \lambda^n \prod_{i=1}^n \left\{ \left( \frac{x_i - a}{b - a} \right)^{\lambda - 1} \frac{b - x_i}{b - a} \frac{1}{b - a} \right\} \exp\left[ \lambda \sum_{i=1}^n \frac{b - x_i}{b - a} \right]$$

Taking logarithm, we obtain

$$\begin{aligned} \ln L. &= n \ln \lambda - n(\lambda + 1) \ln(b - a) + (\lambda - 1) \sum_{i=1}^n \ln(x_i - a) \\ &\quad + \sum_{i=1}^n \ln(b - x_i) + \lambda n \frac{b - \bar{x}}{b - a}, \end{aligned}$$

where  $n\bar{x} = \sum_{i=1}^n x_i$ . In order to obtain the maximum likelihood estimates of  $\lambda$ ,  $a$  and  $b$ , we differentiate the above expression with respect to  $\lambda$ ,  $a$  and  $b$ , respectively, and equate them to zero and solve for  $\hat{\lambda}$ ,  $\hat{a}$  and  $\hat{b}$ , respectively

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - n \ln(b - a) + \sum_{i=1}^n \ln(x_i - a) + n \frac{b - \bar{x}}{b - a}$$

$$\text{or} \quad \frac{n}{\lambda} + \sum_{i=1}^n \ln\left(\frac{x_i - a}{b - a}\right) + n \frac{b - \bar{x}}{b - a} = 0$$

$$\text{or} \quad -\frac{1}{\lambda} = \frac{1}{n} \sum_{i=1}^n \ln\left(\frac{x_i - a}{b - a}\right) + \frac{b - \bar{x}}{b - a},$$

$$\frac{\partial \ln L}{\partial a} = n \frac{\lambda + 1}{b - a} - (\lambda - 1) \sum_{i=1}^n \frac{1}{x_i - a} + \lambda n \frac{b - \bar{x}}{(b - a)^2} = 0$$



$$\text{or } (\lambda - 1) \sum_{i=1}^n \left\{ \frac{1}{b-a} - \frac{1}{x_i - a} \right\} + \frac{2n}{b-a} + \lambda n \frac{b - \bar{x}}{(b-a)^2} = 0$$

$$\text{or } \lambda \frac{b - \bar{x}}{b-a} + 2 = \frac{\lambda - 1}{n} \sum_{i=1}^n \frac{b - x_i}{x_i - a},$$

and finally, the derivative with respect to  $b$  is given by

$$\frac{\partial \ell_m L}{\partial b} = -n \frac{\lambda + 1}{b-a} + \sum_{i=1}^n \frac{1}{b-x_i} + \lambda n \frac{\bar{x} - a}{(b-a)^2} = 0$$

$$\text{or } \frac{n\lambda}{b-a} \left\{ \frac{\bar{x} - a}{b-a} - 1 \right\} + \sum_{i=1}^n \left\{ \frac{1}{b-x_i} - \frac{1}{b-a} \right\} = 0$$

$$\text{or } \lambda \frac{b - \bar{x}}{b-a} = \frac{1}{n} \sum_{i=1}^n \frac{x_i - a}{b - x_i}.$$

Thus, the three equations to be solved for  $\lambda$ ,  $a$  and  $b$  are:

$$\frac{1}{n} \sum_{i=1}^n \ell_m \left\{ \frac{x_i - a}{b-a} \right\} = -\frac{b - \bar{x}}{b-a} - \frac{1}{\lambda}$$

$$\frac{1}{n} \sum_{i=1}^n \frac{b - x_i}{x_i - a} = \frac{\lambda}{\lambda - 1} \frac{b - \bar{x}}{b-a} + \frac{2}{\lambda - 1}$$

$$\frac{1}{n} \sum_{i=1}^n \frac{x_i - a}{b - x_i} = \lambda \frac{b - \bar{x}}{b-a}$$

### PROBLEMS FOR FURTHER CONSIDERATION

There are several problems of interest that are worth considering. One can easily think of a two-parameter version of this hybrid distribution, given by the cumulative distribution function:

$$1 - F_{\lambda, \eta}(w) = w^{\lambda} \exp[\eta(1-w)] \quad , \quad 0 < w < 1 \quad , \quad \lambda > 0 \quad , \quad \eta > 0 .$$

The theory and applications of this model will be very challenging. Of course, the multivariate generalizations of our discussion are nontrivial and need powerful techniques.

Since  $1 - F_{\lambda}(w)$  does not have a simple analytic function, one need to consider the study of computer calculation of hazard rates or force of mortality and mean residual life and other parameters of interest. If one can find some practical applications, such a study may be worth the effort.

The statistical behavior of this distribution, and the moments should be analytically considered for small and large value of  $\lambda$ . The understanding of the role of the parameter  $\lambda$ , may be helpful in this consideration.

The hybrid distribution of  $X$ , when  $X$  ranges from  $a$  to  $b$ , is useful in practical problems. The maximum likelihood estimates of  $\lambda$ ,  $a$  and  $b$  were obtained as solutions of three equations. One has to develop computer algorithms to obtain these solutions.

## 5. SOME EXAMPLES OF FEATURES OF UPPULURI DISTRIBUTION

Graphs and Tables will be given to describe the typical features of the one parameter form of Uppuluri distribution.

### 5.1 Probability Density Function

Figure 1 gives curves of probability density function of Uppuluri's model for  $\lambda=0.02, 1, 4$  and  $50$ . The smaller the value of  $\lambda$  is below  $1$ , the more the L-shaped nature of the probability density function and it approaches the vertical line  $w=0$ . The curve is exponential for  $\lambda=1$ , and is unimodal for  $\lambda > 1$ . The curve has the mode at  $w=0.5$  for  $\lambda=4$  and at  $w$  greater than  $0.8$  for  $\lambda=50$ . The larger the value of  $\lambda$ , the more the curve approaches the vertical line  $w=1$ .

The characteristics of the curve of probability density function of Uppuluri's model is somewhat similar to that of Weibull distribution. The Uppuluri distribution, however, is defined in the interval between  $0$

and 1, more generally defined between parameters  $a$  and  $b$  ( $a < b$ ) in the three parameter form, although the Weibull distribution is defined between zero and positive infinity.

Table 1 is the probability density function for  $\lambda = 0.02, 1, 4$  and  $50$ .

## 5.2 Cumulative Distribution Function

Figure 2 gives curves of cumulative distribution function of Uppuluri's model for  $\lambda = 0.02, 1, 4$  and  $50$ . The smaller the value of  $\lambda$  far below 1, the more the  $\Gamma$ -shaped nature of the cumulative distribution function and it approaches the vertical line at  $w=0$ . The larger the value of  $\lambda$  far greater than 50 is, the more the L-shaped curve approaches the vertical line at  $w=1$ .

Table 2 is the cumulative distribution function for  $\lambda = 0.02, 1, 4$  and  $50$ .

## 5.3 Median

Figure 3 gives the curve of median of Uppuluri distribution for  $\lambda$  from 0.01 to 1000. For  $\lambda \ll 1$ , the median approximates  $\exp\{-\ln 2 / \lambda\}$  which approaches zero for  $\lambda$  close to zero. For a large  $\lambda$  approaching the positive infinity, the median is almost 1.

Define a function of co-hybrid as  $v = \text{coh}(w) = w - \ln w$  and its inverse function as  $w = \text{coh}^{-1}(v)$ , the median is expressed as  $\text{coh}^{-1}(1 + \ln 2 / \lambda)$ . The function  $\text{coh}^{-1}(v)$ , ( $v \geq 1$ ), has two values of  $w$  for a given value of  $v$  above 1. Thus the smaller one is a valid solution in this case. The function subprogram INCOH2(v) written in FORTRAN is to calculate the smaller solution of  $\text{coh}^{-1}(v)$ , which is given in appendix. The FORTRAN function subprogram can be used as a SAS function, whose example is also given in the appendix.

## 5.4 Mode

Figure 4 gives the curve of the mode of Uppuluri distribution according to  $\lambda$  from 1 to 1000. The mode is zero for  $\lambda \leq 1$ . For  $\lambda > 1$ , the mode is  $1 - \lambda^{-1/2}$ . Thus the mode increases to approach 1 as a function of  $\lambda$ . The mode is smaller or greater than the median, respectively, for  $\lambda$  smaller or greater than about 6.1.

The maximum of the probability density function, which occurs at the mode, is shown in Figure 5 in a function of  $\lambda$  from 1 to 1000. The

curve shows that there is a minimum in the maximum of the probability density function. Figure 6 shows an enlargement of the curve near the minimum which is about 1.7014394307594 at  $\lambda$  about 2.138644. Figure 7 is the curve of the mode corresponding to Figure 6.

Table 4 is the mode of Uppuluri distribution as a function of  $\lambda$  from 1 to 16 and from 1 to 1000.

### 5.5 Mean

Figure 8 gives the curve of the mean of Uppuluri distribution as a function of  $\lambda$  going from 0.01 to 1000. The mean approaches zero for  $\lambda$  close to zero and unity for  $\lambda$  approaching the positive infinity. The mean is greater or smaller than the median, respectively, according as  $\lambda$  is smaller or greater than about 5.2. The mean is smaller than the mode for  $\lambda$  above 6.1.

The mean of Uppuluri distribution is numerically calculated by the integral formula of Gauss-Legendre method by changing the integral interval according to the value of parameter  $\lambda$ . The function subprogram UPLAV( $\lambda$ ) written in FORTRAN is to calculate the mean, which is given in the appendix, including its SAS function. Table 4 is the mean of Uppuluri distribution as a function of  $\lambda$  from 0.01 to 1000.

### 5.6 Variance or Standard Deviation

Figure 9 gives the curve of the standard deviation of Uppuluri distribution as a function of  $\lambda$  from 0.01 to 1000. The curve has the peak at  $\lambda$  about 1.266494 and then the mean and standard deviation are, respectively, 0.3217557 and 0.2192925. The more  $\lambda$  departs from about 1.266494 to be smaller or larger, the smaller the standard deviation is. Figure 10 gives another presentation of the curve of standard deviation.

The variance of Uppuluri distribution is numerically calculated like the mean. The function subprogram UPLVR( $\lambda$ ) written in FORTRAN and its SAS function are to calculate the variance as given in appendix. Table 5 gives the variance and standard deviation for  $\lambda$  from 0.01 to 1000.

## 6. APPLICATION OF THE MODEL TO DATA

### 6.1 Distribution of breeding value

Nishida, et al.<sup>6,7)</sup> have been studying the effects of repeated selection on the mean, heritability and distribution of breeding value of animal populations. The breeding value  $G$  is defined as  $G=P-E$ , where  $P$  is the phenotypic value and  $E$  is the environmental effect. Nishida, et al. tried to apply Uppuluri's model to the data of breeding value by generation of selection, because of the similarity of the frequency curve of breeding value as shown in Figure 11 to the probability density function of Uppuluri's model.

The given data is a set of  $(g_i, p_i)$ , ( $i=1$  to  $n$ ), where  $g_i$  is the  $i$ -th breeding value and  $p_i$  is the cumulative probability at the  $i$ -th value. Putting  $u_i = \exp(p_i)$  and  $y_i = \ln w_i + 1 - w_i$ , where  $w_i = (g_i - a)/(b - a)$ ,  $a < g_i < b$ , after selecting appropriate values of both limits of breeding value, we can use the form of linear model  $u_i = \lambda y_i + \epsilon_i$ , where  $\epsilon_i$  is an error term. The SAS (Statistical Analysis System) software<sup>8)</sup> affords us to the REG procedure which fits least-squares estimates to linear models. However, as values of  $a$  and  $b$  are unknown, we used the NLIN procedure of SAS, which produces least-squares estimates of the parameters of a nonlinear model. The parameters to be estimated are  $\lambda$ ,  $a$  and  $b$ , based on the form of non-linear model  $u_i = \lambda [\ln \{(g_i - a)/(b - a)\} + 1 - (g_i - a)/(b - a)] + \epsilon_i$ .

Figure 12 gives the results of fitting the UPL distribution to the data of breeding values for the 2nd, 4th, 6th, 8th and 10th generations of selection by using the NLIN procedure. In Figure 12, the plotted data points  $(y_i, u_i)$  of open square lie well on the straight line  $u = \lambda y$  for each generation of selection. This means the UPL model is likely to be applicable to the data because of the linearity of UPL model,  $u_i = \lambda y_i + \epsilon_i$ , with the estimated values of  $a$  and  $b$ . The presentation of Figure 12 is conveniently called a UPL probability plot.

The applicability of the UPL model to the data was studied more by using the two other presentations to show the goodness of fit and by comparing with that of two other competent models<sup>9)</sup>, the hybrid lognormal (HLN) and the Johnson's  $S_B$  (JSB) distributions. The HLN model has the four- or five-parameter form for the application to the data of breeding values: It is defined as  $\ln \rho X + \rho X \sim N(\mu, \sigma^2)$ , where  $X = G - a$  for the 2nd, 4th and 6th generations and  $X = (G - a)/(b - G)$  for the 8th and 10th generations.

The JSB model is defined as  $\ln\{(G-a)/(b-G)\} \sim N(\mu, \sigma^2)$ . The best model can be selected by Akaike's information criteria (AIC),  $n \ln\{\sum \epsilon_i^2/n\} + 2p$ , where  $n$  is the number of data,  $\sum \epsilon_i^2$  is an error sum of squares for all data, and  $p$  is the number of parameters of the given model.

The other presentations are the normal probability plots shown in Figure 13 and the probability density function of the same data shown in Figure 14.

The presentation of Figure 13 shows that the goodness of fit is better than that of Figure 12 in this case. In Figure 12 the data seems to deviate slightly from the straight line fitted by using the UPL model, close to zero of  $y$  for the earlier generation of selection, but in Figure 13 these deviations are more explicit. However Figure 13 shows clearly the excellent goodness of fit of the UPL model to the data for the 10th generation of selection. Figure 14 also shows the excellent goodness of fit of the UPL model for the 10th generation.

For all generations of selection the JSB model is best among three models, based on the calculation of AIC. The HLN model also gives a good fit but we have to change the model from the four parameter to five parameter form according to the generation of selection. The UPL model is quite applicable for the 10th generation and gives the almost best fit to the data among three models by the calculation of AIC.

## 6.2 Distribution of appearance times of cancers

According to the BEIR V report<sup>1)</sup>, following an instantaneous exposure to radiation, the rates of leukemia and bone cancer appear to follow a wave like pattern, rising within 5 years after exposure and then returning to near baseline rates within 30 years. In contrast to the rates for leukemia and bone cancer, the rates for most other cancers appear to have remained in excess for as long as most exposed populations have been followed. Therefore we applied the UPL model to the rates of leukemia and bone cancer.

There are several data of the onset of leukemia and bone cancer in a function of years after irradiation. We applied the UPL model to the data of Mays and Spiess<sup>10)</sup> whose paper was cited in Figure VII of annex F of the 1988 UNSCEAR report<sup>11)</sup> to show that the distribution of appearance times is remarkably similar for leukemias, following prompt radiation, and for bone sarcomas, following relatively brief radium-224 irradiation. Figure 15 shows it.

The UPL probability plots of the leukemia and bone sarcoma data are

both good straight as shown in Figure 16. Therefore the UPL model is applicable to these data. In addition the UPL model is as proximately equal to the HLN and JSB models to give good fits for these data. In Figure 17 of the normal probability plots, the given data points (open square) are well fitted by the UPL model (solid curve) for both leukemia and bone sarcoma data, as well as by the HLN (dotted curve) and the JSB (dashed curve). The HLN model is the five-parameter form for the leukemia data and four-parameter form for the bone sarcoma data. Figure 18 gives comparisons of the frequency curves among the given data (open square) and the predicted values by the UPL (solid curve), the HLN (dotted curve) and the JSB (dashed curve) models for leukemias and bone sarcomas.

According to the calculation of AIC, the JSB is the best for both data. The UPL is the second best for leukemia data and the HLN is the second for bone sarcoma data. However the UPL model is sufficiently applicable to both data. The means (medians) of appearance times for both data are, respectively, 8.4 (7.8) and 10.2 (9.5) years estimated by the UPL, 8.4 (7.7) and 10.4 (9.2) years by the HLN, and 8.5 (7.9) and 10.5 (9.3) years by the JSB. All estimates of mean (median) are very close to actual values of 8.4 (7.2) and 10.3 (9.2) for the leukemia and bone sarcoma data, respectively.

The leukemia data of the atomic bomb survivors mentioned above was first reported by Bizzozero et al.<sup>12)</sup> and was for the years 1946-1966. The data is shorter in latent period than that detected during 1950-1970 or 1950-1978 and exposed more than one rad reported by Ichimaru, et al.<sup>13, 14)</sup>. The mean of Ichimaru et al. is 11.9 years for survivors detected during 1950-1970 and 13.8 years for those detected during 1950-1978. These two distributions are considerably different. The recent data contain longer periods of the onset of leukemias after irradiation, including some shorter periods because of the possibility due to the improved techniques of detection of leukemias and the longer follow-up period. The UPL model proved to give a good fit to each of these data. However, to determine the true distribution of appearance times of leukemias, we should wait more to have the forthcoming data of the onset of leukemias.

## 7. CONCLUSION

The hybrid distribution derived by Uppuluri in 1986 was studied for the purpose of health risk assessment to low level ionizing radiation. This distribution is conveniently called "Uppuluri distribution" because of many types of potential hybrid distributions. The Uppuluri distribution was successfully applied to the data of appearance times of leukemias and bone cancers after irradiation of ionizing radiation. According to the genesis theorem by Uppuluri and Gröer, the Uppuluri distribution has the potentiality of wide application to various data sets as well as those analyzed here.

The review of Uppuluri distribution stimulated the study of a family of hybrid distributions, whose origin is the hybrid lognormal distribution proposed by Kumazawa and Numakunai in 1980 to formulate the annual dose distributions indicating the effect of dose limits. The common feature of hybrid distributions was proved to be tailored to fit to actual data at both the tails and also proved to explore the underlying mechanisms of the phenomena.

The calculations of features of Uppuluri distribution were almost completed for the basic statistics, including their figures and tables. However, the technique of estimation of the parameters is in the initial stage. In this memorandum we used the SAS procedure of a nonlinear models to fit these distributions to data.

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Table 1 Probability density function  $f(w)$  of Uppuluri's model.

DENSITY FUNCTION OF UPL DISTRIBUTION					
OBS	W	$\lambda = 0.01$	$\lambda = 1$	$\lambda = 4$	$\lambda = 50$
1	0.004	4.54906	2.69660	0.00001	0.00000
2	0.005	3.65178	2.69120	0.00003	0.00000
3	0.006	3.05114	2.68581	0.00005	0.00000
4	0.007	2.62065	2.68043	0.00007	0.00000
5	0.008	2.29684	2.67505	0.00011	0.00000
6	0.009	2.04434	2.66968	0.00015	0.00000
7	0.010	1.84189	2.66432	0.00021	0.00000
8	0.011	1.67591	2.65897	0.00028	0.00000
9	0.012	1.53734	2.65363	0.00036	0.00000
10	0.013	1.41989	2.64829	0.00045	0.00000
11	0.014	1.31906	2.64296	0.00056	0.00000
12	0.015	1.23155	2.63764	0.00068	0.00000
13	0.020	0.92418	2.61117	0.00158	0.00000
14	0.030	0.61468	2.55881	0.00507	0.00000
15	0.040	0.45880	2.50723	0.01143	0.00000
16	0.050	0.36477	2.45642	0.02123	0.00000
17	0.060	0.30181	2.40638	0.03488	0.00000
18	0.070	0.25668	2.35709	0.05265	0.00000
19	0.080	0.22273	2.30855	0.07470	0.00000
20	0.090	0.19625	2.26073	0.10108	0.00000
21	0.100	0.17502	2.21364	0.13175	0.00000
22	0.200	0.07872	1.78043	0.62803	0.00000
23	0.300	0.04620	1.40963	1.24322	0.00000
24	0.400	0.02981	1.09327	1.69316	0.00001
25	0.500	0.01992	0.82436	1.84726	0.00320
26	0.600	0.01330	0.59673	1.71177	0.13072
27	0.620	0.01223	0.55567	1.65633	0.22779
28	0.640	0.01123	0.51600	1.59326	0.37618
29	0.660	0.01029	0.47768	1.52339	0.59035
30	0.680	0.00940	0.44068	1.44755	0.88261
31	0.700	0.00856	0.40496	1.36656	1.25983
32	0.720	0.00777	0.37048	1.28122	1.72006
33	0.740	0.00702	0.33720	1.19233	2.24971
34	0.760	0.00631	0.30510	1.10061	2.82216
35	0.780	0.00564	0.27414	1.00681	3.39835
36	0.800	0.00500	0.24428	0.91158	3.92965
37	0.820	0.00439	0.21550	0.81558	4.36286
38	0.840	0.00381	0.18776	0.71939	4.64655
39	0.860	0.00326	0.16104	0.62357	4.73787
40	0.880	0.00273	0.13530	0.52863	4.60860
41	0.900	0.00222	0.11052	0.43502	4.24938
42	0.920	0.00174	0.08666	0.34315	3.67148
43	0.940	0.00128	0.06371	0.25341	2.90583
44	0.960	0.00083	0.04163	0.16612	1.99944
45	0.980	0.00041	0.02040	0.08157	1.01012
46	1.000	0.00000	0.00000	0.00000	0.00000

Table 2 Cumulative distribution function F(w) of Uppuluri's model.

UPL PROBABILITY DISTRIBUTION FUNCTION

OBS	w	$\lambda = 0.01$	$\lambda = 1$	$\lambda = 4$	$\lambda = 50$
1	0.00000	0.00000	0.00000	0.00000	0.00000
2	0.00001	0.81037	0.00003	0.00000	0.00000
3	0.00010	0.84856	0.00027	0.00000	0.00000
4	0.00100	0.88854	0.00272	0.00000	0.00000
5	0.00200	0.90093	0.00543	0.00000	0.00000
6	0.00300	0.90824	0.00813	0.00000	0.00000
7	0.00400	0.91347	0.01083	0.00000	0.00000
8	0.00500	0.91753	0.01352	0.00000	0.00000
9	0.00600	0.92087	0.01621	0.00000	0.00000
10	0.00700	0.92369	0.01890	0.00000	0.00000
11	0.00800	0.92614	0.02157	0.00000	0.00000
12	0.00900	0.92831	0.02425	0.00000	0.00000
13	0.01000	0.93025	0.02691	0.00000	0.00000
14	0.01100	0.93200	0.02957	0.00000	0.00000
15	0.01200	0.93361	0.03223	0.00000	0.00000
16	0.02000	0.94305	0.05329	0.00001	0.00000
17	0.03000	0.95053	0.07914	0.00004	0.00000
18	0.04000	0.95583	0.10447	0.00012	0.00000
19	0.05000	0.95991	0.12929	0.00028	0.00000
20	0.06000	0.96323	0.15360	0.00056	0.00000
21	0.07000	0.96601	0.17742	0.00099	0.00000
22	0.08000	0.96840	0.20074	0.00162	0.00000
23	0.09000	0.97049	0.22359	0.00250	0.00000
24	0.10000	0.97234	0.24596	0.00366	0.00000
25	0.20000	0.98394	0.44511	0.03925	0.00000
26	0.30000	0.98997	0.60413	0.13320	0.00000
27	0.40000	0.99369	0.72885	0.28219	0.00000
28	0.50000	0.99614	0.82436	0.46182	0.00006
29	0.60000	0.99779	0.89509	0.64191	0.00392
30	0.62000	0.99804	0.90662	0.67561	0.00743
31	0.64000	0.99828	0.91733	0.70812	0.01338
32	0.66000	0.99849	0.92727	0.73929	0.02292
33	0.68000	0.99869	0.93645	0.76901	0.03751
34	0.70000	0.99887	0.94490	0.79716	0.05879
35	0.72000	0.99903	0.95265	0.82364	0.08846
36	0.74000	0.99918	0.95973	0.84839	0.12806
37	0.76000	0.99931	0.96615	0.87132	0.17874
38	0.78000	0.99943	0.97194	0.89240	0.24097
39	0.80000	0.99954	0.97712	0.91158	0.31437
40	0.82000	0.99963	0.98172	0.92885	0.39751
41	0.84000	0.99971	0.98575	0.94420	0.48789
42	0.86000	0.99978	0.98924	0.95763	0.58208
43	0.88000	0.99984	0.99220	0.96915	0.67593
44	0.90000	0.99989	0.99465	0.97879	0.76489
45	0.92000	0.99993	0.99662	0.98656	0.84444
46	0.94000	0.99996	0.99813	0.99253	0.91049
47	0.96000	0.99998	0.99918	0.99672	0.95973
48	0.98000	1.00000	0.99980	0.99919	0.98992
49	1.00000	1.00000	1.00000	1.00000	1.00000

Table 3 Median of Uppuluri distribution.

## MEDIAN OF UPL DISTRIBUTION

OBS	$\lambda$	MED
1	0.01	0.000000
2	0.10	0.000359
3	0.20	0.011631
4	0.30	0.037908
5	0.40	0.069729
6	0.50	0.101828
7	0.60	0.132260
8	1.00	0.231961
9	2.00	0.380620
10	3.00	0.464712
11	4.00	0.520694
12	5.00	0.561518
13	6.00	0.593053
14	7.00	0.618398
15	8.00	0.639367
16	9.00	0.657105
17	16.00	0.733792
18	50.00	0.842600
19	100.00	0.886834
20	300.00	0.933554
21	1000.00	0.963228

Table 4 Mode and f(mode) of Uppuluri distribution.

MODE OF UPL DISTRIBUTION

OBS	$\lambda$	MODE	f(MODE)
1	1.0000	0.000000	.
2	1.0001	0.000050	2.71586
3	1.0010	0.000500	2.70039
4	1.0100	0.004963	2.60363
5	1.1000	0.046537	2.20277
6	1.5000	0.183503	1.78553
7	2.0000	0.292893	1.70376
8	3.0000	0.422650	1.74881
9	4.0000	0.500000	1.84726
10	5.0000	0.552786	1.95356
11	6.0000	0.591752	2.05861
12	7.0000	0.622036	2.16011
13	8.0000	0.646447	2.25758
14	9.0000	0.666667	2.35112
15	16.0000	0.750000	2.91848

MODE OF UPL DISTRIBUTION

OBS	$\lambda$	MODE	f(MODE)
1	1.00	0.000000	.
2	1.00	0.000050	2.7159
3	1.00	0.000500	2.7004
4	1.01	0.004963	2.6036
5	1.10	0.046537	2.2028
6	1.50	0.183503	1.7855
7	2.00	0.292893	1.7038
8	3.00	0.422650	1.7488
9	4.00	0.500000	1.8473
10	5.00	0.552786	1.9536
11	6.00	0.591752	2.0586
12	7.00	0.622036	2.1601
13	8.00	0.646447	2.2576
14	9.00	0.666667	2.3511
15	16.00	0.750000	2.9185
16	30.00	0.817426	3.7870
17	50.00	0.858579	4.7384
18	75.00	0.884530	5.6932
19	100.00	0.900000	6.5006
20	150.00	0.918350	7.8577
21	300.00	0.942265	10.9270
22	500.00	0.955279	13.9800
23	750.00	0.963485	17.0256
24	1000.00	0.968377	19.5938

Table 5 Mean, variance and standard deviation of Uppuluri distribution.

MEAN & STD OF UPL DISTRIBUTION				
OBS	$\lambda$	MEAN	VARIANCE	STD
1	0.01	0.004959	0.0016365	0.040454
2	0.02	0.009837	0.0032145	0.056697
3	0.10	0.046193	0.0139990	0.118318
4	0.50	0.178634	0.0398564	0.199641
5	1.00	0.281718	0.0475076	0.217962
6	2.00	0.402736	0.0460117	0.214503
7	3.00	0.475145	0.0412913	0.203203
8	4.00	0.525043	0.0369379	0.192192
9	5.00	0.562267	0.0332965	0.182473
10	6.00	0.591490	0.0302826	0.174019
11	7.00	0.615265	0.0277689	0.166640
12	8.00	0.635125	0.0256479	0.160149
13	9.00	0.652056	0.0238367	0.154391
14	10.00	0.666725	0.0222729	0.149241
15	15.00	0.718907	0.0168408	0.129772
16	20.00	0.751840	0.0136006	0.116622
17	25.00	0.775120	0.0114384	0.106951
18	30.00	0.792730	0.0098879	0.099438
19	35.00	0.806670	0.0087188	0.093375
20	40.00	0.818069	0.0078044	0.088342
21	45.00	0.827621	0.0070686	0.084075
22	50.00	0.835780	0.0064632	0.080394
23	55.00	0.842858	0.0059559	0.077174
24	60.00	0.849076	0.0055244	0.074327
25	65.00	0.854595	0.0051528	0.071783
26	70.00	0.859540	0.0048292	0.069492
27	75.00	0.864002	0.0045447	0.067415
28	80.00	0.868058	0.0042927	0.065519
29	85.00	0.871765	0.0040678	0.063779
30	90.00	0.875170	0.0038658	0.062176
31	95.00	0.878314	0.0036834	0.060691
32	100.00	0.881228	0.0035177	0.059310
33	150.00	0.902054	0.0024339	0.049334
34	200.00	0.914673	0.0018660	0.043198
35	250.00	0.923373	0.0015153	0.038927
36	300.00	0.929842	0.0012768	0.035732
37	350.00	0.934896	0.0011037	0.033223
38	400.00	0.938988	0.0009724	0.031184
39	450.00	0.942389	0.0008693	0.029484
40	500.00	0.945274	0.0007862	0.028038
41	550.00	0.947762	0.0007176	0.026789
42	600.00	0.949938	0.0006602	0.025695
43	650.00	0.951860	0.0006114	0.024726
44	700.00	0.953576	0.0005693	0.023860
45	750.00	0.955119	0.0005327	0.023080
46	800.00	0.956517	0.0005006	0.022373
47	850.00	0.957792	0.0004721	0.021728
48	900.00	0.958960	0.0004467	0.021136
49	950.00	0.960035	0.0004239	0.020590
50	1000.00	0.961030	0.0004034	0.020085

## DENSITY FUNCTION OF UPL DISTRIBUTION

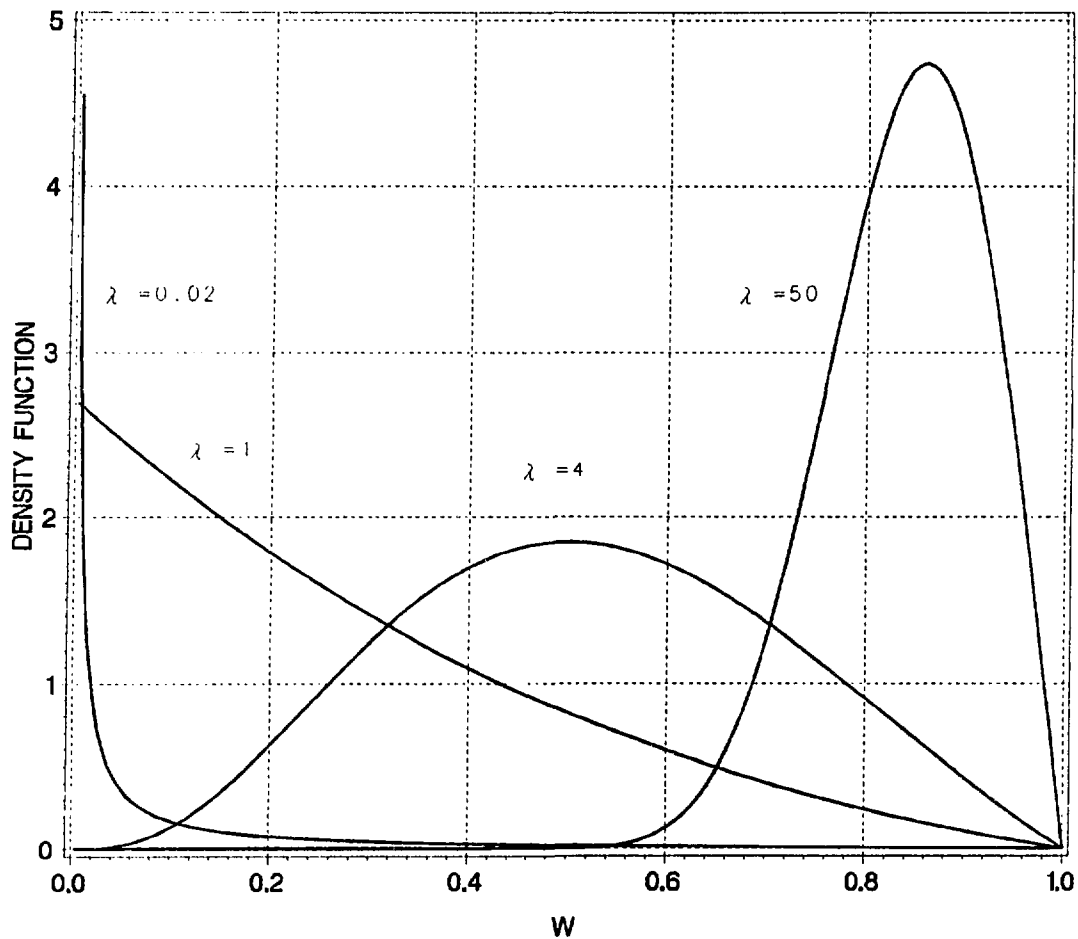


Fig. 1 Probability density function of Uppuluri's model,

$$f(w) = \lambda w^{\lambda-1} (1-w) \exp\{\lambda(1-w)\}, \quad \lambda > 0, \quad 0 < w < 1.$$



## UPL PROBABILITY DISTRIBUTION FUNCTION

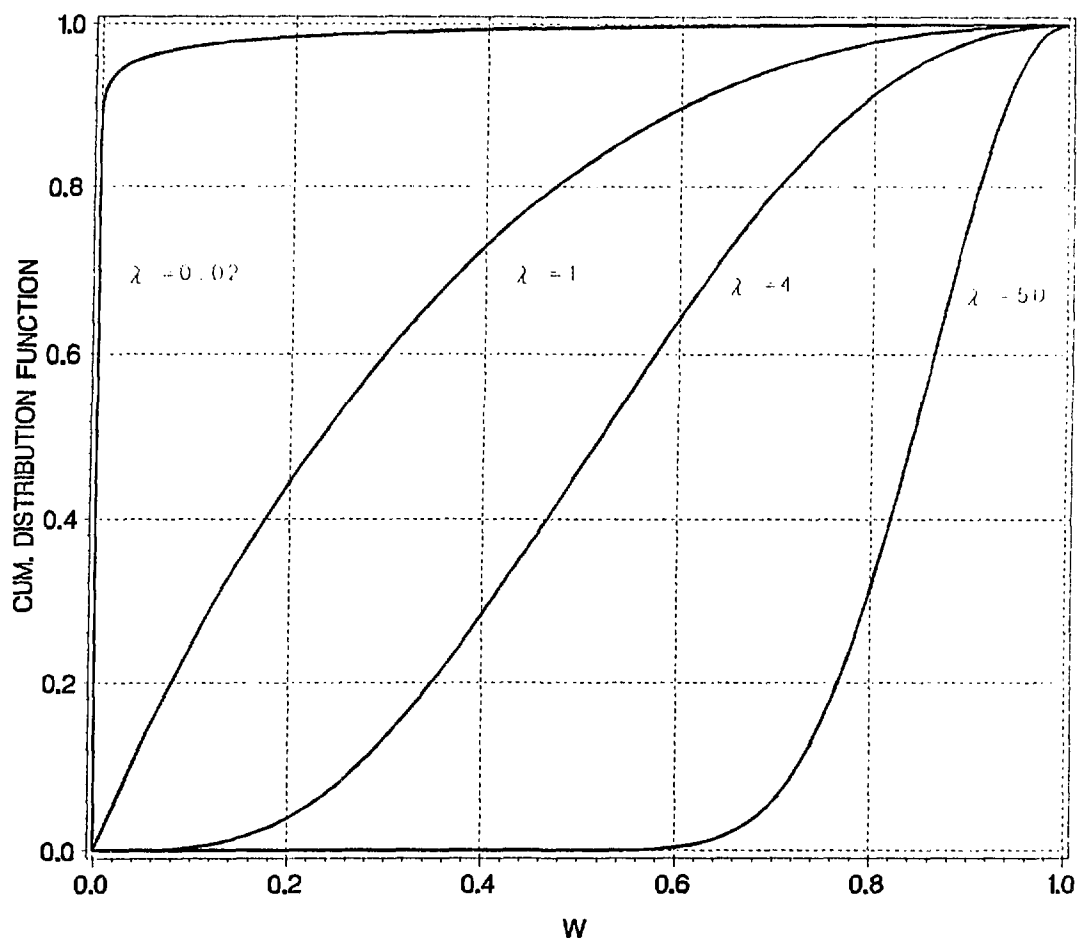


Fig. 2 Cumulative distribution function of Uppuluri's model,

$$F(w) = w^\lambda \exp\{\lambda(1-w)\}, \quad \lambda > 0, \quad 0 < w < 1.$$

## MEDIAN OF UPL DISTRIBUTION

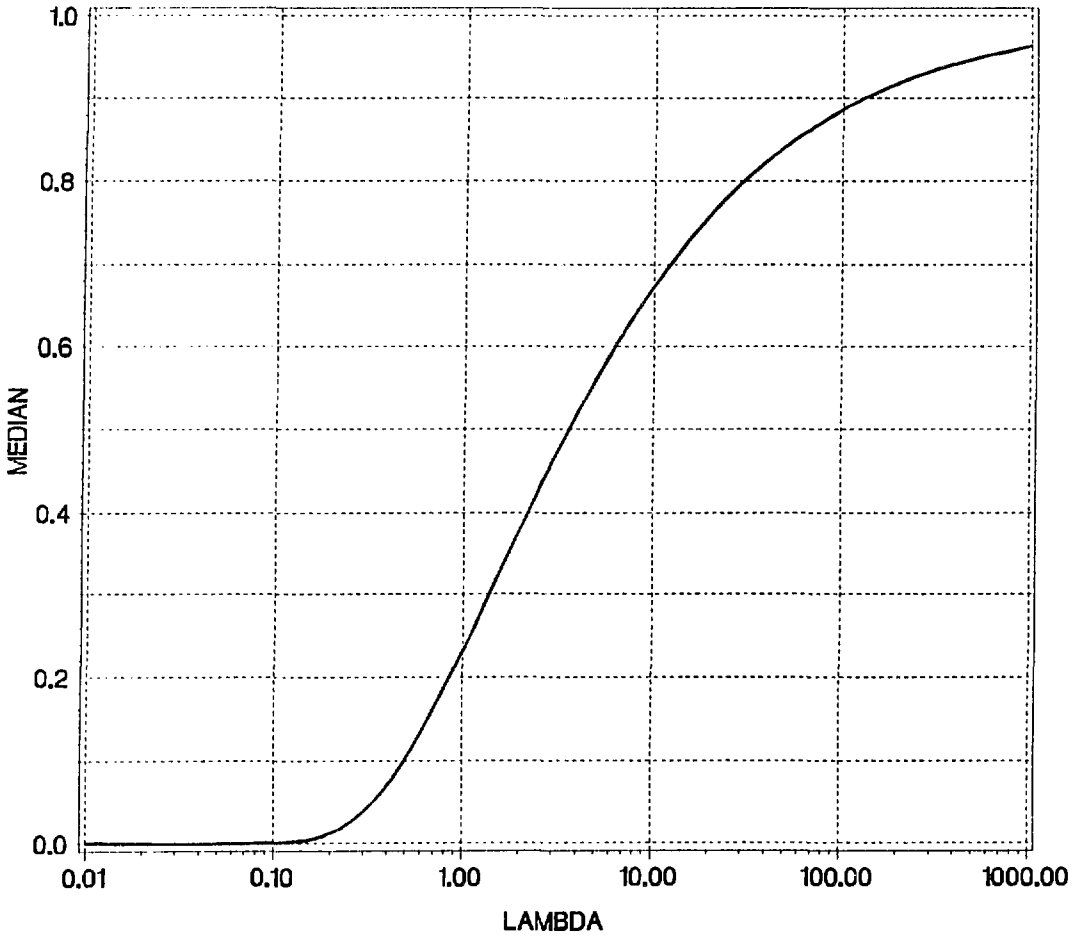


Fig. 3 Median of Uppuluri distribution,  $\text{coth}^{-1}(1 + \ell_m^2 / \lambda)$  or a solution of  $w - \ell_m w = 1 + \ell_m^2 / \lambda$ , for  $\lambda$  from 0.01 to 1000.

## MODE OF UPL DISTRIBUTION

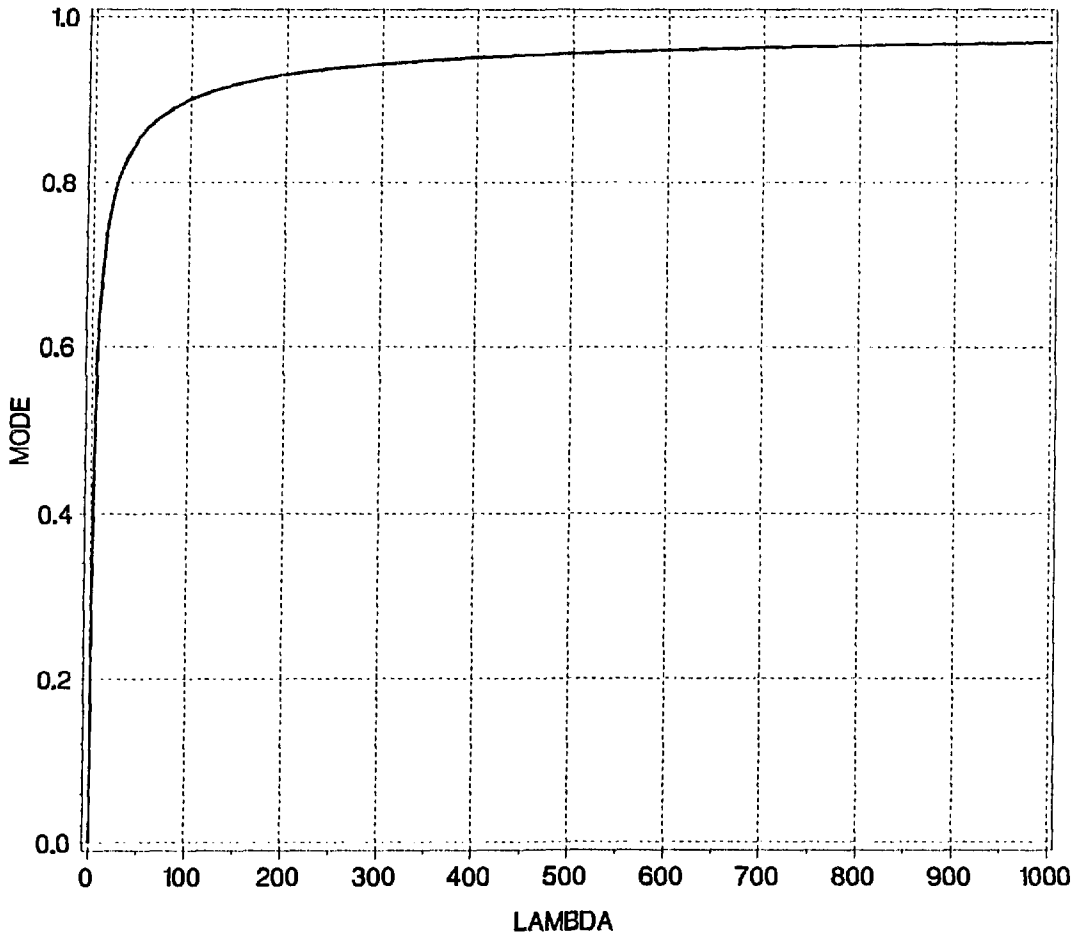


Fig. 4 Mode of Uppuluri distribution,  $1 - \lambda^{-1/2}$ , for  $\lambda$  from 0.01 to 1000.

## MAXIMUM OF UPL PROBABILITY DENSITY FUNCTION

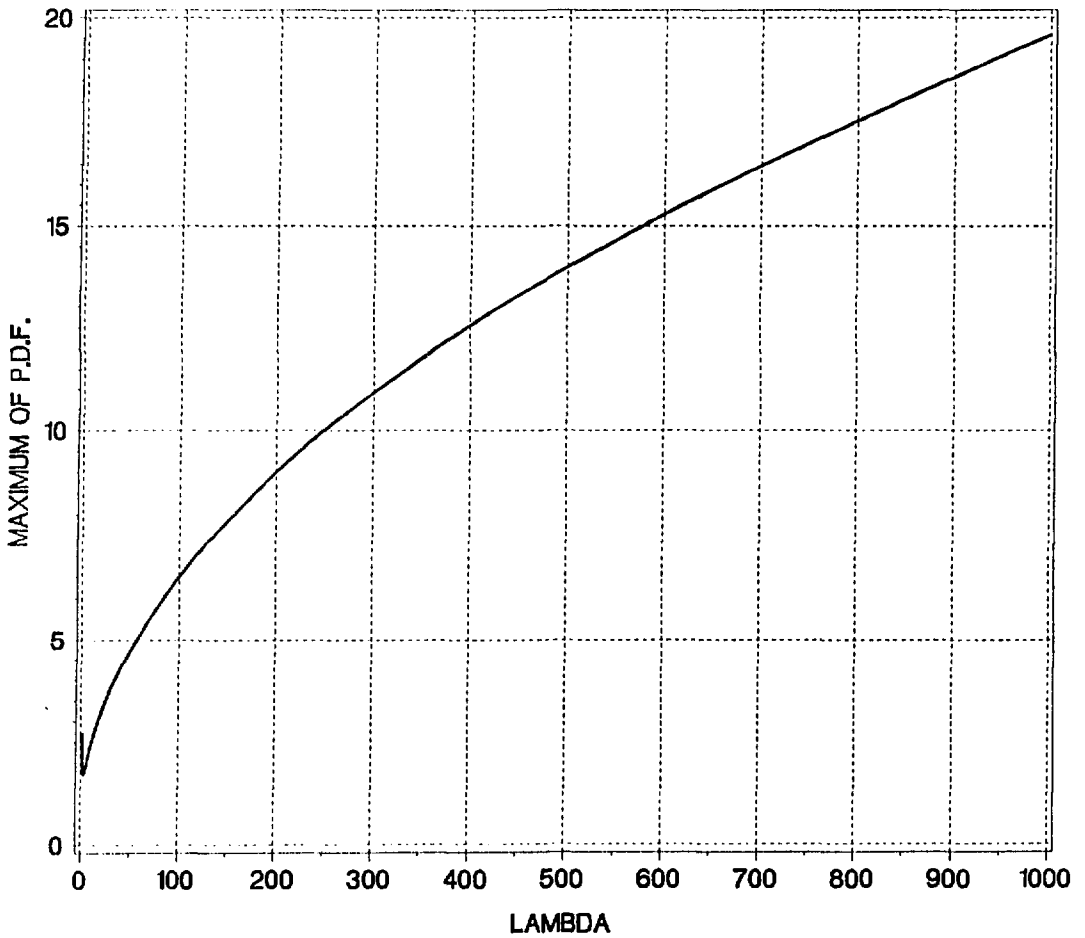


Fig. 5 Maximum of probability density function of Uppuluri's model,  $f(1-\lambda^{-1/2})$ , for  $\lambda$  from 0.01 to 1000.

## MAXIMUM OF UPL PROBABILITY DENSITY FUNCTION

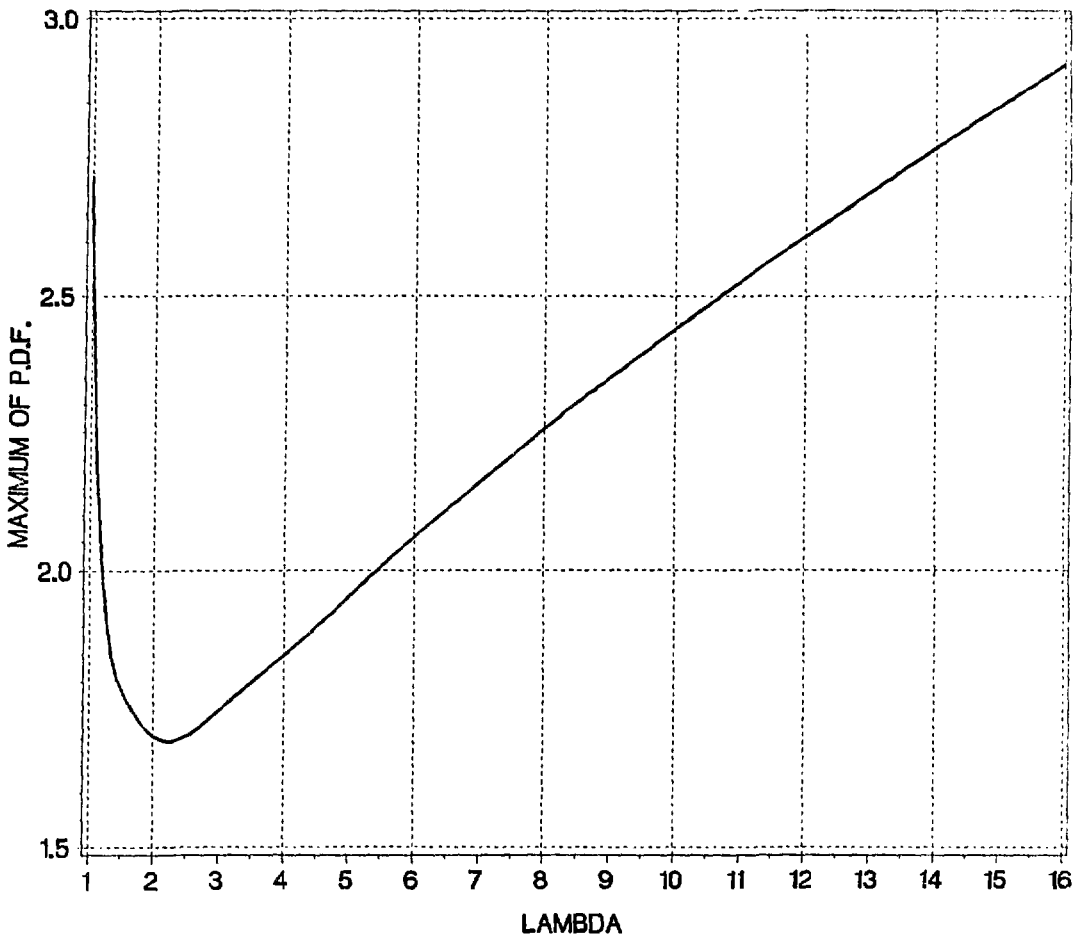


Fig. 6 The enlargement of the graph shown in Figure 5 to see the minimum of the maximum of the probability density function, for  $\lambda$  from 1 to 16.

## MODE OF UPL DISTRIBUTION

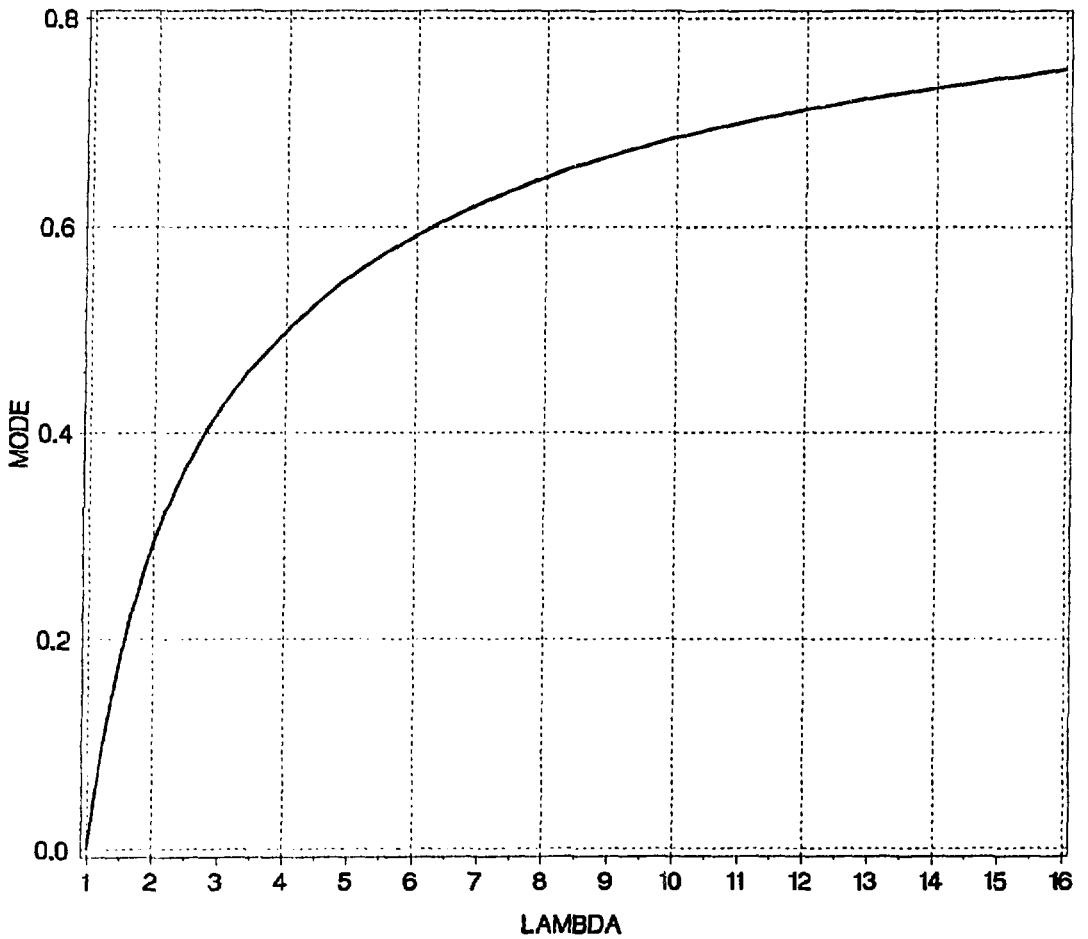


Fig. 7 Mode of Uppuluri distribution  $1 - \lambda^{-1/2}$ , for  $\lambda$  from 1 to 16, corresponding to Figure 6.

## MEAN OF UPL DISTRIBUTION

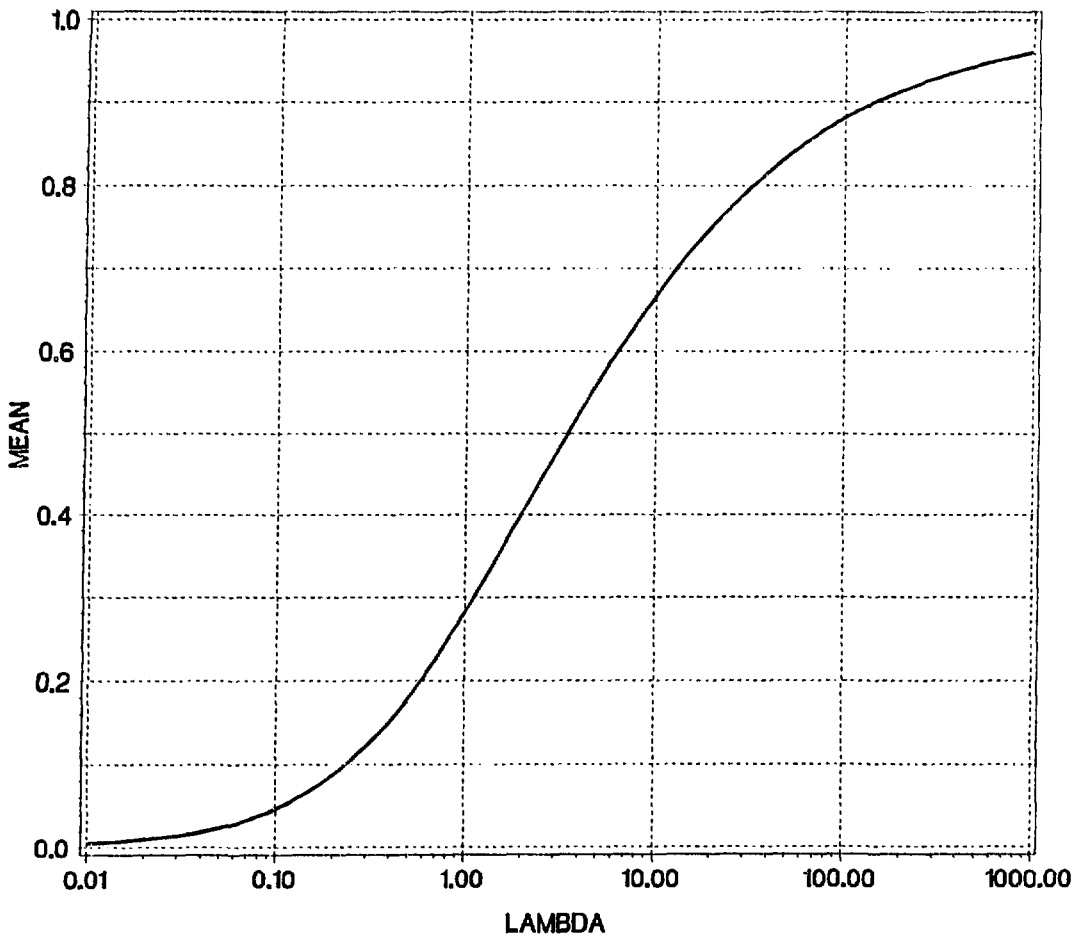


Fig. 8 Mean of Uppuluri distribution for  $\lambda$  from 0.01 to 1000.

## STANDARD DEVIATION OF UPL DISTRIBUTION

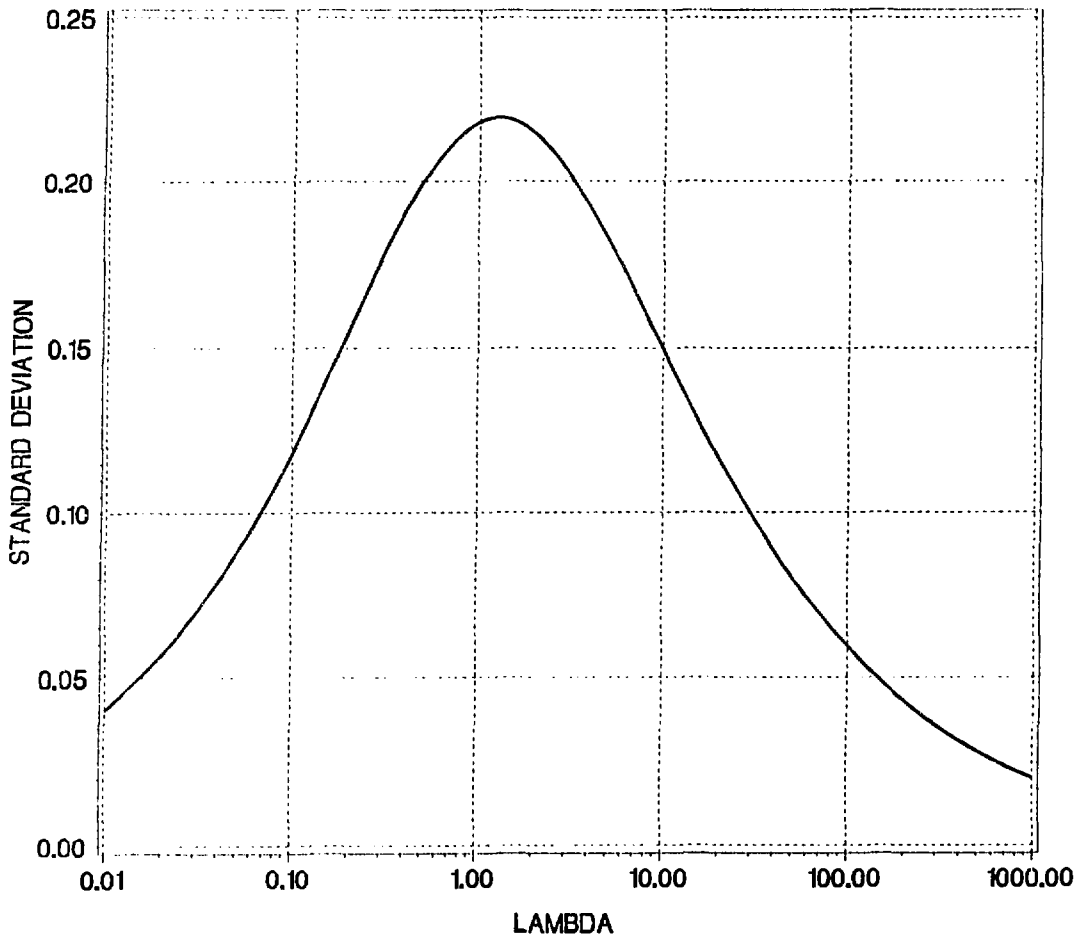


Fig. 9 Standard deviation of Uppuluri distribution for  $\lambda$  from 0.01 to 1000 (log-scale).



## STANDARD DEVIATION OF UPL DISTRIBUTION

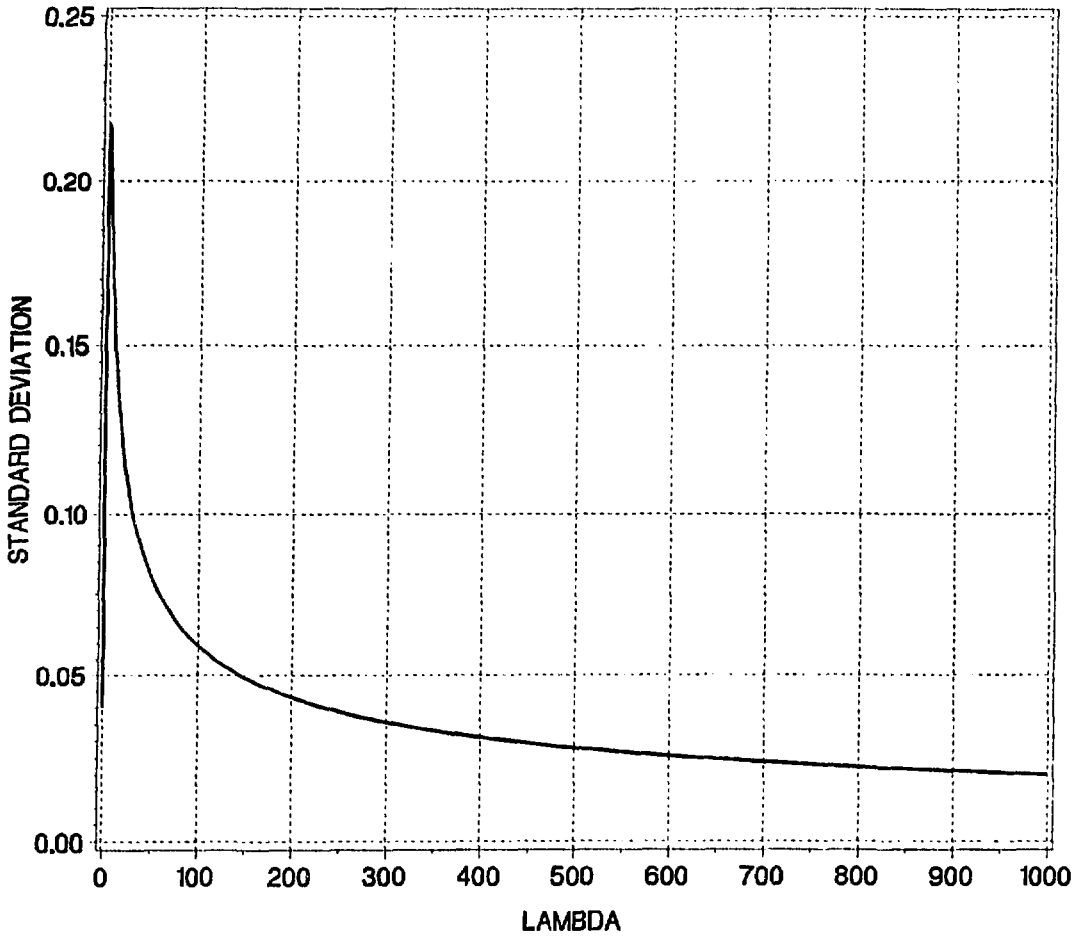


Fig. 10 Standard deviation of Uppuluri distribution for  $\lambda$  from 0.01 to 1000 (linear-scale).

FREQ DISTRIBUTION OF BREEDING VALUE

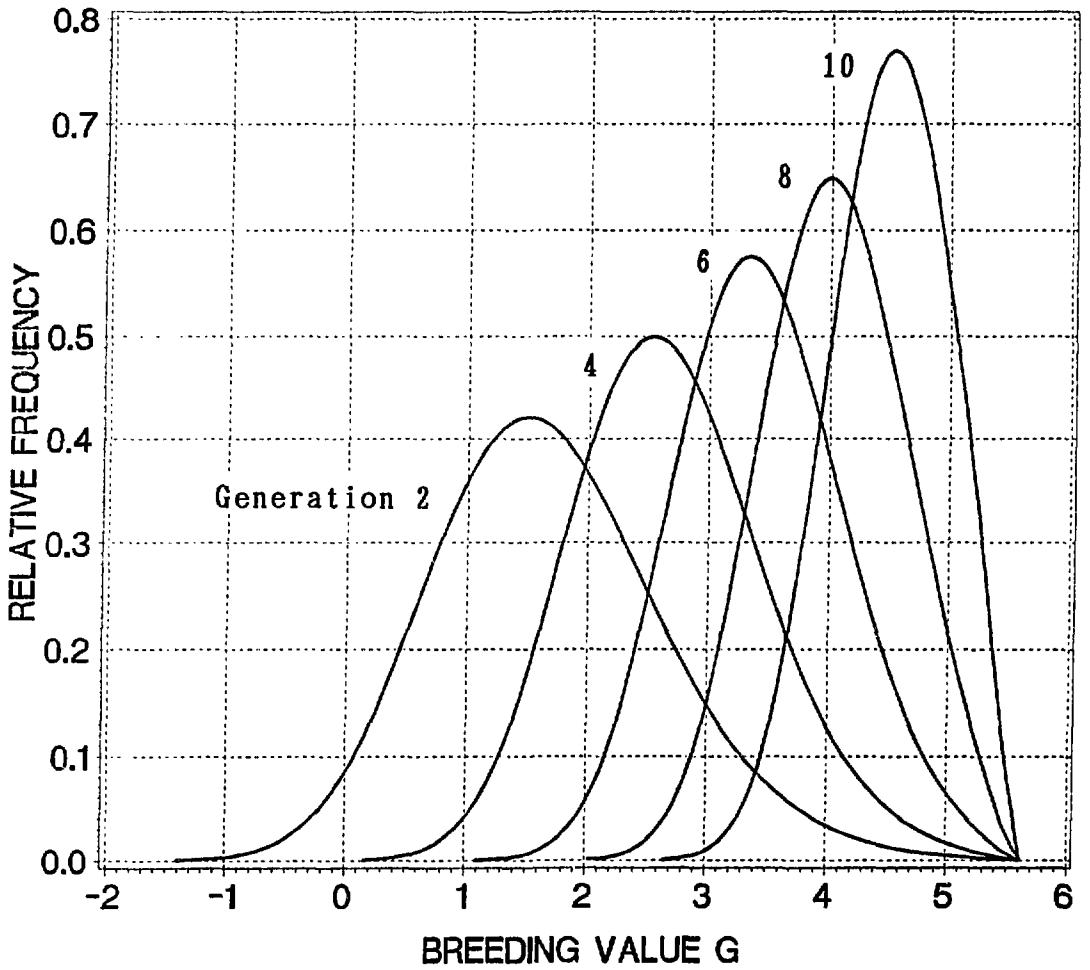


Fig. 11 Change in distribution of breeding value with selection ( $h_0^2=0.5$ ,  $p=0.5$ ). [ref.7]

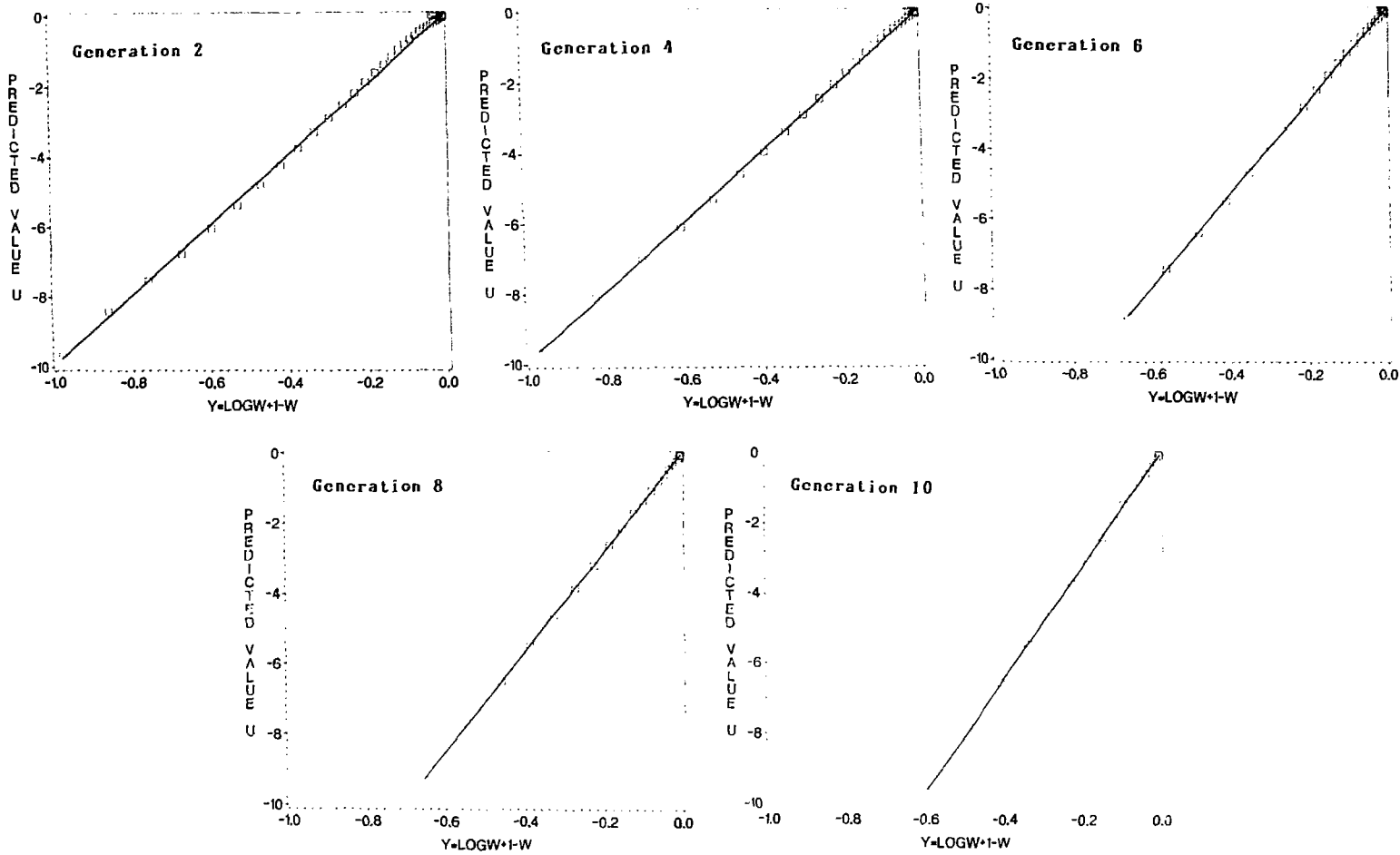


Fig. 12 UPL probability plots of breeding value by generation of selection. Open squares represent given data of breeding value and straight lines are graphs fitted by the UPL model.

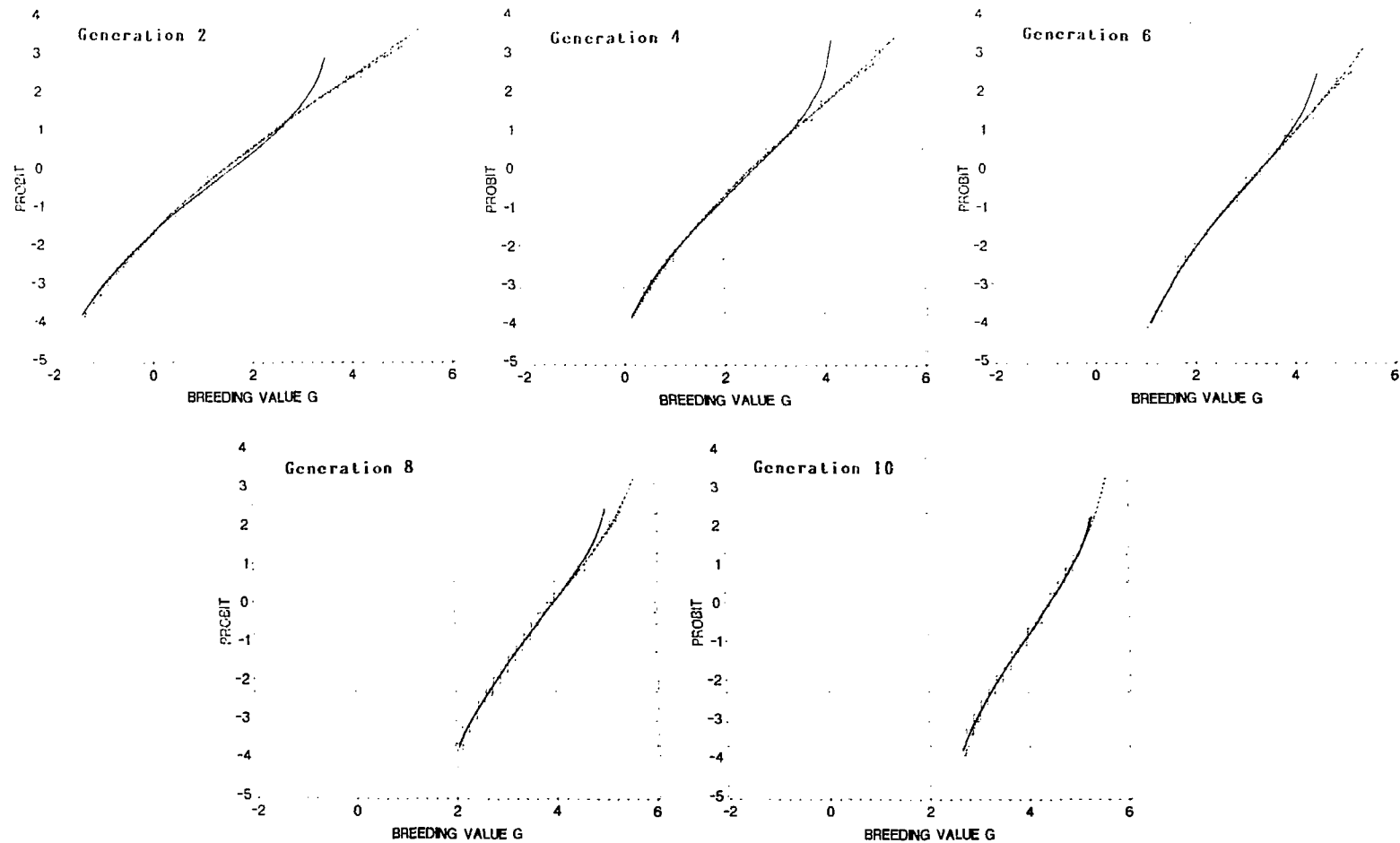


Fig. 13 Normal probability plots of breeding value by generation of selection to compare three types of fitting results among the UPL (solid curve), HLN (dotted curve) and JSB (dashed curve) models. Open squares represent the given data.

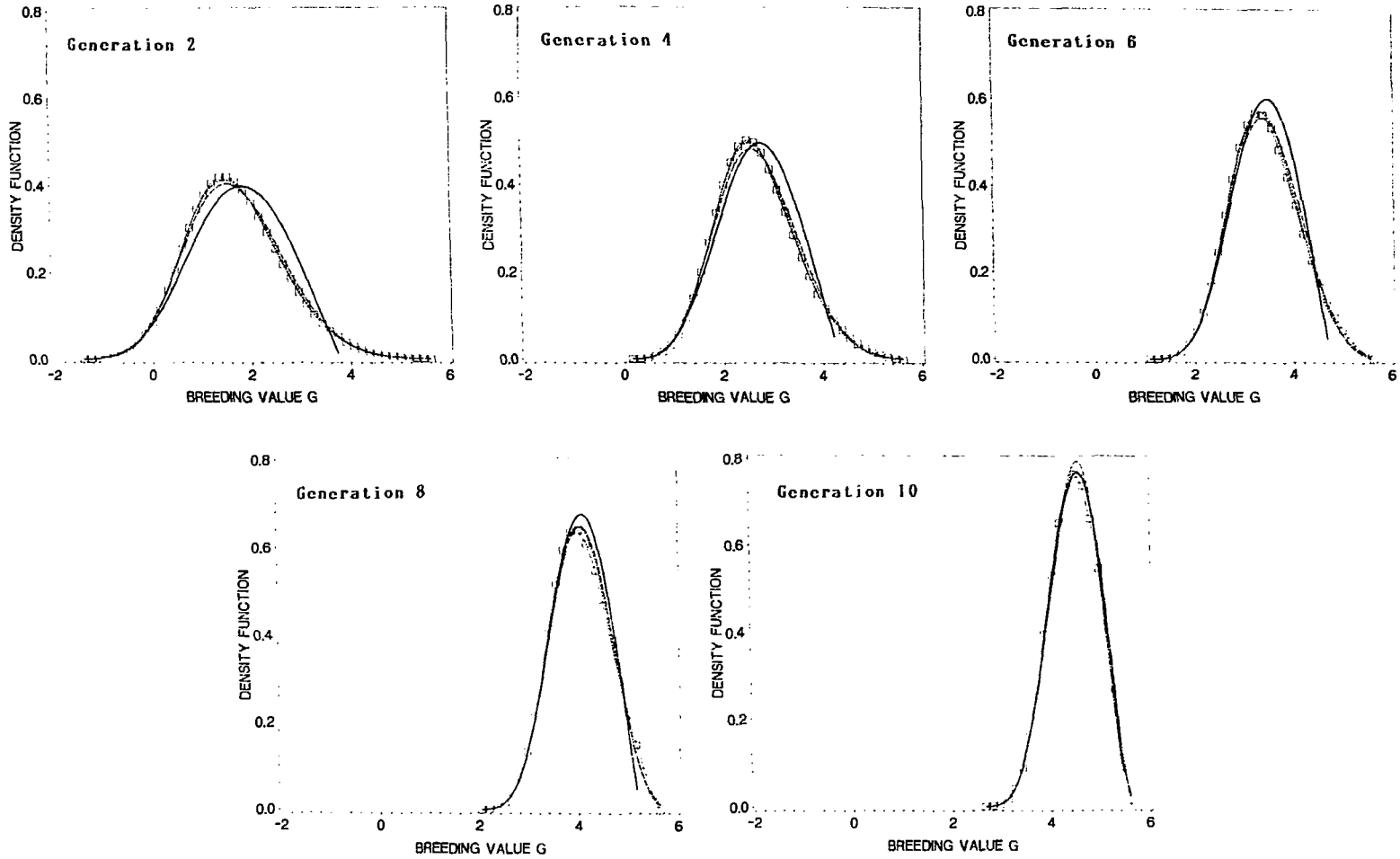


Fig. 14 Probability density functions of breeding value by generation of selection to compare three types of fitting results among the UPL (solid curve), HLN (dotted curve) and JSB (dashed curve) models. Open squares represent the given data.

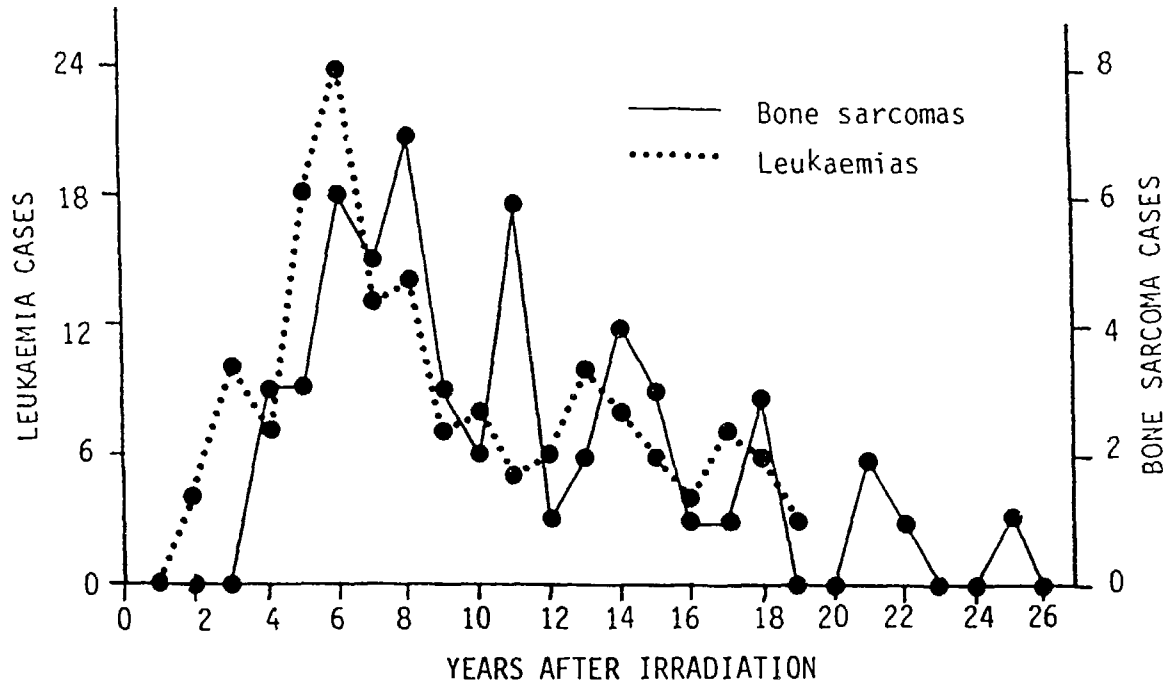


Fig. 15 Bone sarcoma incidence in radium-224 patients and leukemias in the atomic bomb survivors. The distribution of appearance times is remarkably similar for leukemias, following prompt radiation, and for bone sarcomas, following relatively brief radium-224 irradiation. [ref.11]

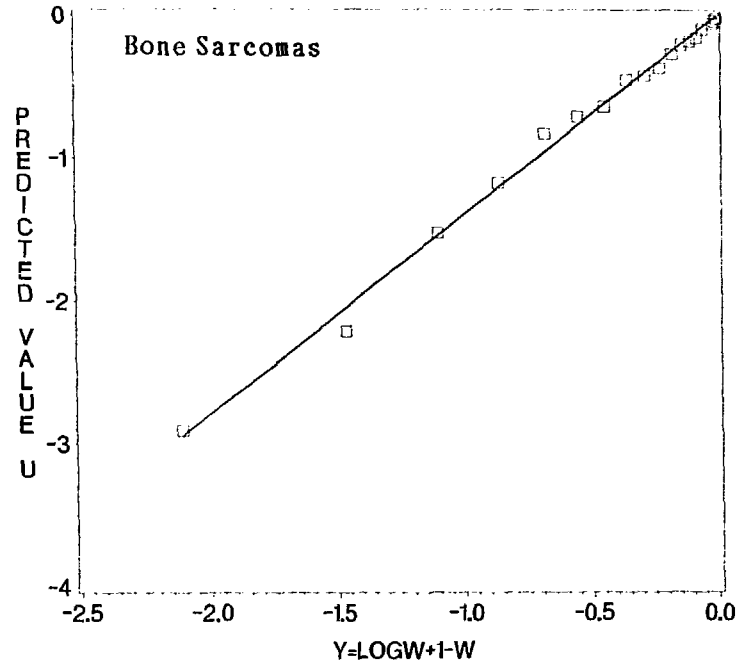
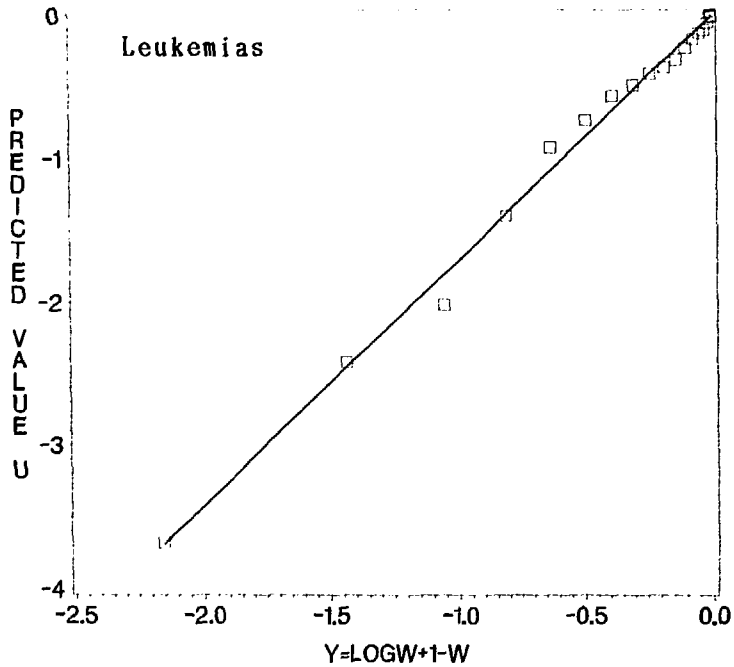


Fig. 16 UPL probability plots of data of the onset of leukemias in the atomic bomb survivors and bone sarcomas in radium-224 patients. Open squares represent the given data and straight lines are graphs fitted by the UPL model.

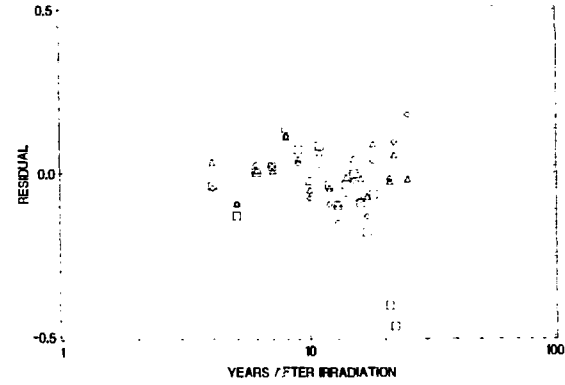
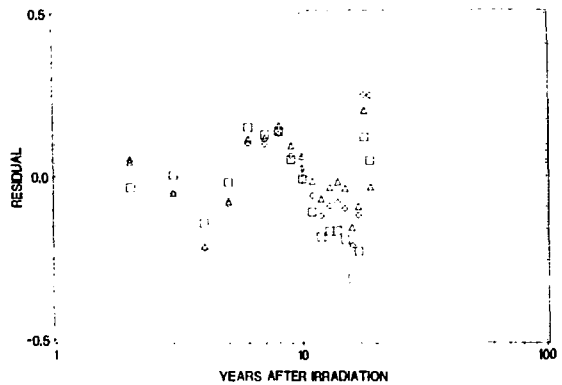
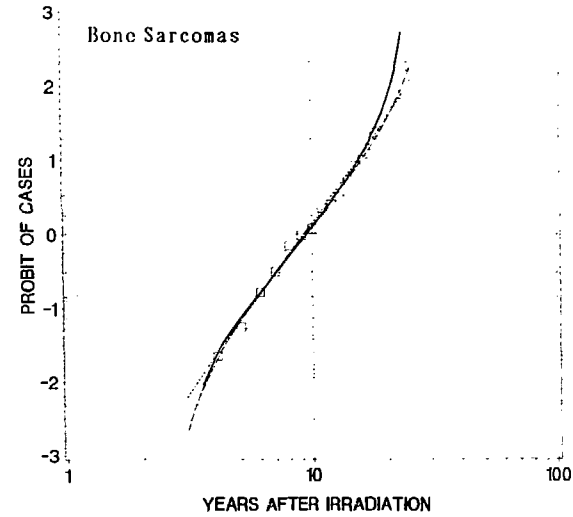
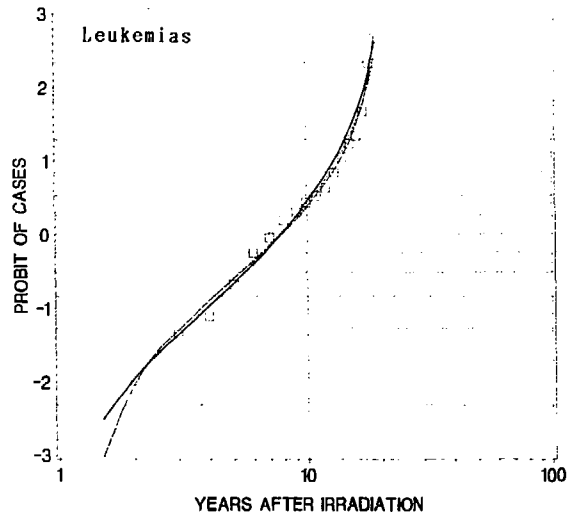


Fig. 17 Normal probability plots of the onset of leukemias and bone sarcomas to compare three types of fitting results among the UPL (solid curve), HLN (dotted curve) and JSB (dashed curve) models. Open squares in upper graphs represent the given data, where in lower graphs squares (UPL), diamonds (HLN) and triangles (JSB).



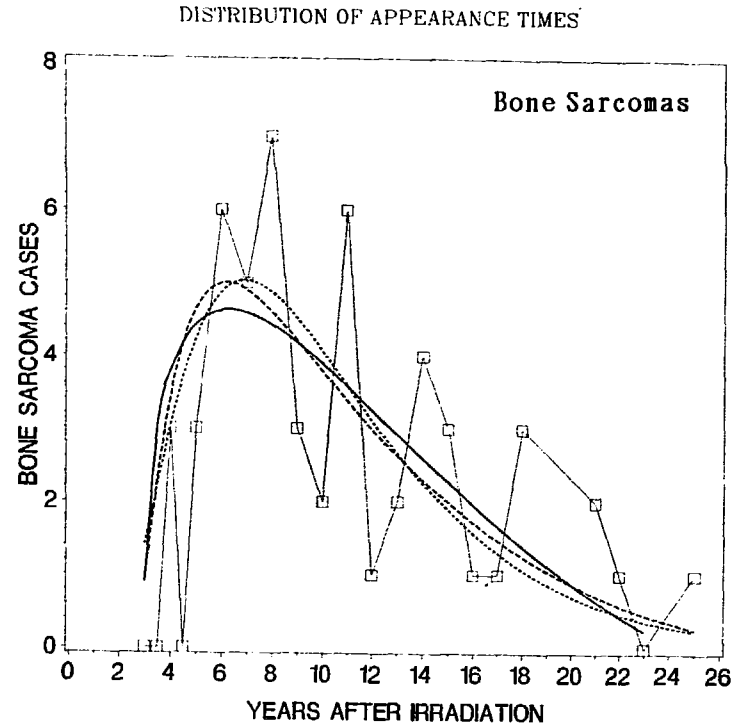
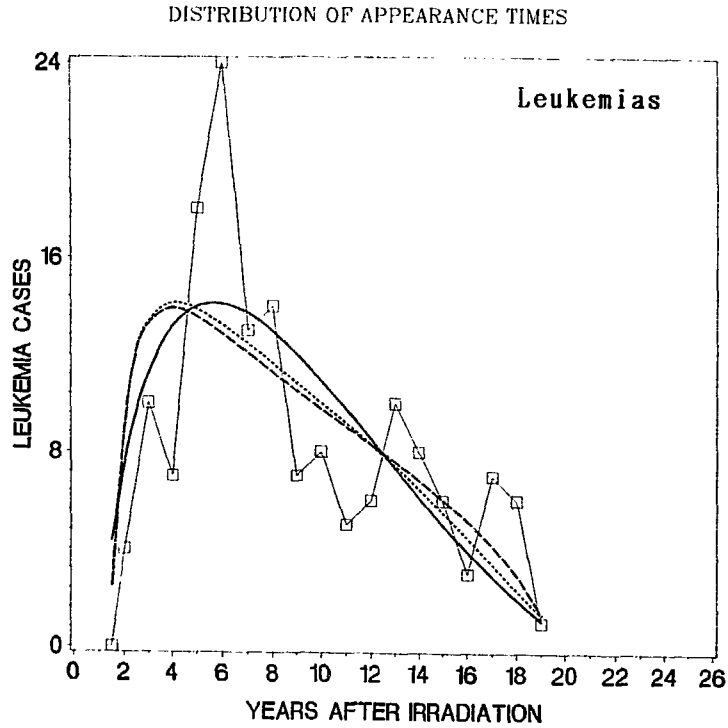


Fig. 18 Leukemia and bone sarcoma incidence (both open square), and their fitting results among UPL (solid curve), HLN (dotted curve) and JSB (dashed curve) models.

## APPENDIX

## SOME FORTRAN FUNCTION SUBPROGRAMS FOR SAS FUNCTIONS

## 1. FORTRAN FUNCTION SUBPROGRAMS

1.1 Inverse co-hybrid function with smaller solution: INCOH2(Y)  
(Median of Uppuluri distribution: INCOH2(1+LOG(2)/λ) )

A function of  $y=w-\ell_n w$  is defined as  $y=\text{coh}(w)$ , and then the inverse co-hybrid function is expressed as  $w=\text{coh}^{-1}(y)$ ,  $y \geq 1$ . There are two solutions  $w_1$  and  $w_2$  ( $w_2 < w_1$ ) for a given value of  $y$  above 1. Kumazawa, et al.<sup>2)</sup> defined the hybrid function of  $y=w+\ell_n w$  as  $y=\text{hyb}(w)$  and its inverse function of  $w=\text{hyb}^{-1}(y)$ ,  $-\infty < y < +\infty$ , whose solution is always determined uniquely. Because of the relationships of  $w=\{\text{hyb}(w)+\text{coh}(w)\}/2$  and  $\ell_n w=\{\text{hyb}(w)-\text{coh}(w)\}/2$ , the name of co-hybrid function has been selected.

The inverse co-hybrid function is required to solve the reference dose distribution<sup>15)</sup>, proposed by the United Nations Scientific Committee on the Effects of Atomic Radiation (UNSCEAR) in 1977, if we use the hybrid lognormal distribution in stead of the lognormal distribution used by UNSCEAR. In this case both solutions are important. However, to calculate the median of Uppuluri distribution, only the smaller solution  $w_2$  ( $0 < w_2 \leq 1$ ) is used.

The inverse co-hybrid function is numerically calculated by the Newton-Raphson method. The starting point was appropriately determined based on the numerical analysis. The relative error is less than  $10^{-6}$ . If one gives a value of  $y$  below 1, there will be an error message. For the value of  $y \geq 1$ , one can obtain a good result.

## 2. MEAN AND VARIANCE OF UPPULURI DISTRIBUTION: UPLAV(λ) AND UPLVR(λ)

The mean and the variance of Uppuluri distribution are numerically calculated by the integral formula of Gauss-Legendre method by changing the integral interval according to the value of parameter  $\lambda$ . The integral intervals for both mean and variance are from 0 to 1 for  $\lambda$  below 100, from 0.5 to 1 for  $\lambda$  above 100 and below 1,000, from 0.8 to 1 for  $\lambda$  above 1,000 and below 5,000, and from 0.93 to 1 for  $\lambda$  above 5,000 but no more than 10,000. The subroutines required to calculate the mean are the Gauss-Legendre integration function subprogram GAUSSA and the

integrand function subprogram WFUNC calculating the integrand function of the first moment of Uppuluri distribution,  $\lambda(1-w)\exp(\lambda(1-w+\log w))$ . The subroutines required to calculate the variance are the function subprograms GAUSSA and WFUNC, and another integrand function subprogram WWFNC calculating the integrand function of the second moment of Uppuluri distribution,  $\lambda w(1-w)\exp(\lambda(1-w+\log w))$ .

### 3. HOW TO USE THESE FUNCTION SUBPROGRAMS

#### 3.1 Median of Uppuluri ditribution UPL(LA)

The median, UMED, is yielded by the INCOH2 function:

$$\text{UMED} = \text{INCOH2}(1+\text{LOG}(2)/\text{LA})$$

#### 3.2 Mean of Uppuluri ditribution UPL(LA)

The mean, UMEAN, is yielded by the UPLAV function:

$$\text{UMEAN} = \text{UPLAV}(\text{LA})$$

#### 3.3 Variance of Uppuluri ditribution UPL(LA)

The variance, UVAR, is yielded by the UPLVR function:

$$\text{UVAR} = \text{UPLVR}(\text{LA})$$

### 4. LIST OF FORTRAN FUNCTION SUBPROGRAMS SUITABLE FOR SAS FUNCTION

- 4.1 Inverse co-hybrid function with smaller solution to calculate the median of Uppuluri distribution
- 4.2 Mean of Uppuluri distribution
- 4.3 Variance of Uppuluri distribution
- 4.4 Gauss-Legendre integration
- 4.5 Integrand function of the first moment of Uppuluri distribution
- 4.6 Integrand function of the second moment of Uppuluri distribution

Using these function subprograms in pure FORTRAN, we delete the statements from "CALL SASFUN" to "... CALL SASFMS".

4.1 Inverse co-hybrid function with smaller solution to calculate the  
median of Uppuluri distribution

```

C      00000100
C      .....00000200
C      .00000300
C      .          FUNCTION INCOH2(ARG)          .00000400
C      .                                MADE BY S. KUMAZAWA 1/31/86 .00000500
C      .                                MODIFIED BY S. KUMAZAWA 8/08/90 .00000600
C      .  ** PURPOSE **                          .00000700
C      .    COMPUTATION OF INVERSED FUNCTION OF Y=T-LN(T) .00000800
C      .    MEDIAN OF UPPULURI DISTRIBUTION : INCOH2(1+LN(2)/LA) .00000900
C      .    UPL P.D.F.: F(W)=LA*W**(LA-1)*(1-W)*EXP(LA*(1-W)) .00001000
C      .                                          .00001100
C      .  ** PARAMETERS **                        .00001200
C      .    ARG - A VALUE OF Y, NOT LESS THAN 1 .00001300
C      .                                          .00001400
C      .  ** REQUIRED SUBROUTINES **              .00001500
C      .                                          .00001600
C      .  ** METHOD OF COMPUTATION **            .00001700
C      .    NEWTON-S ITERATION METHOD IS USED. .00001800
C      .                                          .00001900
C      .....00002000
C      00002100
C      DOUBLE PRECISION FUNCTION INCOH2(ARG)      00002200
C      REAL*8  Y ,T2 ,E ,DEPS,ARG                00002300
C      INTEGER IY                                00002400
C      DATA DEPS/1.D-12/                        00002500
C      EQUIVALENCE (Y,IY)                        00002600
C      Y=ARG                                       00002700
C      CALL SASFUN                                00002800
C      IF(Y.NE.0.0D0) GO TO 1                     00002900
C      IF(IY.NE.0) CALL SASFMS                    00003000
C  1  IF(Y.LT.1.0D0 .OR. Y.GT.1.D50) CALL SASFER 00003100
C      IF ( Y.GT.1.DO ) GO TO 10                  00003200
C      INCOH2=1.DO                               00003300
C      GO TO 80                                  00003400
C  10  IF ( Y - 1.2D0 ) 15 , 15 , 20             00003500
C  15  T2 = 0.7D0                                00003600
C      GO TO 25                                  00003700
C  20  T2 = DEXP(-Y)                             00003800
C      IF ( Y .GE. 27.66D0 ) GO TO 30           00003900
C  25  T2 = T2 - (T2-DLOG(T2)-Y)*T2/(T2-1.DO)   00004000
C      E = 1.DO- (T2-DLOG(T2))/Y                00004100
C      IF ( DABS(E) .GT. DEPS ) GO TO 25        00004200
C  30  INCOH2=T2                                 00004300
C  80  RETURN                                    00004400
C      END                                       00004500

```

4.2 Mean of Uppuluri distribution

```

C                                                     00000100
C .....00000200
C . . . . .00000300
C . . . . .00000400
C . . . . .00000500
C . . . . .00000600
C . . . . .00000700
C . . . . .00000800
C . . . . .00000900
C . . . . .00001000
C . . . . .00001100
C . . . . .00001200
C . . . . .00001300
C . . . . .00001400
C . . . . .00001500
C . . . . .00001600
C . . . . .00001700
C . . . . .00001800
C . . . . .00001900
C .....00002000
C
C DOUBLE PRECISION FUNCTION UPLAV(LA) 00002200
C DOUBLE PRECISION LA,AL,WFUNC 00002300
C INTEGER LL 00002400
C EQUIVALENCE (AL,LL) 00002500
C EXTERNAL WFUNC 00002600
C COMMON/UPL/AL 00002700
C AL=LA 00002800
C CALL SASFUN 00002900
C IF(AL.NE.0.D0) GO TO 1 00003000
C IF(LL.NE.0) CALL SASFMS 00003100
C 1 IF(AL.GE.1D2) GO TO 10 00003200
C CALL GAUSSA(WFUNC,0D0,1D0,48,UPLAV) 00003300
C GO TO 40 00003400
C 10 IF(AL.GE.1D3) GO TO 20 00003500
C CALL GAUSSA(WFUNC,0.5D0,1D0,48,UPLAV) 00003600
C GO TO 40 00003700
C 20 IF(AL.GE.5D3) GO TO 30 00003800
C CALL GAUSSA(WFUNC,0.8D0,1D0,48,UPLAV) 00003900
C GO TO 40 00004000
C 30 CALL GAUSSA(WFUNC,0.93D0,1D0,48,UPLAV) 00004100
C 40 CONTINUE 00004200
C RETURN 00004300
C END 00004400

```

4.3 Variance of Uppuluri distribution

```

C                                                     00000100
C .....00000200
C .
C           FUNCTION UPLVR(LA)                        00000400
C           MADE BY S. KUMAZAWA   8/08/90           00000500
C . ** PURPOSE **                                   00000600
C .   VARIANCE OF UPPULURI PROBABILITY DISTRIBUTION 00000700
C .   P.D.F.: F(W)=LA*W**(LA-1)*(1-W)*EXP(LA*(1-W)) 00000800
C .                                                     00000900
C . ** PARAMETERS **                                00001000
C .   LA - PARAMETER OF UPPULURI DISTRIBUTION < W UPL(LA) > 00001100
C .                                                     00001200
C . ** REQUIRED SUBROUTINES **                       00001300
C .   GAUSSA-GAUSS LEGENDRE INTEGRATION            00001400
C .   WFUNC -W*F(W)                                00001500
C .   WWFNC -W*W*F(W)                              00001600
C .                                                     00001700
C . ** METHOD OF COMPUTATION **                     00001800
C .   GAUSS-LEGENDRE INTERATION METHODD IS USED.    00001900
C .                                                     00002000
C .....00002100
C
DOUBLE PRECISION FUNCTION UPLVR(LA)                00002200
DOUBLE PRECISION LA,AL,WFUNC,WWFNC,UPLAV,UPLVV    00002300
INTEGER LL                                         00002400
EQUIVALENCE (AL,LL)                              00002500
EXTERNAL WFUNC                                    00002600
EXTERNAL WWFNC                                    00002700
COMMON /UPL/AL                                    00002800
AL=LA                                              00002900
CALL SASFUN                                       00003000
IF(AL.NE.0.DO) GO TO 1                            00003100
IF(LL.NE.0) CALL SASFMS                           00003200
1 IF(AL.GE.1D2) GO TO 10                          00003300
CALL GAUSSA(WFUNC,0D0,1D0,48,UPLAV)              00003400
CALL GAUSSA(WWFNC,0D0,1D0,48,UPLVV)              00003500
GO TO 40                                          00003600
10 IF(AL.GE.1D3) GO TO 20                         00003700
CALL GAUSSA(WFUNC,0.5D0,1D0,48,UPLAV)            00003800
CALL GAUSSA(WWFNC,0.5D0,1D0,48,UPLVV)            00003900
GO TO 40                                          00004000
20 IF(AL.GE.5D3) GO TO 30                         00004100
CALL GAUSSA(WFUNC,0.8D0,1D0,48,UPLAV)            00004200
CALL GAUSSA(WWFNC,0.8D0,1D0,48,UPLVV)            00004300
GO TO 40                                          00004400
30 CALL GAUSSA(WFUNC,0.93D0,1D0,48,UPLAV)        00004500
CALL GAUSSA(WWFNC,0.94D0,1D0,48,UPLVV)          00004600
40 UPLVR=UPLVV-UPLAV*UPLAV                        00004700
RETURN                                           00004800
END                                               00004900
00005000

```

4.4 Gauss-Legendre integration

```

C .....00000100
C . .00000200
C . SUBROUTINE GAUSSA(FUNC,A,B,N,S) .00000300
C . .00000400
C . ** PURPOSE ** .00000500
C . INTEGRAL OF A FUNCTION OF FUNC IN A FINITE INTERVAL FROM .00000600
C . A TO B BY GAUSS-LEGENDRE INTEGRATION METHOD .00000700
C . .00000800
C . ** PARAMETERS ** .00000900
C . LA - PARAMETER OF UPPULURI DISTRIBUTION < W UPL(LA) > .00001000
C . FUNC - A DOUBLE PRECISION FUNCTION TO BE INTEGRATED .00001100
C . A - LOWER LIMIT OF INTEGRAL(DOUBLE PRECISION) .00001200
C . B - UPPER LIMIT OF INTEGRAL(DOUBLE PRECISION) .00001300
C . N - NUMBER OF NODE POINTS FOR GAUSS LEGENDRE INTEGRATION .00001400
C . N MUST BE 24 OR 48. ON THE CONTRARY N WILL BE 48 .00001500
C . S - RESULTANT INTEGRAL. . 00001600
C . .00001700
C . ** REQUIRED SUBROUTINES ** .00001800
C . .00001900
C . ** METHOD OF COMPUTATION ** .00002000
C . GAUSS-LEGENDRE INTERATION METHOD /REF.:JAERI-M-8479(1979).00002100
C . .00002200
C .....00002300
C .....00002400
C REAL*8 A,B,S, FUNC, Z24(24), W24(24), Z48(48), W48(48), Z(48), W(48) 00002500
C DATA Z24/ 00002600
C 1 12*0.000 / 0.064056892862605626085D0 / 00002700
C 2 0.191118867473616309159D0 / 0.315042679696163374387D0 / 00002800
C 3 0.433793507626045138437D0 / 0.545421471388839535638D0 / 00002900
C 4 0.648093651936975569252D0 / 0.740124191578554364244D0 / 00003000
C 5 0.820001985973902921954D0 / 0.886415527004401034213D0 / 00003100
C 6 0.938274552002732758524D0 / 0.974728555971309498198D0 / 00003200
C 7 0.995187219997021360180D0 / 00003300
C DATA W24/ 00003400
C 1 12*0.000 / 0.127938195346752156974D0 / 00003500
C 2 0.125837456346828296121D0 / 0.121670472927803391204D0 / 00003600
C 3 0.115505668053725601353D0 / 0.107444270115965634783D0 / 00003700
C 4 0.097618652104113888270D0 / 0.086190161531953275917D0 / 00003800
C 5 0.073346481411080305734D0 / 0.059298584915436780746D0 / 00003900
C 6 0.044277438817419806169D0 / 0.028531388628933663181D0 / 00004000
C 7 0.012341229799987199547D0 / 00004100
C DATA Z48/ 00004200
C 1 24*0.000 / 0.032380170962869362033D0 / 00004300
C 2 0.097004699209462698930D0 / 0.161222356068891718056D0 / 00004400
C 3 0.224763790394689061225D0 / 0.287362487355455576736D0 / 00004500
C 4 0.348755886292160738160D0 / 0.408686481990716729916D0 / 00004600
C 5 0.466902904750958404545D0 / 0.523160974722233033678D0 / 00004700
C 6 0.577224726083972703818D0 / 0.628867396776513623995D0 / 00004800
C 7 0.677872379632663905210D0 / 0.724034130923814654674D0 / 00004900
C 8 0.767159032515740339254D0 / 0.807066204029442627083D0 / 00005000
C 9 0.843588261624393530711D0 / 0.876572020274247885906D0 / 00005100
C A 0.905879136715569672822D0 / 0.931386690706554333114D0 / 00005200
C B 0.952987703160430860723D0 / 0.970591592546247250461D0 / 00005300
C C 0.984124583722826857745D0 / 0.993530172266350757548D0 / 00005400
C D 0.998771007252426118601D0 / 00005500
C DATA W48/ 00005600

```

```

1      24*0.0D0      , 0.064737696812683922503D0 , 00005700
2      0.064466164435950082207D0 , 0.063924238584648186624D0 , 00005800
3      0.063114192286254025657D0 , 0.062039423159892663904D0 , 00005900
4      0.060704439165893880053D0 , 0.059114839698395635746D0 , 00006000
5      0.057277292100403215705D0 , 0.055199503699984162868D0 , 00006100
6      0.052890189485193667096D0 , 0.050359035553854474958D0 , 00006200
7      0.047616658492490474826D0 , 0.044674560856694280419D0 , 00006300
8      0.041545082943464749214D0 , 0.038241351065830706317D0 , 00006400
9      0.034777222564770438893D0 , 0.031167227832798088902D0 , 00006500
A      0.027426509708356948200D0 , 0.023570760839324379141D0 , 00006600
B      0.019616160457355527814D0 , 0.015579315722943848728D0 , 00006700
C      0.011477234579234539490D0 , 0.007327553901276262102D0 , 00006800
D      0.003153346052305838633D0 /      00006900
IF(N,NE,24) GO TO 10      00007000
DO 5 I=13,24      00007100
Z(I) =Z24(I)      00007200
5 W(I) =W24(I)      00007300
NN=12      00007400
GO TO 20      00007500
10 DO 15 I=25,48      00007600
Z(I) =Z48(I)      00007700
15 W(I) =W48(I)      00007800
NN=24      00007900
20 NNN=NN*2      00008000
DO 25 I=1,NN      00008100
II=NNN-I+1      00008200
Z(I) =-Z(II)      00008300
25 W(I) =+W(II)      00008400
C      00008500
S = 0.0D0      00008600
DO 30 I=1,NNN      00008700
S = S + W(I)*FUNC((B-A)*Z(I)*0.5D0 + (B+A)*0.5D0)      00008800
30 CONTINUE      00008900
S = S*(B-A)*0.5D0      00009000
RETURN      00009100
END      00009200

```



## 4.5 Integrand function of the first moment of Uppuluri distribution

```

C                                                     00000100
C .....00000200
C . . . . .00000300
C . . . . .FUNCTION WFUNC(W) .00000400
C . . . . .00000500
C . . . . .** PURPOSE ** .00000600
C . . . . .PRODUCT OF W AND F(W) : W AND F(W) ARE, RESPECTIVELY, A .00000700
C . . . . .VARIATE AND P.D.F. OF UPPULURI PROBABILITY DISTRIBUTION .00000800
C . . . . .P.D.F.: F(W)=LA*W**(LA-1)*(1-W)*EXP(LA*(1-W)) .00000900
C . . . . .00001000
C . . . . .** PARAMETERS ** .00001100
C . . . . .W - A POINT OF UPPULURI DISTRIBUTION .00001200
C . . . . .AL - PARAMETER OF UPPULURI DISTRIBUTION, LA .00001300
C . . . . .00001400
C . . . . .** REQUIRED SUBROUTINES ** .00001500
C . . . . .00001600
C . . . . .** METHOD OF COMPUTATION ** .00001700
C . . . . .00001800
C .....00001900
C . . . . .00002000
C DOUBLE PRECISION FUNCTION WFUNC(W) 00002100
C DOUBLE PRECISION AL,W,YY 00002200
C COMMON /UPL/AL 00002300
C YY=AL*(1.DO-W+DLOG(W)) 00002400
C WFUNC=AL*(1.DO-W)*DEXP(YY) 00002500
C RETURN 00002600
C END 00002700

```

## 4.6 Integrand function of the second moment of Uppuluri distribution

```

C .....00000100
C . .00000200
C . FUNCTION WWFNC(W) .00000300
C . .00000400
C . ** PURPOSE ** .00000500
C . PRODUCT OF W**2 AND F(W) : W AND F(W) ARE, RESPECTIVELY, .00000600
C . A VARIATE AND P.D.F. OF UPPULURI PROBABILITY DISTRIBUTION.00000700
C . P.D.F.: F(W)=LA*W**(LA-1)*(1-W)*EXP(LA*(1-W)) .00000800
C . .00000900
C . ** PARAMETERS ** .00001000
C . W - A POINT OF UPPULURI DISTRIBUTION .00001100
C . AL - PARAMETER OF UPPULURI DISTRIBUTION,LA .00001200
C . .00001300
C . ** REQUIRED SUBROUTINES ** .00001400
C . .00001500
C . ** METHOD OF COMPUTATION ** .00001600
C . .00001700
C .....00001800
C 00001900
DOUBLE PRECISION FUNCTION WWFNC(W) 00002000
DOUBLE PRECISION AL,W,YY 00002100
COMMON /UPL/AL 00002200
YY=AL*(1.DO-W+DLOG(W)) 00002300
WWFNC=AL*W*(1.DO-W)*DEXP(YY) 00002400
RETURN 00002500
END 00002600

```

# 国際単位系 (SI) と換算表

表1 SI基本単位および補助単位

量	名称	記号
長さ	メートル	m
質量	キログラム	kg
時間	秒	s
電流	アンペア	A
熱力学温度	ケルビン	K
物質	モル	mol
光度	カンデラ	cd
平面角	ラジアン	rad
立体角	ステラジアン	sr

表2 SIと併用される単位

名称	記号
分、時、日	min, h, d
度、分、秒	°, ', "
リットル	l, L
トン	t
電子ボルト	eV
原子質量単位	u

$1 \text{ eV} = 1.60218 \times 10^{-19} \text{ J}$   
 $1 \text{ u} = 1.66054 \times 10^{-27} \text{ kg}$

表5 SI接頭語

倍数	接頭語	記号
$10^{18}$	エクサ	E
$10^{15}$	ペタ	P
$10^{12}$	テラ	T
$10^9$	ギガ	G
$10^6$	メガ	M
$10^3$	キロ	k
$10^2$	ヘクト	h
$10^1$	デカ	da
$10^{-1}$	デシ	d
$10^{-2}$	センチ	c
$10^{-3}$	ミリ	m
$10^{-6}$	マイクロ	$\mu$
$10^{-9}$	ナノ	n
$10^{-12}$	ピコ	p
$10^{-15}$	フェムト	f
$10^{-18}$	アト	a

表3 固有の名称をもつSI組立単位

量	名称	記号	他のSI単位による表現
周波数	ヘルツ	Hz	$\text{s}^{-1}$
力	ニュートン	N	$\text{m} \cdot \text{kg} / \text{s}^2$
圧力、応力	パスカル	Pa	$\text{N} / \text{m}^2$
エネルギー、仕事、熱量	ジュール	J	$\text{N} \cdot \text{m}$
工率、放射束	ワット	W	$\text{J} / \text{s}$
電気量、電荷	クーロン	C	$\text{A} \cdot \text{s}$
電位、電圧、起電力	ボルト	V	$\text{W} / \text{A}$
静電容量	ファラド	F	$\text{C} / \text{V}$
電気抵抗	オーム	$\Omega$	$\text{V} / \text{A}$
コンダクタンス	ジーメンズ	S	$\text{A} / \text{V}$
磁束	ウェーバ	Wb	$\text{V} \cdot \text{s}$
磁束密度	テスラ	T	$\text{Wb} / \text{m}^2$
インダクタンス	ヘンリー	H	$\text{Wb} / \text{A}$
セルシウス温度	セルシウス度	$^{\circ}\text{C}$	
光照度	ルーメン	lm	$\text{cd} \cdot \text{sr}$
照射度	ルクス	lx	$\text{lm} / \text{m}^2$
放射能	ベクレル	Bq	$\text{s}^{-1}$
吸収線量	グレイ	Gy	$\text{J} / \text{kg}$
線量当量	シーベルト	Sv	$\text{J} / \text{kg}$

表4 SIと共に暫定的に維持される単位

名称	記号
オングストローム	$\text{\AA}$
バ	b
バール	bar
ガリ	Gal
キュリー	Ci
レントゲン	R
ラド	rad
レム	rem

$1 \text{ \AA} = 0.1 \text{ nm} = 10^{-10} \text{ m}$   
 $1 \text{ b} = 100 \text{ fm} = 10^{-26} \text{ m}^2$   
 $1 \text{ bar} = 0.1 \text{ MPa} = 10^5 \text{ Pa}$   
 $1 \text{ Gal} = 1 \text{ cm} / \text{s}^2 = 10^{-2} \text{ m} / \text{s}^2$   
 $1 \text{ Ci} = 3.7 \times 10^{10} \text{ Bq}$   
 $1 \text{ R} = 2.58 \times 10^{-4} \text{ C} / \text{kg}$   
 $1 \text{ rad} = 1 \text{ cGy} = 10^{-2} \text{ Gy}$   
 $1 \text{ rem} = 1 \text{ cSv} = 10^{-2} \text{ Sv}$

(注)

- 表1 5は「国際単位系」第5版、国際度量衡局1985年刊行による。ただし、1eVおよび1uの値はCODATAの1986年推奨値によった。
- 表4には海里、ノット、アール、ヘクタールも含まれているが日常の単位としてここでは省略した。
- barは、JISでは流体の圧力を表す場合に限り表2のカテゴリに分類されている。
- EC閣僚理事会指令ではbar, barnおよび「血圧の単位」mmHgを表2のカテゴリに入れている。

## 換算表

力	N (=10 <sup>5</sup> dyn)	kgf	lbf
	1	0.101972	0.224809
	9.80665	1	2.20462
	4.44822	0.453592	1

粘 度 1 Pa·s (=N·s/m<sup>2</sup>) = 10 P (ポアズ) (g/(cm·s))  
 動粘度 1 m<sup>2</sup>/s = 10<sup>4</sup> St (ストークス) (cm<sup>2</sup>/s)

圧	MPa (=10 bar)	kgf/cm <sup>2</sup>	atm	mmHg (Torr)	lbf/in <sup>2</sup> (psi)
	1	10.1972	9.86923	7.50062 × 10 <sup>1</sup>	145.038
力	0.0980665	1	0.967841	735.559	14.2233
	0.101325	1.03323	1	760	14.6959
	1.33322 × 10 <sup>-4</sup>	1.35951 × 10 <sup>-3</sup>	1.31579 × 10 <sup>-3</sup>	1	1.93368 × 10 <sup>-2</sup>
	6.89476 × 10 <sup>-3</sup>	7.03070 × 10 <sup>-2</sup>	6.80460 × 10 <sup>-2</sup>	51.7149	1

エネルギー・仕事・熱量	J (=10 <sup>7</sup> erg)	kgf·m	kW·h	cal (計量法)	Btu	ft·lbf	eV
	1	0.101972	2.77778 × 10 <sup>-7</sup>	0.238889	9.47813 × 10 <sup>-4</sup>	0.737562	6.24150 × 10 <sup>18</sup>
	9.80665	1	2.72407 × 10 <sup>-6</sup>	2.34270	9.29487 × 10 <sup>-3</sup>	7.23301	6.12082 × 10 <sup>19</sup>
	3.6 × 10 <sup>6</sup>	3.67098 × 10 <sup>5</sup>	1	8.59999 × 10 <sup>5</sup>	3412.13	2.65522 × 10 <sup>6</sup>	2.24694 × 10 <sup>25</sup>
	4.18605	0.426858	1.16279 × 10 <sup>-6</sup>	1	3.96759 × 10 <sup>-3</sup>	3.08747	2.61272 × 10 <sup>19</sup>
	1055.06	107.586	2.93072 × 10 <sup>-4</sup>	252.042	1	778.172	6.58515 × 10 <sup>21</sup>
	1.35582	0.138255	3.76616 × 10 <sup>-7</sup>	0.323890	1.28506 × 10 <sup>-3</sup>	1	8.46233 × 10 <sup>18</sup>
	1.60218 × 10 <sup>-19</sup>	1.63377 × 10 <sup>-20</sup>	4.45050 × 10 <sup>-26</sup>	3.82743 × 10 <sup>-20</sup>	1.51857 × 10 <sup>-22</sup>	1.18171 × 10 <sup>-19</sup>	1

1 cal = 4.18605 J (計量法)  
 = 4.184 J (熱化学)  
 = 4.1855 J (15 °C)  
 = 4.1868 J (国際蒸気表)  
 仕事率 1 PS (馬馬力)  
 = 75 kgf·m/s  
 = 735.499 W

放射能	Bq	Ci
	1	2.70270 × 10 <sup>-11</sup>
	3.7 × 10 <sup>10</sup>	1

吸収線量	Gy	rad
	1	100
	0.01	1

照射線量	C/kg	R
	1	3876
	2.58 × 10 <sup>-4</sup>	1

線量当量	Sv	rem
	1	100
	0.01	1